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THE AVERAGE ORDERS OF HOOLEY'S Δ_r-FUNCTIONS, II

R.R. Hall and G. Tenenbaum

Introduction

In this paper we determine, to within a factor $(\log n)^{0(1)}$, the average order of the arithmetical function

$$\Delta_r(n) := \max_{u_1,\dots,u_r-1} \sum_{d_1\dots d_{r-1}\mid n} \{1 : u_i < \log d_i \leqq u_i + 1 \text{ for } 1 \leqq i \leqq r\}$$

introduced by Hooley [4] and shown by him to be relevant in the study of Waring's Problem and in certain problems of Diophantine Approximation. It is usual to write Δ instead of Δ_2 and set $\Delta_1 \equiv 1$.

One of us [9] has recently pointed out that the ratio $\delta(n)/\tau(n)$ may be interpreted as a concentration function in the sense of Paul Lévy (1937) and the observation may be extended to $\Delta_r(n)/\tau_r(n)$. Here $\tau_r(n)$ denotes the number of ways of writing n as the product of r factors, $\tau = \tau_2$). In fact such problems in the subject area which has been called the propinquity of divisors [1], [2] are more than 45 years old, dating at least as far back as Erdös'notorious conjecture that almost all integers have divisors d, d' such that $d < d' < 2d$. This has only recently been settled by Maier and Tenenbaum [7].

We prove the following theorems, generalizing the recent result of Tenenbaum [9] concerning the case $r = 2$.

THEOREM 2: *Put* $\gamma_r = (r-1)\sqrt{(\frac{r(r+2)}{2})}$. *We have for* $r \geqq 2$ *and* $x \geqq 16$

$$\sum_{N \leqq x} \Delta_r(N) \ll_r x \exp\left\{\left(\gamma_r + \frac{18}{\log_3 x}\right)\sqrt{(\log_2 x \log_3 x)}\right\}.$$

Throughout the paper we write $\log_2 x$, $\log_3 x$ instead of loglog x, logloglog x. The constant implied by Vinogradov's notation $\ll$ is absolute, or, if there are suscripts, depends at most on these.

If, for instance, we insert the estimate of Theorem 1 with $r = 3$ in Hooley's upper bound method [4] for the number $r_8(n)$ of representations of the integer n as a sum of eight positive cubes, we obtain

$$r_8(n) \leqq n^{5/3} \exp\{\sqrt{((30+\epsilon)\log_2 n \log_3 n)}\}, (n \geqq n_0(\epsilon)).$$

This falls relatively close to the expected true order of magnitude $n^{5/3}$.

Let $\omega(N)$ denote the number of distinct prime factors of N. We now turn our attention to weighted versions of the sum considered in Theorem 1, namely we consider

$$S_r(x, y) = \sum_{N \leqslant x} \Delta_r(N) y^{\omega(N)}$$

for $y > 0$. This generalization appears genuinely in some applications and, taking advantage of the drastic variations of the weights, enables one pratically to evaluate the sum of $\Delta_r(N)$ restricted to those integers $N \leqslant x$ having a prescribed number of prime factors. First, we note that Theorem 1 has the following

COROLLARY: *For $r \geqslant 2$, $x \geqslant 16$ and $y \geqslant 1$ we have*

$$\sum_{N \leqslant x} \Delta_r(N) y^{\omega(N)}$$

$$\ll_{r,y} x(\log x)^{ry-r} \exp\left\{\left(\gamma_r + \frac{18}{\log_3 x}\right)\sqrt{(\log_2 x \log_3 x)}\right\}.$$

This follows very simply from Theorem 1 if we remark that

$$y^{\omega(N)} = \sum_{d \mid N} \mu^2(d)(y-1)^{\omega(d)},$$

$$\Delta_r(md) \leqslant \Delta_r(m)\tau_r(d), \quad (m, d \geqslant 1)$$

(the inequality extends (23) of [9]), just inverting summations.

The case $y < 1$ is more difficult. We have

THEOREM 2: *For $r \geqslant 2$, $x \geqslant 16$, and $0 \leqslant y < 1$ we have*

$$\sum_{N \leqslant x} \Delta_r(N) y^{\omega(N)} \ll_{r,y} x(\log x)^{y-1} \exp\left\{\frac{r \log r}{\alpha(1-y)}(\log_3 x)^2\right\}$$

where α is an absolute constant.

The exponents of $\log x$ in these results are best possible in view of the inequality $\Delta_r(N) \geqslant \max\{1, \tau_r(N)/(\log eN)^{r-1}\}$. Both Theorem 1 and its corollary are special cases ($t = 1$) of the following more general

THEOREM 3: *Let $r \in \mathbb{Z}^+$, t, $y \in \mathbb{R}$ satisfy $r \geqslant 2$, $t \geqslant 1$ and $y \geqslant (r-1)t/(r^t - 1)$. Then*

$$\sum_{N \leqslant x} \Delta_r(N)^t y^{\omega(N)} \ll_{r,t,y} x(\log x)^{\beta(r,t,y)-1}$$

$$\times \exp\left\{\left(\gamma_r + \frac{18 + 2(t-1)r \log r}{\log_3 x}\right)\sqrt{(t \log_2 x \log_3 x)}\right\},$$

where $\beta(r, t, y) = r^t y - (r-1)t$ and γ_r is defined as in Theorem 1.

Again, the exponent of $\log x$ is sharp. For $y < (r-1)t/(r^t - 1)$ this exponent must be at least $y - 1$, and we might expect a generalization of Theorem 2. However, when $t > 1$ our proof fails – instead we deduce from Theorem 3 the

COROLLARY: *Let r, t be as above, $y < (r-1)t/(r^t - 1)$. Then*

$$\sum_{N \leqslant x} \Delta_r(N)^t y^{\omega(N)} \ll_{r,t,y} x(\log x)^{y-1}$$

$$\times \exp\left\{\gamma_r\left(\left(1 + \frac{\log(1/y)}{\log((r-1)/\log r)}\right)\log_2 x \log_3 x\right)^{1/2}\right\}.$$

In previous papers [3,4,5] we defined

$$\alpha(r, y) := \inf\left\{\xi : S_r(x, y) = O\left(x(\log x)^{\xi}\right)\right\}.$$

We see now that

$$\alpha(r, y) = y - 1 + (r-1)\max(0, y-1)$$

for every $r \geqslant 1$, $y > 0$. This was known [3], [5] provided y was outside an excluded interval $(\frac{1}{2}, \Lambda_r^+)$, (where $\Lambda_r^+ \in (1, 1 + 1/\sqrt{3})$ is a function of r). An examination of the proofs shows that for such y the upper bounds for $S_r(x, y)$ which could be obtained were better than the present ones. Thus from [5] we may deduce that

$$S_r(x, y) \ll_{r,y} x(\log x)^{y-1}, \quad (r \geqslant 2,\ y < \tfrac{1}{2})$$

$$S_r(x, \tfrac{1}{2}) \ll_r x(\log x)^{-1/2}(\log\log x)^{r-1}, \quad (r \geqslant 2).$$

The result for $y < \frac{1}{2}$ is sharp. Again in [3], if we re-define

$$\Lambda_r^+ = \inf\{ y := z > y \Rightarrow S_r(x, z) \ll_{r,z} x(\log x)^{rz-r} \},$$

then Theorem 1 [3] holds, as stated. This raises an interesting question about the status of the various methods. It would be useful to know to what extent the present technique (which already has two variants, as in the proofs of Theorems 2 and 3 above), can be sharpened. Certainly the question of lower bounds has been somewhat neglected: as far as we are aware nothing better than $S_r(x, 1) \gg x \log\log x$ is known for any r.

2. Two lemmas

LEMMA 1: *We have*

$$\sum_p \frac{\log p}{p^\sigma} < -\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma - 1}, \quad (\sigma > 1).$$

PROOF: We have

$$\zeta(\sigma) = \frac{1}{\sigma - 1} + \frac{1}{2} + \frac{\sigma(\sigma+1)}{2} \int_1^\infty \frac{\{x\}(1 - \{x\})\, dx}{x^{\sigma+2}}$$

where $\{x\}$ denotes the fractional part of x. Hence

$$\zeta(\sigma) > \frac{1}{\sigma - 1} + \frac{1}{2}, \quad \sigma > 1.$$

Differentiating,

$$-\zeta'(\sigma) = \frac{1}{(\sigma - 1)^2} + \frac{\sigma(\sigma+1)}{2} \int_1^\infty \frac{\{x\}(1 - \{x\}) \log x\, dx}{x^{\sigma+2}}$$

$$- \frac{(2\sigma+1)}{2} \int_1^\infty \frac{\{x\}(1 - \{x\})\, dx}{x^{\sigma+2}}.$$

The third term on the right is negative. The second does not exceed

$$\frac{\sigma(\sigma+1)}{8} \int_1^\infty \frac{\log x\, dx}{x^{\sigma+2}} = \frac{\sigma}{8(\sigma+1)}.$$

Hence

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{\dfrac{1}{(\sigma-1)^2} + \dfrac{\sigma}{8(\sigma+1)}}{\dfrac{1}{\sigma-1} + \dfrac{1}{2}} < \frac{1}{\sigma-1}, \quad (1 < \sigma \leqslant 5).$$

The proof is completed by the estimate

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \sum_{n=2}^{\infty} \frac{\log n}{n^\sigma} < \frac{\log 2}{2^\sigma} + \int_2^\infty \frac{\log x}{x^\sigma} \mathrm{d}x < \frac{1}{\sigma-1}, \quad (\sigma > 5).$$

LEMMA 2: *Let* $L(\sigma)$, $X(\sigma)$ *be continuously differentiable for* $1 < \sigma \leqslant \sigma_0$ *and satisfy respectively*

$$-L'(\sigma) \leqslant \varphi(\sigma, L(\sigma))$$

$$-X'(\sigma) = \varphi(\sigma, X(\sigma))$$

where $\varphi(\sigma, x)$ *is a non-decreasing function of* x *for each fixed* σ. *Let* $L(\sigma_0) < X(\sigma_0)$. *Then* $L(\sigma) < X(\sigma)$ *throughout the range* $1 < \sigma \leqslant \sigma_0$.

PROOF: Suppose the contrary. Then there exists $\sigma_1 \in (1, \sigma_0)$ such that $L(\sigma_1) = X(\sigma_1)$, $L(\sigma) < X(\sigma)$ for $\sigma \in (\sigma_1, \sigma_0)$. Then

$$L(\sigma_1) = L(\sigma_0) - \int_{\sigma_1}^{\sigma_0} L'(\sigma)\, \mathrm{d}\sigma \leqslant L(\sigma_0) + \int_{\sigma_1}^{\sigma_0} \varphi(\sigma, L(\sigma))\, \mathrm{d}\sigma$$

$$< X(\sigma_0) + \int_{\sigma_1}^{\sigma_0} \varphi(\sigma, X(\sigma))\, \mathrm{d}\sigma = X(\sigma_1)$$

which is a contradiction.

3. The main lemma

We define, for $u_1, u_2, \ldots, u_{r-1} \in \mathbb{R}$.

$$\Delta(n;\, u_1, \ldots, u_{r-1})$$
$$= \operatorname{card}\{\mathrm{d}_1\mathrm{d}_2\ldots\mathrm{d}_{r-1} \mid n : u_i < \log \mathrm{d}_i \leqslant u_i + 1 \text{ for all } i\}$$

so that

$$\Delta_r(n) = \max_{u_1, \ldots, u_{r-1}} \Delta(n;\, u_1, \ldots, u_{r-1})$$

the maximum being attained at $(v_1, v_2, \dots, v_{r-1})$ say. For positive integers q we define

$$M_q(n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Delta(n;\ u_1, \dots, u_{r-1})^q \,\mathrm{d}u_1 \dots \mathrm{d}u_{r-1}$$

so that in particular $M_1(n) = \tau_r(n)$ and

$$M_q(n) \leqslant \Delta_r(n)^{q-1} M_1(n) = \Delta_r(n)^{q-1}\tau_r(n) \leqslant \tau_r(n)^q.$$

We need an inequality in the opposite direction. Let $\epsilon_i = 0$ or 1, $v_i \leqslant u_i \leqslant v_i + 1$, $1 \leqslant i < r$. Then

$$\Delta(n;\ v_1, \dots, v_{r-1}) \leqslant \sum \Delta(n;\ u_1 - \epsilon_1,\ u_2 - \epsilon_2, \dots, u_{r-1} - \epsilon_{r-1})$$

where the sum is over all 2^{r-1} choices of the ϵ_i. Thus

$$\Delta_r(n)^q \leqslant 2^{(r-1)(q-1)} \sum \Delta(n;\ u_1 - \epsilon_1,\ u_2 - \epsilon_2, \dots, u_{r-1} - \epsilon_{r-1})^q$$

and we integrate this over the cube $v_i \leqslant u_i \leqslant v_i + 1$. Since there is no overlapping we get

$$\Delta_r(n)^q \leqslant 2^{(r-1)(q-1)} M_q(n). \tag{1}$$

We have to consider more general integrals than $M_q(n)$. Let $a_0, a_1, \dots, a_{r-1}$ be non-negative integers whose sum is q, and $w \in \mathbb{R}$. We define

$$N(n;\ w;\ a_0, \dots, a_{r-1}) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Delta(n;\ u_1, \dots, u_{r-1})^{a_0}$$

$$\times \Delta(n;\ u_1 - w,\ u_2, \dots, u_{r-1})^{a_1} \dots$$

$$\times \Delta(n;\ u_1, \dots, u_{r-1} - w)^{a_{r-1}} \,\mathrm{d}u_1 \dots \mathrm{d}u_{r-1}$$

where w is subtracted, in turn, from each u_i in the last $r-1$ factors of the integrand. Our objective in this section is

LEMMA 3: *Provided* $\max\{a_0, a_1, \dots, a_{r-1}\} \leqslant q - 1$ *we have*

$$\sum_p N(m;\ \log p;\ a_0, \dots, a_{r-1}) \frac{\log p}{p}$$

$$\leqslant C 2^{q-1} D_{r-1}(m) \{\tau_r(m)\}^{1/(q-1)} M_q(m)^{(q-2)/(q-1)}$$

where C is an absolute constant and for $s \geqslant 1$ we define

$$D_s(m) = \sum_{d \mid m} \Delta_s(d).$$

(We write $\Delta_1(d) \equiv 1$ so that $D_1(m) = \tau(m)$).

PROOF: By successive applications of Hölder's inequality we may reduce to the case where there are just two non-zero exponents a_j and a_k say, where $1 \leqslant a_j,\ a_k \leqslant q-1,\ a_j + a_k = q$. There are then two cases according as $\min(j,k) = 0$ or not. Of course when $r = 2$ only the first case occurs.

In the first case we may suppose without loss of generality that $j = 0$, $k = 1$ and we write $a_0 = a,\ a_1 = b$. We begin by estimating the sum

$$\sum_p \Delta(m;\ u_1 - \log p,\ u_2, \dots, u_{r-1})^b \frac{\log p}{p}$$

$$= \sum^*_{d_1^{(1)} \dots d_{r-1}^{(1)} \mid m} \ \sum^*_{d_1^{(2)} \dots d_{r-1}^{(2)} \mid m} \cdots \sum^*_{d_1^{(b)} \dots d_{r-1}^{(b)} \mid m} \ \sum^*_p \frac{\log p}{p}$$

where * denotes the following conditions of summation:

(*†) $\quad u_i < \log d_i^{(s)} \leqslant u_i + 1, \quad (s \leqslant b,\ 2 \leqslant i < r)$,

(*) $\quad u_1 - \log p < \log d_1^{(s)} \leqslant u_1 - \log p + 1, \quad (s \leqslant b)$,

so that we must have

(†) $\quad \max_s \log d_1^{(s)} - \min_s \log d_1^{(s)} \leqslant 1$

$$u_1 - \min_s \log d_1^{(s)} < \log p \leqslant u_1 + 1 - \max_s \log d_1^{(s)}.$$

It follows that

$$\sum_p^* \frac{\log p}{p} \leqslant C := \sup_z \sum_{z \leqslant p \leqslant ez} \frac{\log p}{p}.$$

Hence the sum under consideration does not exceed

$$C \sum^\dagger_{d_1^{(1)} \dots d_{r-1}^{(1)} \mid m} \cdots \sum^\dagger_{d_1^{(b)} \dots d_{r-1}^{(b)} \mid m} 1$$

where the conditions of summation are marked † above. For positive integers b we define

$$V_b(m;\ u_2,\dots,u_{r-1}) = \int_{-\infty}^{\infty} \Delta(m;\ u_1,\ u_2,\dots,u_{r-1})^b \,\mathrm{d}u_1.$$

We consider the integral

$$\int_{-\infty}^{\infty} \{\Delta(m;\ u_1-1,\ u_2,\dots,u_{r-1}) + \Delta(m;\ u_1,\ u_2,\dots,u_{r-1})\}^b \,\mathrm{d}u_1$$

which does not exceed $2^b V_b(m;\ u_2,\dots,u_{r-1})$. On the other hand this last integral is

$$\sum_{d_1^{(1)}\dots d_{r-1}^{(1)}\mid m} \cdots \sum_{d_1^{(b)}\dots d_{r-1}^{(b)}\mid m} \max\left\{0,\ 2-\log\left(\max_s d_1^{(s)}/\min_s d_1^{(s)}\right)\right\},$$

where the condition marked $(*\dagger)$ applies to the sums. Hence

$$\sum_{d_1^{(1)}\dots d_{r-1}^{(1)}\mid m}^{\dagger} \cdots \sum_{d_1^{(b)}\dots d_{r-1}^{(b)}\mid m}^{\dagger} 1 \leqslant 2^b V_b(m;\ u_2,\dots,u_{r-1})$$

and we have

$$\begin{aligned}
&\sum_p N(m;\ \log p;\ a,\ b,\ 0,\dots,0)\frac{\log p}{p}\\
&\quad\leqslant C2^b \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \Delta(m;\ u_1,\ u_2,\dots,u_{r-1})^a\\
&\qquad\times V_b(m;\ u_2,\dots,u_{r-1})\,\mathrm{d}u_1\dots\mathrm{d}_{r-1}\\
&\quad\leqslant C2^b \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} V_a(m;\ u_2,\dots,u_{r-1})\\
&\qquad\times V_b(m;\ u_2,\dots,u_{r-1})\,\mathrm{d}u_2\dots\mathrm{d}_{r-1},
\end{aligned}$$

performing the integration w.r.t. u_1. Recall that $1 \leqslant a,\ b \leqslant q-1$. Hölder's inequality gives

$$\begin{aligned}
V_a(m;\ u_2,\dots,u_{r-1}) &\leqslant V_1(m;\ u_2,\dots,u_{r-1})^{(q-a)/(q-1)}\\
&\qquad\times V_q(m;\ u_2,\dots,u_{r-1})^{(a-1)/(q-1)}.
\end{aligned}$$

Since $a + b = q$, the last integral above does not exceed

$$C2^{q-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} V_1(m;\ u_2,\ldots,u_{r-1})^{q/(q-1)}$$

$$\times V_q(m;\ u_2,\ldots,u_{r-1})^{(q-2)/(q-1)}\, du_2\ldots du_{r-1}.$$

Notice that

$$V_1(m;\ u_2,\ldots,u_{r-1}) = \sum_{d_1 \mid m} \Delta\left(\frac{m}{d_1};\ u_2,\ u_3,\ldots,u_{r-1}\right) \leqslant D_{r-1}(m).$$

We apply this upper bound to the factor V_1 in the above integral, leaving an exponent $1/(q-1)$, and then apply Hölder's inequality to the remaining multiple integral, with exponents $q-1$, $(q-1)/(q-2)$. Since

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} V_1(m;\ u_2,\ldots,u_{r-1})\, du_2\ldots du_{r-1} = \tau_r(m)$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} V_q(m;\ u_2,\ldots,u_{r-1})\, du_2\ldots du_{r+1} = M_q(m)$$

we obtain the result stated in this case.

Now suppose $\min(j,\ k) > 0$. We make a further reduction, by means of Hölder, to the case where the non-zero exponents are 1 and $q-1$, say $a_1 = 1$, $a_2 = q-1$.

We have to estimate

$$\sum_p \frac{\log p}{p}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \Delta(m;\ u_1 - \log p,\ u_2,\ldots,u_{r-1})$$

$$\times\Delta(m;\ u_1,\ u_2 - \log p,\ u_3,\ldots,u_{r-1})^{q-1}\, du_1\ldots du_{r-1}$$

and we make the substitution $u_2 = w + \log p$. Now consider the sum

$$\sum_p \Delta(m;\ u_1 - \log p,\ w + \log p,\ u_3,\ldots,u_{r-1})\frac{\log p}{p}$$

$$= \sum_p \sum_{d_1 d_2 \ldots d_{r-1} \mid m} \frac{\log p}{p}$$

with the summation conditions

$$u_1 - \log p < \log \mathrm{d}_1 \leqslant u_1 - \log p + 1$$

$$w + \log p < \log d_2 \leqslant w + \log p + 1$$

$$u_i < \log \mathrm{d}_i \leqslant u_i + 1, \quad (3 \leqslant i < r).$$

Notice that we must have $u_1 + w < \log(d_1 d_2) \leqslant u_1 + w + 2$. Hence the sum does not exceed

$$\sum_{\substack{d_1 d_2 \mid m \\ u_1 + w < \log(d_1 d_2) \leqslant u_1 = w + 2}} \Delta(m/d_1 d_2;\ u_3, \dots, u_{r-1}) \sum \frac{\log p}{p}$$

where in the inner sum, we require

$$\max\{u_1 - \log d_1,\ w - \log d_2 - 1\}$$
$$< \log p < \min\{u_1 - \log d_1 + 1,\ w - \log d_2\}$$

so that this does not exceed C. We now have

$$\leqslant C \sum_{d_1 \mid m} \left\{ \Delta\left(\frac{m}{d_1};\ u_1 + w - \log d_1,\ u_3, \dots, u_{r-1}\right) \right.$$
$$\left. + \Delta\left(\frac{m}{d_1};\ u_1 + w + 1 - \log d_1,\ u_3, \dots, u_{r-1}\right)\right\}$$
$$\leqslant 2CD_{r-1}(m).$$

Therefore

$$\sum_p N(m;\ \log p, 0, 1, q-1, \dots, 0) \frac{\log p}{p}$$
$$\leqslant 2CD_{r-1}(m) \int_{-\infty}^{\infty} \cdots$$
$$\times \int_{-\infty}^{\infty} \Delta(m;\ u_1,\ w,\ u_3, \dots, u_{r-1})^{q-1}\, \mathrm{d}u_1\, \mathrm{d}w \dots \mathrm{d}u_{r-1}$$
$$\leqslant 2CD_{r-1}(m) \left\{ \int_{-\infty}^{\infty} \cdots \right.$$
$$\left. \times \int_{-\infty}^{\infty} \Delta(m;\ u_1,\ w,\ u_3, \dots, u_{r-1})\, \mathrm{d}u_1 \dots \mathrm{d}u_{r-1} \right\}^{1/(q-1)}$$
$$\times \left\{ \int_{-\infty}^{\infty} \cdots \right.$$
$$\left. \times \int_{-\infty}^{\infty} \Delta(m;\ u_1,\ w,\ u_3, \dots, u_{r-1})^q\, \mathrm{d}u_1 \dots \mathrm{d}u_{r-1} \right\}^{(q-2)/(q-1)},$$

and the required result is established in this case too. This completes the proof of Lemma 3.

4. Proof of Theorem 3

We are going to show that for $l,\ r \in \mathbb{Z}^+$, $t \geqslant 1$ and

$$y \geqslant \begin{cases} 0, & (r=1), \\ (r-1)t/(r^t-1), & (r \geqslant 2), \end{cases}$$

there exists a constant $C(l,\ r,\ t,\ y)$ such that

$$\sum_{n=1}^{\infty}{}' \frac{\Delta_r(n)^t y^{\omega(n)}}{n^\sigma} \leqslant \frac{C(l,\ r,\ t,\ y)}{(\sigma-1)^{\beta(r,t,y)+tr(r-1)/2(l+r)}}, \quad (1<\sigma \leqslant 2)$$

where $\beta(r,\ t,\ y) := r^t y - (r-1)t$ and the sum is restricted to square free n. This will be done by induction on r, with an explicit upper bound for $C(l,\ r,\ t,\ y)$. We may then optimize our choice of l on the right hand side for given r, t, y and σ. It is then a straightforward matter to change over from the sum on the left to $S_r(x;\ t,\ y)$.

Let $l \in \mathbb{Z}^+$ be initially fixed, and for $r \in \mathbb{Z}^+$ set $q = l + r$, and define

$$L(\sigma) = \sum_{n=1}^{\infty}{}' \frac{M_q(n)^{t/q}}{n^\sigma} y^{\omega(n)}, \quad (\sigma > 1)$$

so that differentiating w.r.t. σ we have

$$\begin{aligned} -L'(\sigma) &= \sum_{n=1}^{\infty}{}' \frac{M_q(n)^{t/q}}{n^\sigma} y^{\omega(n)} \log n \\ &= \sum_{n=1}^{\infty}{}' \frac{M_q(n)^{t/q}}{n^\sigma} y^{\omega(n)} \sum_{p \mid n} \log p \\ &= y \sum_p \frac{\log p}{p^\sigma} \sum_{\substack{m=1 \\ p \nmid m}}^{\infty}{}' \frac{M_q(mp)^{t/q}}{m^\sigma} y^{\omega(m)}. \end{aligned}$$

For integers m and primes p such that $p \nmid m$, we have

$$\begin{aligned} \Delta(pm;\ u_1, \dots, u_{r-1}) = \Delta(m;\ u_1, \dots, u_{r-1}) \\ + \Delta(m;\ u_1 - \log p,\ u_2, \dots, u_{r-1}) + \cdots \\ + \Delta(m;\ u_1,\ u_2, \dots, u_{r-1} - \log p) \end{aligned}$$

and we raise to the power q and integrate, to obtain

$$M_q(pm) = \sum_{a_0+a_1+\cdots+a_{r-1}=q} \binom{q}{a_0,\, a_1, \ldots, a_{r-1}}$$

$$\times N(m;\, \log p;\, a_0,\, a_1, \ldots, a_{r-1}).$$

We pick out the diagonal terms in which some $a_\iota = q$, noting that

$$N(m;\, \log p;\, 0, 0, \ldots, q, 0, \ldots 0) = M_q(m).$$

There are r such terms and we deduce that for $q \geqslant t$,

$$M_q(pm)^{t/q} \leqslant r^{t/q} M_q(m)^{1/q}$$

$$+ r^t \sum_{a_0+\cdots+a_{r-1}=q}^{*} N(m;\, \log p;\, a_0, \ldots, a_{r-1})^{t/q}$$

where the * signifies that $\max a_\iota \leqslant q-1$ and the factor r arises as an upper bound for the q-th root of any multinomial coefficient. Hence

$$-L'(\sigma) \leqslant y \sum_p \frac{\log p}{p^\sigma} r^{t/q} L(\sigma) + y r^t \sum_{a_0+\cdots+a_{r-1}=q}^{*} \sum{}' \frac{1}{m^\sigma}$$

$$\times \sum_p N(m;\, \log p;\, a_0, \ldots, a_{r-1})^{1/q} \frac{\log p}{p^\sigma}.$$

In the first term on the right we estimate the sum over p from above using Lemma 1. We apply Hölder's inequality with exponents q/t, $q/(q-t)$ to the sum over p in the second term. This yields

$$-L'(\sigma) \leqslant \frac{y r^{t/q} L(\sigma)}{\sigma - 1} + \frac{y r^t}{(\sigma-1)^{1-t/q}} \sum_{a_0+\cdots+a_{r-1}=q}^{*} \sum{}' \frac{y^{\omega(m)}}{m^\sigma}$$

$$\times \left\{ \sum_p N(m;\, \log p;\, a_0, \ldots, a_{r-1}) \frac{\log p}{p^\sigma} \right\}^{t/q}.$$

We drop the exponent σ in the inner sum and apply Lemma 3, which yields

$$-L'(\sigma) - \frac{y r^{t+q} L(\sigma)}{\sigma - 1} \leqslant \frac{y r^t \binom{q+r-1}{r-1}}{(\sigma-1)^{1-t/q}} \sum{}' \frac{y^{\omega(m)}}{m^\sigma}$$

$$\times\left\{C2^{q-1}D_{r-1}(m)r^{\omega(m)/(q-1)}M_q(m)^{(q-2)/(q-1)}\right\}^{t/q}$$

$$\leqslant \frac{C^{t/q}y(2r)^t\binom{q+r-1}{r-1}}{(\sigma-1)^{1-t/q}}$$

$$\times\left\{\sum{}' \frac{y^{\omega(m)}}{m^\sigma}D_{r-1}(m)^{t(1-1/q)}r^{t\omega(m)/q}\right\}^{1/(q-1)} L(\sigma)^{(q-2)/(q-1)}$$

by Hölder's inequality with exponents $q-1$, $(q-1)/(q-2)$. Recall that

$$D_{r-1}(m)=\sum_{d\mid m}\Delta_{r-1}(d)=\sum_{d\mid m}\zeta^{\omega(d)}\frac{\Delta_{r-1}(d)}{\zeta^{\omega(d)}}\cdot$$

We apply Hölder's inequality to the sum on the right, with exponents $t/(t-1)$, t and $\zeta=(r-1)^{1-1/t}$, which yields (for squarefree m)

$$D_{r-1}(m)^t\leqslant r^{(t-1)\omega(m)}\sum_{d\mid m}\frac{\Delta_{r-1}(d)^t}{(r-1)^{(t-1)\omega(d)}}.$$

Hence we have

$$\sum_{m=1}^{\infty}{}' D_{r-1}(m)^t\frac{y^{\omega(m)}}{m^\sigma}\leqslant\sum_{m=1}^{\infty}{}'\frac{\left(r^{t-1}y\right)^{\omega(m)}}{m^\sigma}\sum_{d\mid m}\frac{\Delta_{r-1}(d)^t}{(r-1)^{(t-1)\omega(d)}}$$

$$\leqslant\sum_{d=1}^{\infty}{}'\frac{\Delta_{r-1}(d)^t}{d^\sigma}\left(\frac{r^{t-1}y}{(r-1)^{t-1}}\right)^{\omega(d)}$$

$$\times\sum_{(n,d)=1}{}'\frac{\left(r^{t-1}y\right)^{\omega(n)}}{n^\sigma}.$$

Now let $r\geqslant 2$, $y\geqslant(r-1)t/(r^t-1)$. We assume our induction hypothesis is valid for $r-1$, viz.

$$\sum_{d=1}^{\infty}{}'\frac{\Delta_{r-1}(d)^t}{d^\sigma}z^{\omega(d)}\leqslant\frac{C(l,\,r-1,\,t,\,z)}{(\sigma-1)^{\beta(r-1,t,z)+t(r-1)(r-2)/2(l+r-1)}}$$

provided $z\geqslant(r-2)t/((r-1)^t-1)$, ($z\geqslant 0$ when $r=2$). We apply this with $z=r^{t-1}y/(r-1)^{t-1}$ which is permissible because for $r\geqslant 2$, $t\geqslant 1$, we have

$$\frac{r^{t-1}}{(r-1)^{t-1}}\frac{(r-1)^t-1}{r^t-1}\geqslant\frac{r-2}{r-1}.$$

We also have that

$$\sum{}' \frac{(r^{t-1}y)^{\omega(n)}}{n^\sigma} = \prod_p \left(1 + \frac{r^{t-1}y}{p^\sigma}\right) < \zeta(\sigma)^{r^{t-1}y} < \left(\frac{2}{\sigma-1}\right)^{r^{t-1}y}$$

for $1 < \sigma \leqslant 2$. Hence in this range,

$$\sum{}' D_{r-1}(m)^t \frac{y^{\omega(m)}}{m^\sigma} \leqslant \frac{2^{r^{t-1}y} C\left(l,\ r-1,\ t,\ r^{t-1}y/(r-1)^{t-1}\right)}{(\sigma-1)^{r^t y-(r-2)t+t(r-1)(r-2)/2(l+r-1)}}.$$

Recalling that $l + r = q$, we deduce that for $1 < \sigma \leqslant 2$, and for the relevant values of y,

$$\sum{}' \frac{y^{\omega(m)}}{m^\sigma} D_{r-1}(m)^{t(1-1/q)} r^{t\omega(m)/q}$$

$$\leqslant \left(\frac{y^{\omega(m)}}{m^\sigma} D_{r-1}(m)^t\right)^{1-1/q} \left(\sum{}' \frac{(r^t y)^{\omega(m)}}{m^\sigma}\right)^{1/q}$$

$$\leqslant \frac{2^{r^t y} C\left(l,\ r-1,\ t,\ r^{t-1}y/(r-1)^{t-1}\right)^{1-1/q}}{(\sigma-1)^{r^t y-(1-1/q)(r-2)t+t(r-1)(r-2)/2q}},$$

and hence that

$$-L'(\sigma) - \frac{y r^{t/q} L(\sigma)}{\sigma - 1}$$

$$\leqslant \frac{6C^{t/q} y (2r)^{t-1} q^{r-1} 2^{r^t y/(q-1)} C(l,\ r-1,\ \text{etc}\ldots,)^{1/q}}{(\sigma-1)^{1+\beta(r,t,y)/(q-1)+\text{tr}(r-1)/2q(q-1)}}$$

$$\times L(\sigma)^{(q-2)/(q-1)}$$

where we have used the inequality

$$r\binom{q+r-1}{r-1} \leqslant 3q^{r-1}, \quad (q \geqslant r \geqslant 2),$$

and simplified the exponent of $\sigma - 1$. Next we apply Lemma 2. The corresponding differential equation has the solution $X(\sigma) = K(\sigma-1)^{-\gamma}$ where γ satisfies

$$\gamma + 1 = 1 + \beta(r,\ t,\ y)/(q-1) + \text{tr}(r-1)/2q(q-1)$$

$$+ \gamma(q-2)/(q-1)$$

whence

$$\gamma = \beta(r, t, y) + \mathrm{tr}(r-1)/2q.$$

Next,

$$K = \left\{ \frac{6yq^{r-1}(2r)^{t-1}}{\gamma - yr^{t/q}} \right\}^{q-1} 2^{r^t y}\{C^t C(l, r-1, \text{etc}\ldots)\}^{1-1/q}.$$

We put $\sigma_0 = 2$ and we require that $L(2) < X(2) = K$. Recall that $M_q(n) < \tau_r(n)^q$ whence

$$L(2) \leqslant \sum{}' \tau_r(n)^t \frac{y^{\omega(n)}}{n^2} = \prod_p \left(1 + \frac{r^t y}{p^2}\right) < 2^{r^t y}.$$

Obviously $C \geqslant 1$ and we assume that $C(l, r-1, t, z) \geqslant 1$ for relevant values of l, t, z. Since $\gamma \leqslant r^t y$ it is clear that $K > 2^{r^t y}$. Lemma 2 yields

$$L(\sigma) < \frac{K}{(\sigma - 1)^{\beta(r,t,y) + \mathrm{tr}(r-1)/2q}}, \quad (1 < \sigma \leqslant 2),$$

with K as above, and we deduce from (1) that

$$\sum{}' \frac{\Delta_r(n)^t}{n^\sigma} y^{\omega(n)} < \frac{C(l, r, t, y)}{(\sigma - 1)^{\beta(r,t,y) + \mathrm{tr}(r-1)/2(l+r)}}, \quad (1 < \sigma \leqslant 2),$$

with $C(l, r, t, y) = 2^{r-1}K$. Since we may take $C(l, 1, t, y) = 2^y$ for every l, t and y, the induction is complete.

We require an upper bound for $C(l, r, t, y)$. In the formula for K, we have

$$y/\{\beta(r, t, y) + \mathrm{tr}(r-1)/2q - yr^{t/q}\}$$
$$\leqslant 2q/\{r(r^t - 1) - 2q(r^{t/q} - 1)\}$$

because the function on the left decreases as y increases beyond $(r-1)t/(r^t - 1)$. The function on the right decreases as t increases, and since $r^{1/q} \leqslant 1 + 3r(r-1)/7q$ for $q \geqslant r \geqslant 2$, when $t = 1$ it does not exceed $14q/r(r-1)$. So we have

$$C(l, r, t, y) \leqslant 2^{r-1} \left\{ \frac{84q^r(2r)^{t-1}}{r(r-1)} \right\}^{q-1}$$
$$\times 2^{r^t y} C^t C\left(l, r-1, t, \left(\frac{r}{r-1}\right)^{t-1} y\right)$$

$$\leqslant 2^{r^2}\left\{\frac{84^{r-1}q^{(r-1)(r+2)/2}(2^{r-1}r!)^{t-1}}{r!(r-1)!}\right\}^{l+r-1}$$

$$\times 2^{r^{t+1}y}C^{rt}C(l,1,t,r^{t-1}y)$$

$$\ll_{r,t,y} B^l r^{r(r-1)l}(l+r)^{(r-1)(r+2)(l+r-1)/2},$$

where $B=\max\{84^{r-1}/r!(r-1)!\}<e^{12.5}$, noticing that $2^{r-1}r!\leqslant r^r$. We choose l to be the least integer for which

$$l+r\geqslant\left\{\frac{2\,tr}{r+2}\cdot\frac{\log(1/(\sigma-1))}{\log_2^+(1/(\sigma-1))}\right\}^{1/2}$$

so that provided $1<\sigma\leqslant\sigma_1(r,t)$ say, we obtain

$$\sum{}'\frac{\Delta_r(n)^t}{n^\sigma}y^{\omega(n)}\ll_{r,t,y}(\sigma-1)^{-\beta(r,t,y)}$$

$$\times\exp\left\{\left(\gamma_r+\frac{18+2(t-1)r\log r}{\log_2^+\left(\frac{1}{\sigma-1}\right)}\right)\right.$$

$$\left.\times\sqrt{\left(t\log\left(\frac{1}{\sigma-1}\right)\log_2^+\left(\frac{1}{\sigma-1}\right)\right)}\right\},$$

and clearly this is valid over the whole range $1<\sigma\leqslant 2$.

We extend the sum on the left to all integers following Hooley [6] and Tenenbaum [9]. Any integer N may be written in the form $N=nd^2$ where n is squarefree, and we have $\Delta_r(N)\leqslant\Delta_r(n)\tau_r(d^2)$. Hence

$$\sum_{N=1}^{\infty}\Delta_r(N)^t\frac{y^{\omega(N)}}{N^\sigma}\leqslant\sum{}'\Delta_r(n)^t\frac{y^{\omega(n)}}{n^\sigma}\sum_{d=1}^{\infty}\frac{\tau_r(d^2)^t}{d^{2\sigma}}\max(1,y^{2\omega(d)}).$$

The sum over d on the right is $\ll_{r,t,y}1$. We put $\sigma=1+1/\log x$ so that for $x\geqslant 16$ we have

$$\sum_{N\leqslant x}\Delta_r(N)^t\frac{y^{\omega(N)}}{N}\ll_{r,t,y}(\log x)^{\beta(r,t,y)}$$

$$\times\exp\left\{\left(\gamma_r+\frac{18+2(t-1)r\log r}{\log_3 x}\right)\sqrt{(\log_2 x\,\log_3 x)}\right\},$$

which is clearly valid for all x. Finally, as in [9] we have

$$\sum_{N\leqslant x} \Delta_r(N)^t y^{\omega(N)} \log x = \sum_{N\leqslant x} \Delta_r(N)^t y^{\omega(N)}\left(\log\frac{x}{N} + \sum_{m\mid N}\Lambda(m)\right)$$

$$\leqslant x \sum_{N\leqslant x} \Delta_r(N)^t \frac{y^{\omega(N)}}{N}$$

$$+ \sum_{m\leqslant x}\Lambda(m) \sum_{n\leqslant x/m} \Delta_r(nm)^t y^{\omega(nm)}.$$

For the first sum on the right we use the upper bound proved above. In the second, we apply the inequality $\Delta_r(nm) \leqslant \Delta_r(n)\tau_r(m)$ and reverse the order of summation, noting that

$$\max(1,\, y) \sum_{m\leqslant x/n} \Lambda(m)\tau_r(m)^t \ll_{r,t,y} x/n.$$

This yields a sum $\ll_{r,t,y}$ that already considered. The proof is complete.

5. Proof of the corollary

We have $y<(r-1)t/(r^t-1)$. Since the right hand side is a strictly decreasing function of t, tending to zero as $t\to\infty$, there exists $T>t$ such that $y=(r-1)T/(r^T-1)$, and for this T, $\beta(r,\,T,\,y)=y$. We put $p=T/t>1$ and apply Hölder's inequality with exponents p, $p/(p-1)$. Thus

$$\sum_{N\leqslant x}\Delta_r(N)^t y^{\omega(N)} \leqslant \left(\sum_{N\leqslant x}\Delta_r(N)^T y^{\omega(N)}\right)^{1/p}\left(\sum_{N\leqslant x} y^{\omega(N)}\right)^{1-1/p}$$

$$\ll_{r,t,y} x(\log x)^{y-1}$$

$$\times\exp\left\{\frac{1}{p}\left(\gamma_r + \frac{18+2(T-1)r\log r}{\log_3^+ x}\right)\sqrt{(T\log_2^+ x\log_3^+ x)}\right\}.$$

Next, we require an upper bound for T. Hölder's inequality, which in this instance is strict, gives

$$\int_1^r \mathrm{d}\lambda < \left(\int_1^r \lambda^{T-1}\,\mathrm{d}\lambda\right)^{1/T}\left(\int_1^r \lambda^{-1}\,\mathrm{d}\lambda\right)^{1-1/T}$$

whence

$$\left(\frac{r-1}{\log r}\right)^{T-1} < \frac{r^T-1}{T(r-1)},\quad (r>1,\, T>1).$$

But T was chosen to make the right hand side equal to $1/y$ and so

$$T < 1 + \frac{\log(1/y)}{\log((r-1)/\log r)}.$$

Hence for $x \geqslant x_0(r, t, y)$, we have

$$\frac{1}{p}\left(\gamma_r + \frac{18 + 2(T-1)r \log r}{\log_3 x}\right)\sqrt{T}$$

$$< \gamma_r \sqrt{\left(1 + \frac{\log(1/y)}{\log((r-1)/\log r)}\right)}.$$

This gives the result stated, (which is clearly valid for $x < x_0$).

6. Proof of Theorem 2

For squarefree n, we denote by $p_j(n)$ the j-th prime factor of n in order of magnitude and set

$$n_k = \begin{cases} \prod_{j \leqslant k} p_j(n), & (\text{if } \omega(n) \geqslant k) \\ n, & (\text{if } \omega(n) < k). \end{cases}$$

and

$$L_k(\sigma) = \sum_{n=1}^{\infty}{}' M_q(n_k)^{1/q} \frac{y^{\omega(n)}}{n^\sigma}, \quad (\sigma > 1),$$

where q will ultimately be a function of k. We are going to prove by induction on r that for a suitable function $B(r, y)$, $(r \in \mathbb{Z}^+, y < 1)$ which we estimate, we have for $k \geqslant 1$, $\sigma > 1$,

$$\sum_{n=1}^{\infty}{}' \Delta_r(n_k) \frac{y^{\omega(n)}}{n^\sigma} \ll_{r,y} B(r, y)^{\log^2 k} (\sigma - 1)^{-y}.$$

for $r \geqslant 1$, $k \geqslant 1$, $1 < \sigma \leqslant 2$, $0 \leqslant y < 1$. This is true for $r = 1$, with $B(1, y) \equiv 1$. Also, because $\Delta_r(n) \leqslant 2^{r-1} M_q(n)^{1/q}$, it will be sufficient to obtain this upper bound for the function $L_k(\sigma)$. Now let $r \geqslant 2$: our induction hypothesis is that for every $z \in [0, 1)$ we have

$$\sum_{n=1}^{\infty}{}' \Delta_{r-1}(n_k) \frac{z^{\omega(n)}}{n^\sigma} \ll_{r,z} B(r-1, z)^{\log^2 k} (\sigma - 1)^{-z}.$$

The left hand side is at least

$$\sum_{\substack{m=1\\ \omega(m)=k}}^{\infty}{}' \Delta_{r-1}(m)\frac{z^k}{m^\sigma}\sum{}'\left\{\frac{z^{\omega(h)}}{h^\sigma} : P^-(h) > P^+(m)\right\}$$

where $P^-(h)$, $P^+(m)$ denote the least, greatest prime factor of h, m respectively. Since the inner sum is

$$\prod_{p>P^+(m)}\left(1+\frac{z}{p^\sigma}\right) \gg_z (\sigma-1)^{-z}(\log P^+(m))^{-z},$$

we deduce from the above that for $0 \leqslant z < 1$,

$$\sum_{\substack{m=1\\ \omega(m)=k}}^{\infty}{}' \frac{\Delta_{r-1}(m)z^k}{m(\log P^+(m))^z} \ll_{r,z} B(r-1,\, z)^{\log^2 k}$$

(where we have let $\sigma \to 1+$). Now set

$$R_k(\sigma) = \sum_{\substack{n=1\\ \omega(n)\geqslant k}}^{\infty}{}' \frac{D_{r-1}(n_k)}{\log p_k(n)}\frac{y^{\omega(n)}}{n^\sigma}$$

where D_{r-1} was defined in Lemma 3. We require an upper bound for this. We decompose the variable n in the sum above in the form $n = dmh$, where $dm = n_k$. We have

$$R_k(\sigma) \leqslant y^k \sum_{j=0}^{k} \sum_{\substack{d=1\\ \omega(d)=j}}^{\infty}{}' \frac{\Delta_{r-1}(d)}{d^\sigma} \sum_{\substack{m=1\\ \omega(m)=k-j}}^{\infty}{}' \frac{1}{m^\sigma \log P^+(md)}$$

$$\times \sum\left\{\frac{y^{\omega(h)}}{h^\sigma} : P^-(h) > P^+(md)\right\}$$

The innermost sum is

$$\ll (\sigma-1)^{-y}P^+(md)^{\sigma-1}(\log P^+(md))^{-y}$$

and, since $P^+(m\mathrm{d}) \leqslant m\mathrm{d}$ we now have

$$R_k(\sigma) \ll y^k(\sigma-1)^{-y}\sum_{j=0}^{k} \sum_{\substack{d=1\\ \omega(d)=j}}^{\infty}{}' \frac{\Delta_{r-1}(d)}{d}$$

$$\times \sum_{\substack{m=1\\ \omega(m)=k-j}}^{\infty}{}' \frac{1}{m(\log P^+(md))^{1+y}}.$$

We write $1+y=2z$, so that $z<1$, and note that $(\log P^+(md))^{1+y} \geqslant (\log P^+(m))^z(\log P^+(d))^z$, where $\log P^+(1)$ is to be taken equal to 1. It is not difficult to show that (uniformly in l),

$$\sum_{\substack{m=1\\ \omega(m)=l}}^{\infty}{}' \frac{1}{m(\log P^+(m))^z} \ll z^{-l}$$

and so we have

$$R_k(\sigma) \ll y^k(\sigma-1)^{-y} \sum_{j=0}^{k} \sum_{\substack{d=1\\ \omega(d)=j}}^{\infty}{}' \frac{\Delta_{r-1}(d) z^{j-k}}{d(\log P^+(d))^z}$$

$$\ll_{r,z} (y/z)^k(\sigma-1)^{-y} \sum_{j=0}^{k} B(r-1,\ z)^{\log^2 j}$$

(where log O is to be interpreted as O) whence

$$R_k(\sigma) \ll_{r,y} \left(\frac{2y}{y+1}\right)^k (\sigma-1)^{-y} kB\left(r-1,\ \frac{1+y}{2}\right)^{\log^2 k}.$$

Next, we obtain an inequality connecting $L_{k+1}(\sigma)$ and $L_k(\sigma)$. Let $\omega(n)>k$. As in the proof of Theorem 3 we have

$$M_q(n_{k+1})^{1/q} \leqslant r^{1/q} M_q(n_k)^{1/q}$$
$$+ r \sum_{a_0+\cdots+a_{r-1}=q}^{*} N(n_k;\ \log p_{k+1}(n);\ a_0,\dots,a_{r-1})^{1/q}$$

where the * denotes as before that $\max a_i < q$. For $\omega(n) \leqslant k$ there is no second term on the right. Hence

$$L_{k+1}(\sigma) \leqslant r^{1/q} L_k(\sigma)$$
$$+ r\sum{}^{*} \sum_{\substack{n=1\\ \omega(n)>k}}^{\infty}{}' N(n_k;\ \log p_{k+1}(n);\ a_0,\dots,a_{r-1})^{1/q} \frac{y^{\omega(n)}}{n^\sigma}.$$

The inner sum does not exceed

$$\sum_{\substack{m=1\\ \omega(m)=k}}^{\infty}{}' \frac{y^k}{m^\sigma} \sum_{p>P^+(m)} \frac{1}{p^\sigma} N(m;\ \log p;\ a_0,\dots,a_{r-1})^{1/q}$$
$$\times \sum_{h=1}^{\infty}{}' \left\{\frac{y^{\omega(h)}}{h^\sigma} : P^-(h)>p\right\}$$

and the innermost sum here is

$$\ll (\sigma-1)^{-y} p^{\sigma-1} (\log p)^{-y}.$$

We apply Hölder's inequality, with exponents q, $q/(q-1)$ to the sum over p. We have

$$\sum_{p>P^+(m)} \frac{1}{p} N(m; \log p; a_0, \ldots, a_{r-1})^{1/q}$$

$$\leqslant \left(\sum_p \frac{\log p}{p} N(m; \log p; a_0, \ldots, a_{r-1} \right)^{1/q}$$

$$\times \left(\sum_{p>P^+(m)} \frac{1}{p} (\log p)^{-1/(q-1)} \right)^{1-1/q}$$

$$\ll q \left(\frac{D_{r-1}(m)}{\log P^+(m)} \right)^{1/q} r^{k/q(q-1)} M_q(m)^{(q-2)/q(q-1)}$$

by Lemma 3, noting that $\tau_r(m) = r^k$. Putting all this together we deduce that

$$L_{k+1}(\sigma) \leqslant r^{1/q} L_k(\sigma) + C_1 q^r r^{k/q(q-1)} \sideset{}{'}\sum_{\substack{m=1 \\ \omega(m)=k}}^{\infty} \frac{y^k}{m^\sigma} \left(\frac{D_{r-1}(m)}{\log P^+(m)} \right)^{1/q}$$

$$\times M_q(m)^{(q-2)/q(q-1)} ((\sigma-1) \log P^+(m))^{-y},$$

where C_i, $i = 1, 2, 3 \ldots$ denotes an absolute constant. It is not difficult to show that

$$\sideset{}{'}\sum_{\substack{n=1 \\ n_k=m}}^{\infty} \frac{y^{\omega(n)}}{n^\sigma} \gg \frac{y^k}{m^\sigma} (\sigma-1)^{-y} (\log P^+(m))^{-y}$$

and so the sum on the right above is

$$\ll \sideset{}{'}\sum_{\substack{n=1 \\ \omega(n) \geqslant k}}^{\infty} \left(\frac{D_{r-1}(n_k)}{\log p_k(n)} \right)^{1/q} M_q(n_k)^{(q-2)/(q(q-1))} \frac{y^{\omega(n)}}{n^\sigma}$$

$$\ll R_k(\sigma)^{1/q} L_k(\sigma)^{(q-2)/(q-1)} (\sigma-1)^{-y/q(q-1)}$$

after two applications of Hölder's inequality: we have already defined, and estimated, $R_k(\sigma)$. So we have

$$L_{k+1}(\sigma) \leqslant r^{1/q} L_k(\sigma) + C_2 q^r r^{k/q(q-1)} R_k(\sigma)^{1/q} L_k(\sigma)^{(q-2)/(q-1)} \times (\sigma-1)^{-y/q(q-1)}.$$

We now replace $L_k(\sigma)$ in the above by

$$L_k^*(\sigma) := L_k(\sigma) + r^{k/q} \sum_{n=1}^{\infty}{}' \frac{y^{\omega(n)}}{n^\sigma},$$

and $L_{k+1}(\sigma)$ be the corresponding $L_{k+1}^*(\sigma)$. This step is plainly permissible, and it is useful because the lower bound $L_k^*(\sigma) \gg r^{k/q}(\sigma-1)^{-y}$ is easily available. We use this to deduce that

$$L_{k+1}^*(\sigma) \leqslant L_k^*(\sigma)\left\{r^{1/q} + C_3 q^r R_k(\sigma)^{1/q}(\sigma-1)^{y/q}\right\}$$
$$\leqslant L_k^*(\sigma)\left\{r^{1/q} + C_4 q^r k^{1/q} B^{(\log k)^2/q}\left(\frac{2y}{1+y}\right)^{k/q}\right\},$$

using the upper bound for $R_k(\sigma)$ obtained above. Here $B = B(r-1, (1+y)/2)$. For $k \geqslant k_0(r, y)$ we have

$$kB^{\log^2 k}\left(\frac{2y}{1+y}\right)^k \leqslant \exp\{-\tfrac{1}{3}k(1-y)\},$$

and provided $q \leqslant \alpha k(1-y)/(r \log k)$ where α is a sufficiently small absolute constant, we also have

$$C_4 q^r \exp\left\{-\frac{k}{3q}(1-y)\right\} \leqslant (r+1)^{1/q} - r^{1/q},$$

whence

$$L_{k+1}^*(\sigma) \leqslant L_k^*(\sigma)(r+1)^{1/q}.$$

We use this inequality to derive an upper bound for $L_{k+1}^*(\sigma)$ in terms of $L_k^*(\sigma)$ which is itself $\ll_{r,y} (\sigma-1)^{-y}$, because $k_0 = k_0(r, y)$. There is a slight complication in that $q = [\alpha k(1-y)/r \log k)]$ varies with k, and $L_k'(\sigma)$ is a function of q. Let $q' < q$. We have

$$M_q(n) \leqslant \Delta_r(n)^{q-q'} M_{q'}(n), \quad \Delta_r(n) \leqslant 2^{r-1} M_{q'}(n)^{1/q'}.$$

The inequality on the left is trivial, that on the right is derived from (1). Hence

$$M_q(n)^{1/q} \leqslant 2^{(r-1)(1-q'/q)} M_{q'}(n)^{1/q'}$$

and so a change from q' to q increases $L_k^*(\sigma)$ by at most the factor given on the right. We may assume the values taken by q are consecutive so that the contribution from the factor above is

$$\leqslant 2^{(r-1)(1+1/2+1/3+\ldots+1/q)} \leqslant 2^{(r-1)(\log q+1)},$$

and

$$L_{k+1}^*(\sigma) \leqslant 2^{(r-1)(\log q+1)}(r+1)^{\lambda} L_{k_0}^*(\sigma)$$

where

$$\lambda = \sum_{j=k_0}^{k} 1/[\alpha j(1-y)/(r\log j)]$$

$$\leqslant \frac{1.5r}{\alpha(1-y)} \sum_{j=k_0}^{k} \frac{\log j}{j} \leqslant \frac{3r\log^2 k}{4\alpha(1-y)},$$

provided $\alpha k_0(1-y)/(r\log k_0) \geqslant 2$, as we may assume. Since $\log q = \log k + O(1)$ we may deduce that

$$L_{k+1}^*(\sigma) \leqslant L_{k_0}^*(\sigma) \exp\left\{\frac{4r\log r}{5\alpha(1-y)}\log^2 k\right\},$$

provided k_0 is sufficiently large. Replacing $k+1$ by k, and using the inequality $\Delta_r(n) \leqslant 2^{r-1} M_q(n)^{1/q}$, we deduce that

$$\sum_{n=1}^{\infty}{}' \Delta_r(n_k)\frac{y^{\omega(n)}}{n^{\sigma}} \ll_{r,y} (\sigma-1)^{-y} \exp\left\{\frac{4r\log r}{5\alpha(1-y)}\log^2 k\right\},$$

thereby completing our induction, with $B(r, y) = \exp\{\frac{4}{5}r\log r/\alpha(1-y)\}$. Now we are in a position to estimate from above the sum

$$\sum_{n=1}^{\infty}{}' \Delta_r(n)\frac{y^{\omega(n)}}{n^{\sigma}} = \sum_{\substack{n=1\\ \omega(n)\leqslant k}}^{\infty}{}' \Delta_r(n)\frac{y^{\omega(n)}}{n^{\sigma}} + \sum_{\substack{n=1\\ \omega(n)>k}}^{\infty}{}' \Delta_r(n)\frac{y^{\omega(n)}}{n^{\sigma}}.$$