COMPOSITIO MATHEMATICA

SHELDON KATZ

# On the finiteness of rational curves on quintic threefolds

*Compositio Mathematica*, tome 60, nº 2 (1986), p. 151-162

<http://www.numdam.org/item?id=CM_1986__60_2_151_0>

[1] Compositio Mathematica 60: 151–162 (1986)
© Martinus Nijhoff Publishers, Dordrecht – Printed in the Netherlands

# ON THE FINITENESS OF RATIONAL CURVES ON QUINTIC THREEFOLDS

Sheldon Katz

## Introduction

This paper addresses Clemens' conjecture [2] that a generic quintic threefold contains a finite non-zero number of smooth rational curves of any given degree.

The paper is organized as follows.

In §1, the conjecture is reduced to producing an infinitesimally rigid curve, and proving the irreducibility of the incidence correspondence of smooth rational curves on quintics. Irreducibility is proven for $d \leqslant 7$.

The construction of a rational curve on a generic quintic threefold (Theorem 2.1) proceeds by finding one on a quartic K3 surface, then using Clemens' argument given in [1] to deform the curve to a generic quintic. As a corollary (2.4), Clemens' conjecture is true for $d \leqslant 7$.

The number (609,250) of conics on a generic quintic is computed in §3.

Finally, in three appendices, we explicitly compute the normal bundle of any rational curve $C$ of degree $d \leqslant 3$ on any quintic threefold, smooth along $C$. This explicitly exhibits a curve for $d \leqslant 3$ whose existence was guaranteed by Theorem 2.1.

I'd like to express my thanks here to Ron Donagi and Bob Friedman for helpful conversations, to the referee for suggesting a simplification of Theorem 2.1, and especially to Herb Clemens for sharing his numerous insights and suggestions with me.

## 1. Formulation of the problem and a method of attack

In this paper, the term 'quintic threefold' refers to a hypersurface of degree 5 in $\mathbf{P}^4_{\mathbf{C}}$.

As stated in the introduction, this paper is primarily concerned with the

CONJECTURE 1.1: *Let $d$ be a positive integer. Then the scheme of smooth rational curves of degree $d$ on a generic quintic threefold is finite, non-empty, and reduced.*

REMARK: Here, 'generic' means that the accompanying statement is true for all quintics except those parameterized by a proper subvariety of the moduli space of quintics.

Let $C \subset \mathbf{P}^4$ be an irreducible rational curve of degree $d$. Then $C$ can be parameterized by 5 forms $\alpha_0(t, u), \dots, \alpha_4(t, u)$, homogeneous of degree $d$ on $\mathbf{P}^1$.

Let $\mathcal{M}_d$ be the moduli space of smooth rational curves of degree $d$. Taking into account the above description and the ambiguity arising from the GL(2) action of $\mathbf{P}^1$, we see that $\mathcal{M}_d$ is irreducible of dimension $5(d+1) - 4 = 5d + 1$.

Let $\mathbf{P} = \mathbf{P}H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(5))$ be the projective space of quintics in $\mathbf{P}^4$.

Let $\mathcal{I}$ denote the incidence variety

$$\mathcal{I} = \{(C, F) \in \mathcal{M}_d \times \mathbf{P} \mid C \subset F\}$$

with projection maps $\pi_1 : \mathcal{I} \to \mathcal{M}_d$, $\pi_2 : \mathcal{I} \to \mathbf{P}$. If we use the same letter $F$ to describe the equation of the quintic, then $\mathcal{I}$ is defined inside $\mathcal{M}_d \times \mathbf{P}$ by the equation

$$F(\alpha_0(t, u), \dots, \alpha_4(t, u)) = 0 \qquad [1.2]$$

The left hand side of [1.2] is a polynomial of degree $5d$, hence its vanishing imposes at most $5d + 1$ conditions on $\mathcal{M}_d \times \mathbf{P}$ comparing with $\dim \mathcal{M}_d = 5d + 1$ gives that $\dim \mathcal{I} \geq \dim \mathbf{P}$.

CLAIM 1.3: Conjecture [1.2] is true provided that

(a) $\mathcal{I}$ is irreducible

(b) There exists a smooth rational curve $C$ of degree $d$ on some smooth quintic threefold $F$ with normal bundle $N = N_{C/F} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$.

REMARK: $\mathcal{O}_C(k)$ denotes the unique invertible sheaf of degree $k$ on $\mathbf{P}^1$. In particular, this notation does not refer to the projective embedding $C \hookrightarrow \mathbf{P}^4$.

PROOF of claim: Define the open set $\mathcal{I}_0 \subset \mathcal{I}$ by

$$I_0 = \{(C, F) \in \mathcal{I} \mid F \text{ is smooth}\}$$

For any $(C, F) \in \mathcal{I}_0$, the exact sequence

$$0 \to T_C \to T_{F|C} \to N_{C/F} \to 0$$

together with $K_F \simeq \mathcal{O}_F$ and $\deg T_C = 2$ shows that $\deg N_{C/F} = -2$. Thus, either $N_{C/F} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$, or $H^0(N_{C/F}) \neq 0$.

Let $\mathscr{C} = \{(C, F, p) \in \mathscr{I}_0 \times \mathbf{P}^4 \mid p \in C\}$ be the "universal curve", and $\mathscr{F} = \{(C, F, p) \in \mathscr{I}_0 \times \mathbf{P}^4 \mid p \in F\}$ be the "universal quintic". Then $N_{\mathscr{C}/\mathscr{F}}$ is coherent and flat over $\mathscr{I}_0$, so that dim $H^0(N_{C/F})$ is an upper semicontinuous function on $\mathscr{I}_0$. This says that $\mathscr{I}_0$ has the stratification $\mathscr{I}_0 \supset \mathscr{I}_1 \supset \dots$, where

$$\mathscr{I}_i = \left\{(C, F) \in \mathscr{I}_0 \mid \dim H^0\left(N_{C/F}\right) \geqslant i\right\}$$

Now (b) says that $\mathscr{I}_i$ is a proper subvariety of $\mathscr{I}_0$.

Suppose that $\pi_2(\mathscr{I}_1)$ were dense in $\mathbf{P}$. Then for generic $F \in \mathbf{P}$, $\pi_2^{-1}(F) \cap \mathscr{I}_1$ would be a non-empty proper subvariety of $\pi_2^{-1}(F) \cap \mathscr{I}_0$ of positive codimension. But $\pi_2^{-1}(F) \cap \mathscr{I}_0 - \pi_2^{-1}(F) \cap \mathscr{I}_1$ is finite, since any $(C, F)$ in this set would have $H^0(N_{C/F}) = 0$, which says that $C$ is rigid in $F$. But then $\pi_2^{-1}(F) \cap \mathscr{I}_1$ could not be a nonempty proper subvariety of positive codimension. This contradiction proves finiteness for any smooth $F$ in the dense open subset $\mathbf{P} - \overline{\pi_2(\mathscr{I}_1)}$. The non-emptiness follows by (b) and by dim $\mathscr{I}_0 \geqslant$ dim $\mathbf{P}$. Reducedness follows since the tangent space is $H^0(N) = 0$.

We conclude this section by showing that hypothesis (a) of (1.3) is valid for all $d \leqslant 7$.

LEMMA 1.4: *$\mathscr{I}$ is irreducible if $d \leqslant 7$.*

PROOF: By [5], if $C \subseteq \mathbf{P}^4$ is rational of degree $d$, then $C$ is 6-regular, hence $H^0(\mathbf{P}^4, \mathscr{O}_{\mathbf{P}^4}(5)) \to H^0(C, \mathscr{O}_C(5))$ is surjective. So the fibers of $\pi_1$ are (irreducible) projective spaces of constant dimension 124-5$d$. Since $\mathscr{M}_d$ is also irreducible, we conclude that $\mathscr{I}$ is irreducible.

## 2. Smooth rational curves of arbitrary degree

The goal of this section is to prove that hypothesis (b) of (1.3) is valid for all $d$.

THEOREM 2.1: *Let $F$ be a generic quintic threefold, $d$ a positive integer. Then there exists a smooth rational curve $C \subset F$ of degree $d$, with $N_{C/F} = \mathscr{O}_C(-1) \oplus \mathscr{O}_C(-1)$.*

REMARK: Theorem (2.1) was proven for infinitely many integers $d$ in [1].

We first assert that there exists a smooth quartic surface $S \subset \mathbf{P}^3$ which contains a smooth rational curve $D$ of degree $d$. This is just a special case of a theorem of Mori [11], which says that you can find such a configuration $D \subset S$ if

$$g = \text{genus}(D) < d^2/8, \quad (g, d) \neq (3, 5).$$

The proof concludes by applying the deformation argument of [1] to the curve $D$ and quartic surface $S$ to yield a curve with the required properties on the generic quintic threefold.

Remark: For $d = 1, 2, 3$, explicit examples of $C \subset F$ having the properties of Theorem (2.1) will be given in the appendices. It is hoped that the computational methods given there will find application elsewhere.

Corollary 2.2: *Clemens' conjecture* (1.1) *is true for* $d \leqslant 7$.

## 3. The number of conics on the generic quintic threefold

Theorem 3.1: *Let F be a generic quintic threefold. Then F contains* 609,250 *smooth conics.*

Remark: The formula for conics on quartic threefolds analogous to $c_{11}(B)$ given below can be found in [3]. A different line of reasoning is given here, which gives an intrinsic interpretation of the vector bundle $B$.

We review some of the notation and results of [3].

Let $\mathscr{M}$ be the moduli space of conics in $\mathbf{P}^4$. Then $\mathscr{M}$ may be identified with $\mathbf{P}(S^2U^*)$, where $U$ denotes the universal bundle on the Grassmanian $G = G(3, 5)$ of planes in $\mathbf{P}^4$, and $\mathbf{P}(E)$ denotes the projective bundle of lines in the vector bundle $E$.

We describe the Chow ring of $\mathscr{M}$. Let $z \in A^1(\mathscr{M})$ denote the class of $\mathscr{O}_{\mathbf{P}(E)}(1)$. Then the Chow ring of $\mathscr{M}$ is generated over the Chow ring of $G$ by $z$.

The Chow ring of $G$ is well-known. It is generated multiplicatively by Schubert cycles. Let $x$ = class in $A^1(G)$ of $\sigma_1$ = set of 2-planes meeting a given line, and $y$ = class in $A^2(G)$ of $\sigma_2$ = set of 2-planes passing through a given point. By the canonical injection of $A^*(G)$ in $A^*(\mathscr{M})$, we can view $x$ and $y$ as classes in $A^*(\mathscr{M})$. Thus every element of $A^*(\mathscr{M})$ can be represented as polynomial in $x$, $y$, $z$.

For enumerative purposes, it is necessary to work with $A^{11}(\mathscr{M}) = \mathbf{Z} \cdot$ [point]. The non-zero monomials of weight 11 in $x$, $y$, and $z$ are [3]:

$$\begin{aligned}
&z^5x^6 = 5 \quad z^5x^4y = 2, \quad z^5x^2y^2 = 1, \quad z^5y^3 = 1,\\
&z^6x^5 = -20, \quad z^6x^3y = -8, \quad z^6xy^2 = -4,\\
&z^7x^4 = 40, \quad z^7x^2y = 17, \quad z^7y^2 = 11,\\
&z^8x^3 = -50, \quad z^8xy = -23,\\
&z^9x^2 = 45, \quad z^9y = 27,\\
&z^{10}x = -30\\
&z^{11} = 20.
\end{aligned} \tag{3.2}$$

Next, we let $W$ be the scheme of conics on $F$, and proceed to compute the class of $W$ in $A^*(\mathcal{M})$.

Let $\pi: \mathcal{M} \to G$ denote the natural map. Then an equation for $F$ induces a section of $S^5U^*$, hence of $\pi^*S^5U^*$. At a point of $\mathcal{M}$ representing a conic $C$, this section represents the plane quintinc cut out by the 2-plane supporting $C$. $C$ will lie on $F$ if and only if this quintic factors into $C$ and a cubic. The set of quintics factoring in this way globalizes to the vector bundle $T \otimes \pi^*S^3U^*$, where $T$ is the tautological bundle on $\mathcal{M} = \mathbf{P}(S^2U^*)$ (whose sheaf of sections is $\mathcal{O}(-1)$). There is a natural inclusion of $T \otimes \pi^*S^3U^*$ into $\pi^*S^5U^*$; consider the quotient bundle $B = \pi^*S^5U^*/(T \otimes \pi^*S^3U^*)$. Then $W$ is precisely the zero locus of the section of $B$ induced by an equation for $F$. Since $F$ is generic, $W$ is finite. Thus, the number of conics on $F$ (including multiplicity) is $c_{11}(B)$. Using standard techniques for calculating Chern classes (and a computer), this class is

$$
\begin{aligned}
&85{,}117{,}950z^5x^6 - 132{,}042{,}420z^5x^4y + 19{,}397{,}940z^5x^2y^2\\
&\quad -744{,}200z^5y^3 + 65{,}076{,}835z^6x^5 - 80{,}271{,}040z^6x^3y\\
&\quad +6{,}894{,}855z^6xy^2 + 38{,}046{,}080z^7x^4 - 32{,}530{,}740z^7x^2y\\
&\quad +1{,}004{,}880z^7y^2 + 17{,}138{,}550z^8x^3\\
&\quad -7{,}956{,}000z^8xy + 5{,}856{,}500z^9x^2 - 884{,}000z^9y\\
&\quad +1{,}385{,}670z^{10}x + 167{,}960z^{11} = 609{,}250
\end{aligned}
$$

This number accurately represents the number of smooth conics on $F$, since: (a) There are no pairs of lines on $F$ that meet, hence there are no reducible conics on $F$: Let $I$ be the variety of pairs of distinct intersecting lines in $\mathbf{P}^4$. $I$ is 10 dimensional. Let $Q$ be the projective space of all quintics. Let $C = \{(\ell_1, \ell_2, f) \in I \times Q \mid \ell_1 \cup \ell_2 \subset F\}$. Since all pairs of distinct intersecting lines are equivalent under PGL(5), we may choose coordinates $(X_0, \ldots, X_4)$ for $\mathbf{P}^4$ so that $\ell_1$ is defined by $X_1 = X_3 = X_4 = 0$, and $\ell_2$ is defined by $X_2 = X_3 = X_4 = 0$. A necessary and sufficient condition for a quintic $F$ to lie in $\pi_1^{-1}(\ell_1, \ell_2)$ is that $F$ have an equation of the form

$$X_3 f_4 + X_4 F_4' + X_1 X_2 F_3 = 0$$

where $f_4$, $f_4'$ are quartics, and $f_3$ is a cubic. This shows that the fibers of $\pi_1$ are all 114-dimensional. Thus $C$ is 124 dimensional. Since $Q$ is 125-dimensional, the generic quintic cannot lie in the image of $\pi_2$.

(b) The scheme of lines on $F$ is reduced, hence here are no non-reduced conics.

(c) The scheme of conics on $F$ is reduced, hence each conic counts exactly once in the above Chern class computation.

## Appendix A. Lines on quintic threefolds

In these appendices, we compute the normal bundle of rational curves with $d \leqslant 3$ on a quintic threefold, and give an example of a curve with $N_{C/F} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. In this appendix, we consider $d = 1$.

Let $L \subset F$ be a line on a smooth quintic threefold. We can assume that $L$ is given parametrically by $(t, u, 0, 0, 0)$, with homogeneous coordinates $(X_0, \ldots, X_4)$ on $\mathbf{P}^4$, and $(t, u)$ on $L \simeq \mathbf{P}^1$. Since $L$ is the complete intersection of hyperplanes $X_2 = X_3 = X_4 = 0$, we can write the equation of $F$ as

$$F = X_2 f_2 + X_3 f_3 + X_4 f_4 = 0 \qquad [4.1]$$

where the $f_i$ are homogeneous in the $X_i$ $(i = 0, \ldots, 4)$ of degree 4.

To compute $N = N_{L/F}$, we use the standard exact sequences

$$\begin{aligned} &\text{a)} \quad 0 \to \mathcal{O}_C \to \mathcal{O}_{\mathbf{P}^4}(1)^5 \,|\, _C \to T_{\mathbf{P}^4} \,|\, _C \to 0 \qquad [7] \\ &\text{b)} \quad 0 \to T_F \,|\, _C \to T_{\mathbf{P}^4} \,|\, _C \to N_{F/\mathbf{P}^4} \,|\, _C \to 0; \quad N_{F/\mathbf{P}^4} \cong \mathcal{O}_F(5) \\ &\text{c)} \quad 0 \to T_C \to T_F \,|\, _C \to N \to 0 \end{aligned} \qquad [4.2]$$

valid for any $C \subset F$, where $T$ denotes a tangent sheaf. We know that every vector bundle on $L$ splits into a sum of line bundles [4]. By the syzygy theorem [10] applied to $\mathbf{P}^1$, we know that given generators $(f_i)$ of an ideal, homogeneous in two variables, we can find a set of relations among the $f_i$ which generate all of the relations, without higher syzygies. This technique will allow us to compute all of the bundles in [4.2].

We start with [4.2a]. The map $\mathcal{O}_L \to \mathcal{O}_{\mathbf{P}^4}(1)^5 \,|\, _L$ has matrix ${}^t(X_0 \; X_1 \; X_2 \; X_3 \; X_4) = {}^t(t \; u \; 0 \; 0 \; 0)$. The cokernel must be described by the relations, which are given by the matrix

$$\begin{pmatrix} u & -t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathcal{O}_{\mathbf{P}^4}(1)^5 \,|\, _L \to T_{\mathbf{P}^4} \,|\, _L. \qquad [4.3]$$

Comparing degrees, we see that $T_{\mathbf{P}^4} \,|\, _L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$.

Turning next to [4.2b], we observe that $T_{\mathbf{P}^4} \,|\, _L \to \mathcal{O}(5) \,|\, _L$ is induced by $\Sigma \ell_i \partial/\partial X_i \to \Sigma \ell_i \partial F/\partial X_i$. Now $\partial F/\partial X_0 \,|\, _L = \partial F/\partial X_1 \,|\, L = 0$; $\partial F/\partial X_i \,|\, _L = f_i \,|\, _L$ for $i = 2, 3, 4$. We abuse notation by omitting the restriction to

$L$, writing $f_i$ in place of $f_i \mid_L$, thinking of the $f_i$ as quartics in $t$, $u$. So if $T_{\mathbf{P}^4} \mid L \to \mathcal{O}(5) \mid_L$ has matrix $(a_1\ a_2\ a_3\ a_4)$ in the basis for $T_{\mathbf{P}^4} \mid_L$ just described, then we have

$$(a_1\ a_2\ a_3\ a_4)\begin{pmatrix} u & -t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (0\ 0\ f_2\ f_3\ f_4) \qquad [4.4]$$

Solving, $a_1 = 0$, and $a_i = f_i$ for $i = 2, 3, 4$. So we get the matrix

$$(0\ f_2\ f_3\ f_4) : T_{\mathbf{P}^4} \mid_L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1) \to \mathcal{O}_L(5). \qquad [4.5]$$

Again, the syzygy theorem gives the matrix of the kernel $T_F \mid_L \to T_{\mathbf{P}^4} \mid_L$. Because of the 0 in the matrix [4.5] there is the relation given by ${}^t(1\ 0\ 0\ 0)$, but there must also be two others among the $f_i$.

There are always at least two independent sextic relations $\Sigma q_i f_i = 0$, with the $q_i$ quadratic; this is because the $q_i$ depend on $3 \cdot 3 = 9$ parameters, while sextics depend on 7. So if there are no relations of lower weight than 6 (the generic case), these two relations will complete the description of the kernel. In this case, $T_F \mid_L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1)$. If there happens to be a quintic relation $\Sigma\, \ell_i f_i = 0$ with linear $\ell_i$, but no quartic relation, then there will be a factor of $\mathcal{O}_L$ in $T_F \mid_L$. Considering degrees in [4.2b] shows that $\det T_F \mid_L = \deg T_{\mathbf{P}^4} \mid_L - \deg \mathcal{O}_L(5) = 5 - 5 = 0$, hence in this case, $T_F \mid_L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L \oplus \mathcal{O}_L(-2)$. Finally, if there is a quartic relation $\Sigma\, c_i f_i = 0$ with sçalars $c_i$, we get $T_F \mid_L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(-3)$.

Having computed $T_F \mid_L$, we now turn to [4.2c]. We could compute explicitly as before, but we merely observe that since $T_L \simeq \mathcal{O}_L(2)$, then the matrix of $T_L \to T_F \mid_L$ must contain zeros at the locations corresponding to factors of $\mathcal{O}_L(a)$ in $T_F \mid_L$, with $a < 2$. With this observation, we conclude that

$$N \simeq \begin{cases} \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1) & \text{if there are no quintic or quartic relations among the } f_i \\ \mathcal{O}_L \oplus \mathcal{O}_L(-2) & \text{if there is a quintic relation among the } f_i\text{, but no quartic relations} \\ \mathcal{O}_L(1) \oplus \mathcal{O}_L(-3) & \text{if there is a quartic relation among the } f_i. \end{cases} \qquad [4.6]$$

Now choose $F$ so that $f_2 = t^4$, $f_3 = t^2u^2$, $f_4 = u^4$. Then there are clearly no relations of weight 5 or 4 among the $f_i$, hence $N \simeq \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1)$ by [4.6].

**Remark**: The number of lines on a generic $F$ has been computed to be 2875 [6]. Since $N \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, it follows that $H^0(N) = 0$, so the $L$ has no first order deformations in $F$, implying that $L$ has multiplicity one in $F$. This gives another proof that all lines occur with multiplicity one.

## Appendix B. Conics on quintic threefolds

Let $C \subset F$ be a smooth conic on a smooth quintic threefold. We can assume that $C$ is given parametrically by $(t^2, tu, u^2, 0, 0)$. Since $C$ is defined by the equations $X_1^2 - X_0X_2 = X_3 = X_4 = 0$, we can write the equation of the quintic $F$ in the form

$$F = \left(X_1^2 - X_0X_2\right)f + X_3f_3 + X_4f_4 = 0. \tag{5.1}$$

For some cubic $f$, and quartics $f_3$, $f_4$.

Now the matrix of $\mathcal{O}_C \to \mathcal{O}(1)^5|_C$ in [4.2a] is ${}^t(t^2\ tu\ u^2\ 0\ 0)$, so the cokernel may be described by the matrix of relations

$$\begin{pmatrix} u & -t & 0 & 0 & 0 \\ 0 & u & -t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \underset{\mathcal{O}_C(2)^5}{\mathcal{O}(1)^5|_C} \to T_{\mathbf{P}^4}|_C \tag{5.2}$$

so that $T_{\mathbf{P}^4}|_C \simeq \mathcal{O}_C(3) \oplus \mathcal{O}_C(3) \oplus \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$.

Turning next to [4.2b] the matrix $(a\ b\ c\ d)$ of $T_{\mathbf{P}^4}|_C \to \mathcal{O}(5)|_C \simeq \mathcal{O}_C(10)$ is such that

$$(a\ b\ c\ d)\begin{pmatrix} u & -t & 0 & 0 & 0 \\ 0 & u & -t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \left(-u^2f\ \ 2tuf\ \ -t^2f\ \ f_3\ \ f_4\right) \tag{5.3}$$

where the right hand side is the vector of partial derivatives of $F$. Here, we think of $f$ as a sextic, and $f_3$, $f_4$ and octics, in the variables $t$, $u$. Solving [5.3] for $(a\ b\ c\ d)$, we get the matrix

$$(-uf\ \ tf\ \ f_3\ \ f_4) : \mathcal{O}_C(3) \oplus \mathcal{O}_C(3) \oplus \mathcal{O}_C(2) \oplus \mathcal{O}_C(2) \to \mathcal{O}_C(10). \tag{5.4}$$

so $T_F|_C$ is computed by finding relations among them. For now, assume that $f \neq 0$. The relation $t(-uf)+u(tf)=0$ gives an $\mathcal{O}_C(2)$ factor. Looking for additional relations is equivalent to looking for relations between $f$, $f_3$, and $f_4$. There are always at least 2 independent relations of weight 11 $qf + c_3 f_3 + c_4 f_4 = 0$ with $q$ a quintic, and $c_i$ cubics, since $(q, c_3, c_4)$ depends on $6+4+4=14$ parameters, while degree 11 polynomials depend on 12 parameters; if these are the relations of lowest weight, this gives $T_F|_C \simeq \mathcal{O}_C(2) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$; a relation of weight 10 would give $T_F|_C \simeq \mathcal{O}_C(2) \oplus \mathcal{O}_C \oplus \mathcal{O}_C(-2)$; a relation of weight 9 would give $T_F|_C \simeq \mathcal{O}_C(2) \oplus \mathcal{O}_C(1) \oplus \mathcal{O}_C(-3)$; a relation of weight 8 would give $T_F|_C \simeq \mathcal{O}_C(2) \oplus \mathcal{O}_C(2) \oplus \mathcal{O}_C(-4)$.

Finally, considering [4.2c], we conclude as before that if $f \neq 0$

$$N \simeq \begin{cases} \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1) \\ \quad \text{if the lowest weight relation between } f,\ f_3,\ f_4 \text{ is } 11 \\ \mathcal{O}_C \oplus \mathcal{O}_C(-2) \\ \quad \text{if the lowest weight relation between } f,\ f_3,\ f_4 \text{ is } 10 \\ \mathcal{O}_C(1) \oplus \mathcal{O}_C(-3) \\ \quad \text{if the lowest weight relation between } f,\ f_3,\ f_4 \text{ is } 9 \\ \mathcal{O}_C(2) \oplus \mathcal{O}_C(-4) \\ \quad \text{if the lowest weight relation between } f,\ f_3,\ f_4 \text{ is } 8 \end{cases} \qquad [5.5]$$

Now, let $F$ be such that $f = t^6$, $f_3 = t^3 u^5$, $f_4 = u^8$. By [5.5], $N \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$.

Remark: If $f = 0$, then the plane $P$ defined by $X_3 = X_4 = 0$ is tangent to $F$ along $C$. In that case, $C$ deforms to first order to $F$ by moving in any direction in $P$, and $N_{C/F} \simeq \mathcal{O}_C(4) \oplus \mathcal{O}_C(-6)$.

## Appendix C. Twisted cubics on quintic threefolds

The case of the twisted cubic is complicated somewhat by the fact that it is not a complete intersection.

Let $C \subset F$ be a twisted cubic curve on a quintic threefold. We can assume that $C$ is given parametrically by $(t^3, t^2u, tu^2, u^3, 0)$. Let

$$Q_0 = X_0X_3 + X_1X_3 - X_1X_2 - X_2^2$$

$$Q_1 = X_0X_2 + X_0X_3 - X_1^2 - X_1X_2$$

$$Q_2 = X_0X_3 - X_1X_2$$

Then $C$ is given by the equations $Q_0 = Q_1 = Q_2 = X_4 = 0$. We can write the equation of the quintic $F$ in the form

$$F = Q_0 f_0 + Q_1 f_1 + Q_2 f_2 + X_4 g = 0 \qquad [6.1]$$

where the $f_i$ are cubics, and $g$ is a quartic. As before, we consider [4.2a] and compute a matrix of relations for ${}^t(t^3 \;\; t^2u \;\; tu^2 \;\; u^3 \;\; 0)$ to get

$$\begin{pmatrix} u & -t & 0 & 0 & 0 \\ 0 & u & -t & 0 & 0 \\ 0 & 0 & u & -t & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathcal{O}(1)^5 \,|_C \to T_{\mathbf{P}^4} \,|_C \qquad [6.2]$$
$$\mathcal{O}_C(3)^5$$

giving $T_4 \,|_C \simeq \mathcal{O}_C(4) \oplus \mathcal{O}_C(4) \oplus \mathcal{O}_C(4) \oplus \mathcal{O}_C(3)$, which in turn yields as a matrix for the map $T_{\mathbf{P}^4} \,|_C \to \mathcal{O}(5) \,|_C \simeq \mathcal{O}_C(15)$ of [4.2b]

$${}^t(a \;\; b \;\; c \;\; d)$$
$$= {}^t\big(u^2 f_0 + (tu + u^2) f_1 + u^2 f_2 \quad u^2 f_0 - t^2 f_1 \quad -(t^2 + tu) f_0 - t^2 f_1 - t^2 f_2 \quad g\big) \qquad [6.3]$$

Here deg $a$ = deg $b$ = deg $c = 11$, deg $g = 12$. So $T_F \,|_C$ is computed by finding relations among $a$, $b$, $c$, $g$. The relation $t^2 a + tub + u^2 c = 0$, which has weight 13, gives an $\mathcal{O}_C(2)$ factor. This relation may reduct to lower weight, in special cases. Rather than include the exhaustive analysis here, we content ourselves with a few observations.

1. $a$, $b$, $c$ cannot be all zero. For then the three weight 11 relations $1 \cdot a = 1 \cdot b = 1 \cdot c = 0$ would give $T_F \,|_C \simeq \mathcal{O}_C(4) \oplus \mathcal{O}_C(4) \oplus \mathcal{O}_c(4)$, which contradicts deg $T_F \,|_C = 0$.
2. There cannot be two independent weight 11 relations. For if there were, then there would be a non-zero polynomial $\alpha$ of degree 11 and scalars $a'$, $b'$, $c'$, not all zero, such that $a = a'\alpha$, $b = b'\alpha$, $c = c'\alpha$. But then $t^2 a + tub + u^2 c = 0$ becomes $(a't^2 + b'tu + c'u^2)\alpha = 0$, which is impossible.
3. There are always at least two independent weight 16 relations $q_1 a + q_2 b + q_3 c + qg = 0$, where the $q_i$ are quintics and $q$ is a quartic. This is because the data $(q_1, q_2, q_3, q)$ depends on 23 parameters, but needs to be reduced to 19 because of the set of trivial relations $et^2 a + etub + eu^2 c = 0$ for any cubic e, while degree 16 polynomials depend on 17 parameters.

Summarizing, we get the following chart for the relations of lowest weight

| Weight of relations | $T_F\|_C$ | $N$ |
|---|---|---|
| 13, 16 | $\mathcal{O}_C(2)\oplus\mathcal{O}_C(-1)\oplus\mathcal{O}_C(-1)$ | $\mathcal{O}_C(-1)\oplus\mathcal{O}_C(-1)$ |
| 13, 15 | $\mathcal{O}_C(2)\oplus\mathcal{O}_C\oplus\mathcal{O}_C(-2)$ | $\mathcal{O}_C\oplus\mathcal{O}_C(-2)$ |
| 13, 14 | $\mathcal{O}_C(2)\oplus\mathcal{O}_C(1)\oplus\mathcal{O}_C(-3)$ | $\mathcal{O}_C(1)\oplus\mathcal{O}_C(-3)$ |
| 13, 13 | $\mathcal{O}_C(2)\oplus\mathcal{O}_C(2)\oplus\mathcal{O}_C(-4)$ | $\mathcal{O}_C(2)\oplus\mathcal{O}_C(-4)$ |
| 12, 13 | $\mathcal{O}_C(3)\oplus\mathcal{O}_C(2)\oplus\mathcal{O}_C(-5)$ | $\mathcal{O}_C(3)\oplus\mathcal{O}_C(-5)$ |
| 12, 12 | $\mathcal{O}_C(3)\oplus\mathcal{O}_C(3)\oplus\mathcal{O}_C(-6)$ | $\mathcal{O}_C(4)\oplus\mathcal{O}_C(-6)$ |
| 11, 13 | $\mathcal{O}_C(4)\oplus\mathcal{O}_C(2)\oplus\mathcal{O}_C(-6)$ | $\mathcal{O}_C(4)\oplus\mathcal{O}_C(-6)$ |
| 11, 12 | $\mathcal{O}_C(4)\oplus\mathcal{O}_C(3)\oplus\mathcal{O}_C(-7)$ | $\mathcal{O}_C(5)\oplus\mathcal{O}_C(-7)$ |

[6.4]

If we have $f_0=t^9$, $f_1=t^6u^3$, $f_2=t^3u^6$, $g=u^{12}$, then $N_C|_F\simeq\mathcal{O}_C(-1)\oplus\mathcal{O}_C(-1)$.

REMARK: It would be very interesting to know what the number of twisted cubics on a generic quintic is. A more detailed understanding of the moduli space of twisted cubics is necessary. In [9], we see that this number is divisible by 5, by studying the degeneration of the quintic to a union of 5 hyperplanes, along the lines of [8].

## References


[1] H. CLEMENS: Homological equivalence, modulo algebraic equivalence, is not finitely generated. *Publ. Math IHES* 58 (1983) 19–38.

[2] H. CLEMENS: Some results on Abel-Jacobi mappings. In: *Topics in transcendental algebraic Geometry*. Princeton Univ. Press: Princeton, NJ (1984).

[3] A. COLLINO, J. MURRE and G. WELTERS: On the family of conics lying on a quartic threefold. *Rend. Sem. Mat. Univ. Pol.* Torino 38 (1980).

[4] P. GRIFFITHS and J. HARRIS: *Principles of algebraic Geometry*. Wiley: New York (1978).

[5] L. GRUSON, R. LAZARSFELD and C. PESKINE: On a theorem of Castelnuovo and the equations defining space curves. *Inv. Math.* 72 (1983) 491–506.

[6] J. HARRIS: Galois groups of enumerative problems. *Duke Math. Jour.* 46 (1979) 685–724.

[7] R. HARTSHORNE: *Algebraic Geometry*. Springer-Verlag: New York (1977).

[8] S. KATZ: Degenerations of quintic threefolds and their lines. *Duke Math. Jour.* 50 (1983) 1127–1135.

[9] S. KATZ: Lines on complete intersection threefolds with $K=0$, to appear in *Math. Zeit.*

[10] H. MATSUMURA: *Commutative Algebra*. Benjamin/Cummings: Reading, Mass. (1980).

[11] S. MORI: On degrees and genera of curves on smooth quartic surfaces in $\mathbf{P}^3$. *Nagoya Math. Jour.* 96 (1984).
[12] I. PETROVSKII and S. NIKOLSKI, eds: Algebraic surfaces. *Proc. Steklov Inst. Math.* 75 (1965) AMS, Prov. RI 1967.

(Oblatum 6-VI-1985 & 4-X-1985)

Department of Mathematics
University of Oklahoma
Norman, OK 73019
USA