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ON THE HYPERSURFACES CONTAINING A GENERAL PROJECTIVE CURVE

E. Ballico and Ph. Ellia

If C is a smooth curve in $\mathbb{P}^N$ a natural question to ask is the number of hypersurfaces of degree k containing the curve C. This turns out to the study of the natural map of restriction $r_C(k)$: $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(C, \mathcal{O}_C(k))$. We say that C has maximal rank if for every $k \geqslant 1$ $r_C(k)$ has maximal rank as a map between vector spaces. In this paper we prove the following theorem.

THEOREM 1: *Fix integers N, d, g with $N \geqslant 3$, $g \geqslant 0$, $d \geqslant \max(2g-1, g+N)$. Then a general non degenerate embedding of degree d in $\mathbb{P}^N$ of a general curve of genus g has maximal rank.*

The proof of Theorem 1 gives as a byproduct the following result.

THEOREM 2: *Fix an integer $N \geqslant 3$. There exists a function e_N: $\mathbb{N} \to \mathbb{N}$ with $\lim_{g \to +\infty} e_N(g) = +\infty$ and with the following property: for all integers d, g with $g \geqslant 0$, $d \geqslant 2g - e_N(g)$, a general embedding of degree d in $\mathbb{P}^N$ of a general curve of genus g has maximal rank.*

Both theorems are particular cases of the maximal rank conjecture, which states that a general embedding of a curve with general moduli has maximal rank.

Previously we proved stronger results for $N = 4$ ([2]) and $N = 3$ ([3]). We use in an essential way reducible curves and the general methods introduced in [5] and [7]. The smoothing theorems we use were proved in [9] and [6].

Notations

We work over an algebraically closed field. Fix a closed subscheme X of a projective space K. Let $r_{X,K}(n)$: $H^0(K, \mathcal{O}_K(n)) \to H^0(X, \mathcal{O}_X(n))$ be the restriction map and let $\mathcal{I}_{X,K}$ be the ideal sheaf of X in K. If $K = \mathbb{P}^N$, we will write often $r_X(n)$ and $\mathcal{I}_X$ instead of $r_{X,K}(n)$ and $\mathcal{I}_{X,K}$. Fix integers d, g, N with $N \geqslant 3$, $g \geqslant 0$, $d > 0$. Let $Z(d, g; N)$ be the closure in the Hilbert scheme Hilb $\mathbb{P}^N$ of the set of smooth, connected curves C in $\mathbb{P}^N$ with deg $C = d$, C of genus g, $h^1(C, \mathcal{O}_C(1)) = 0$, and spanning a linear space of dimension $\min(N, d-g)$. Obviously $Z(d, g; N)$ is irreducible.

Fix a curve C and a line L in $\mathbb{P}^N$; L is a k-secant to C, $k = 1, 2$, if it intersects C exactly at k points, all smooth points of C, and quasi-transversally.

§1. Preliminaries

As in [7], [1], [2], [3] we use in an essential way the existence of suitable reducible curves in $Z(d, g; N)$. Fix a curve $X \in Z(d, g; N)$ with at most ordinary nodes as singularities and $h^1(X, N_X) = 0$, where N_X is the normal bundle of X in $\mathbb{P}^N$ and a line L which is k-secant to X with $k = 1$ or 2. If $d < g + N$ and $k = 1$, assume that L is not contained in the linear space spanned by X. Then $X \cup L$ is in $Z(d+1, g+k-1; N)$ ([9] or [6]).

Fix integers d, g, N with $g \geqslant 0$, $N \geqslant 3$ and $d \geqslant g + N$. If $d = N$, we say that $(N, 0; N)$ has critical value 1. If $d > N$, let n be the first integer $m \geqslant 2$ such that

$$md + 1 - g \leqslant \binom{N+m}{N}; \tag{1}$$

in this case we say that $(d, g; N)$ (or (d, g) for short) has critical value n. Note that if (1) is satisfied, then

$$d(m+1) + 1 - g \leqslant \binom{N+m+1}{N},$$

because (1) implies

$$d \leqslant \binom{N+m}{N} / (m-1)$$

and the inequality we have to check follows from the inequality:

$$d < \binom{N+m}{N-1}.$$

We say that the surjective part of Theorem 1 holds in $\mathbb{P}^N$ for a datum (d, g) with critical value n if for a general $Y \in Z(d, g; N)$ the restriction map $r_Y(n)$ is surjective. We say that the injective part of Theorem 1 holds for the datum $(d, g; N)$ with critical value n if for a general $X \in Z(d, g; N)$ the map $r_X(n-1)$ is injective. By Castelnuovo's lemma

([8], p. 99) Theorem 1 holds if for all data the injective and the surjective parts of Theorem 1 are true. Theorem 1 is trivial for all data with critical value 1. The injective part of Theorem 1 is trivial for all data with critical value 2.

In 1.1 we show in particular that the surjective part of Theorem 1 is true for all data with critical value 2. The next result can be considered as a partial extension to non-complete linear systems of [1].

PROPOSITION 1.1: *Fix integers d, g, N with $N \geqslant 3$, $g \geqslant 0$, $d \geqslant g + N$ and $2d + 1 - g \leqslant (N + 1)(N + 2)/2$. Then a general element of $Z(d, g; N)$ has maximal rank.*

PROOF: If $d = g + N$, the result was proved in [1]. Assume $d > g + N$ and the result true for $(d - 1, g; N)$. Fix $X \in Z(d - 1, g; N)$ with maximal rank, hence with $r_X(2)$ surjective. It is sufficient to prove that for a general line L intersecting X, we have $\dim \operatorname{Ker} r_{X \cup L}(2) \leqslant \dim \operatorname{Ker} r_X(2) - 2$. We may assume X irreducible. Fix a point P which is not a base point of $H^0(\mathbb{P}^N, \mathcal{J}_X(2))$. If L is a line containing P we have $\dim \operatorname{Ker} r_{X \cup L}(2) \leqslant \dim \operatorname{Ker} r_X(2) - 1$. Fix a quadric Q containing X and P. If $L \not\subset Q$, then $\dim \operatorname{Ker} r_{X \cup L}(2) < \dim \operatorname{Ker} r_{X \cup \{P\}}(2)$: we won. If P' is a point of Q, P' near P, then P' is not a base point of $H^0(\mathbb{P}^N, \mathcal{J}_X(2))$. Hence we won if for a fixed $A \in X$ and a general P' in Q, the line $[AP']$ is not contained in Q. If for all such P', $[AP']$ is contained in Q, then Q is a cone with vertex A. But since X is non-degenerate, Q cannot be a cone with vertex containing X. □

§2. Intersection with a hyperplane

The following easy lemma is the heart of this paper.

LEMMA 2.1: *Fix $N \geqslant 3$, $n \geqslant 1$. Let $C \subset \mathbb{P}^N$ be a nondegenerate, irreducible curve and $H \subset \mathbb{P}^N$ a hyperplane. Fix a vector subspace V of $H^0(H, \mathcal{O}_H(n))$. For a curve A in $\mathbb{P}^N$, A intersecting transversally H, set $V(A) := \{f \in V : f(P) = 0$ for each P in $A \cap H\}$. Then for a general reducible conic S such that each of the irreducible components of S intersects C, we have* $\dim V(S) = \max(0, \dim V - 2)$.

PROOF: For a general line intersecting C, we have $\dim V(L) = \max(0, \dim V - 1)$. Hence we may assume $\dim V \geqslant 2$. Suppose that the lemma is false. Then for every line R intersecting C and L but not contained in H, $V(L)$ has $R \cap H$ in the base locus. But if R is near to L, $R \cap H$ is not in the base locus of V, hence $L \cap H$ is in the base locus of $V(R)$ and we have $V(R) = V(L)$. For a general line B intersecting C and R we have $V(B) = V(R)$. In a finite number of steps we obtain that

$V(L)$ has H in the base locus, because C is not degenerate: contradiction. □

This lemma is the key difference between this paper and [2]. Now the proofs are easier and shorter, but the result weaker. To show how we will use this lemma we state an immediate Corollary of 2.1.

COROLLARY 2.2: *Fix non negative integers n, d, g, x, n, j with $N \geqslant 3$, $n \geqslant 1$, $x \leqslant d$. Fix a hyperplane H in $\mathbb{P}^N$ and a curve W in H with $r_{W,H}(n)$ surjective. Let j be the dimension of the linear space spanned by W; if $j \leqslant N-2$ assume $x \leqslant j+1$ and set $j' = j$; otherwise set $j' = j+1$. Assume $d \geqslant 2g + \max(0, x - j' - 1)$. Then there exists $Y \in Z(d, g; N)$, Y intersecting transversally H, with card$(Y \cap W) = x$ and $r_{W \cup (Y \cap H), H}(n)$ of maximal rank.*

PROOF: Note that Aut(H) acts transitively on the set of $N+1$ ordered points of H such that any N of them span H. Hence the case $d = N$ is trivial and we assume $d > N$. For the same reason there is a curve $C \in Z(N+1, \min(1, g); N)$ intersecting H transversally with card$(C \cap W) = \min(N+1, x)$ and $r_{W \cup (C \cap H), H}(n)$ of maximal rank. Then we take $\max(0, N+1-x)$ lines L_i, each L_i intersecting both C and W. Then we apply 2.1. □

§3. Proof of Theorem 1

In section 1 we proved Theorem 1 for curves with critical value at most 2. Since Theorem 1 is known to be true in $\mathbb{P}^3$ and $\mathbb{P}^4$ ([3],[2]), it is sufficient to prove the following two lemmas.

LEMMA 3.1: *Fix $N \geqslant 5$, $n \geqslant 3$. Assume that theorem 1 hold in $\mathbb{P}^s$ for all s with $3 \leqslant s \leqslant N-1$ and that theorem 1 holds in $\mathbb{P}^N$ for all data with critical value $< n$. Then the surjectivity part of theorem 1 holds in $\mathbb{P}^N$ for all data with critical value n.*

LEMMA 3.2: *Fix $N \geqslant 5$, $n \geqslant 3$. Assume that theorem 1 holds in $\mathbb{P}^s$ for all s with $3 \leqslant s \leqslant N-1$ and that theorem 1 holds in $\mathbb{P}^N$ for all data with critical value $< n$. Then the injectivity part of theorem 1 holds in $\mathbb{P}^N$ for all data with critical value n.*

In this section we prove 3.1 and 3.2, hence Theorem 1. Fix a datum (d, g) with $d \geqslant \max(g+N, 2g-1)$ and critical value $n \geqslant 3$ in $\mathbb{P}^N$, $N \geqslant 5$.

PROOF OF LEMMA 3.1: Fix natural numbers p, g' with $p \leqslant g$, $g' \leqslant g$ and maximal with the following properties

$$(2p+N)(n-1)+1-p \leqslant \binom{N+n-1}{N}$$

$$(n-1)(\max(g'+N, 2g'-1))+1-g' \leqslant \binom{N+n-1}{N}$$

The integers p, g' exist because $(N, 0; N)$ has critical value $1 \leqslant n-1$. Define integers $f \geqslant 2p+N$, $d' \geqslant \max(g'+N, 2g'-1)$ by the relations

$$\binom{N+n-1}{N} - n + 2 \leqslant (n-1)f + 1 - p \leqslant \binom{N+n-1}{N} \tag{2}$$

$$\binom{N+n-1}{N} - n + 2 \leqslant (n-1)d' + 1 - g' \leqslant \binom{N+n-1}{N} \tag{3}$$

Note that $p \leqslant g'$ and $f \leqslant d' < d$ because (d, g) has critical value n. Set $d'' = d - d'$, $g'' = g - g'$, $x = \min([(d-f+1)/2], g-p)$, $j = g - p - x$, $e = \binom{N+n}{N} - nd - 1 + g$,

$$k = \binom{N+n-1}{N} - (n-1)f - 1 + p,$$

$$k' = \binom{N+n-1}{N} - (n-1)d' - 1 + g'.$$

By (2) and (3) we have $0 \leqslant k \leqslant n-2$ and $0 \leqslant k' \leqslant n-2$. By the definition of k and e we obtain

$$(d-f)n + 1 - (g-p) + (f-1) + (e-k) = \binom{N+n-1}{N-1} \tag{4}$$

By the maximality of p we have either $p = g$ or $f \leqslant 2p + N + 1$, hence $d - f \geqslant 2(g-p) - N - 2$. Hence we have $j \leqslant (N+3)/2$. By the maximality of g' we have either $g' = g$ or $d' \leqslant 2g'$ or $g' + N \geqslant 2g' - 1$ and $d' \leqslant g' + N + 1$. Assume $g' + N \geqslant 2g' - 1$, hence $g' \leqslant N + 1$. Since $k' \leqslant n - 2$ we obtain

$$(n-1)(g'+N+1) + 1 - g' + (n-2) \leqslant \binom{N+n-1}{N}$$

which is false for $N \geqslant 5$, $n \geqslant 3$. Hence we have $d'' \geqslant 2g'' - 1$.

We need two numerical lemmas:

SUBLEMMA 3.3: *If $N \geqslant 5$ and $n \geqslant 3$, we have $f \geqslant 2n - 4 + N$.*

PROOF: Since

$$(n-1)f \geqslant \binom{N+n-1}{N} - 1,$$

the lemma is trivial. □

SUBLEMMA 3.4: *Assume $k > e$. Then*
(a) $d-f \geqslant 2n-1+N$ if $N \geqslant 5$, $n \geqslant 4$ or $N \geqslant 6$, $n \geqslant 3$.
(b) $d-f \geqslant 9$ if $N=5$, $n=3$ and if $d-f=9$, then $g-p \leqslant 4$.
(c) $d'' \geqslant n-1$; $d-f \geqslant 2N-2$, hence $d-f \geqslant x+N-1$.

PROOF: (a) By (2) we have

$$f \leqslant \binom{N+n}{N} / (n-3/2).$$

Then (4) gives the contradiction if $N \geqslant 5$, $n \geqslant 6$ or $N \geqslant 6$, $n=5$ or $N \geqslant 7$, $n=4$ or $N \geqslant 12$, $n=3$. The remaining cases for (a) and (b) have to be checked directly. For example assume $N=5$, $n=3$. By the definitions of p and f we obtain $p \leqslant 3$ and $f \leqslant 11$. From (4) we get $d-f \geqslant 9$ and if $d-f=9$, then $g-p \leqslant 4$. Part (c) is easier. □

We distinguish 5 cases.
Case (A): $k \leqslant e$, $d-f \geqslant g-p+1$, $d-f \geqslant 6$. Take a hyperplane H. We claim the existence of $W \subset H$, $W \in Z(d-f,\ x;\ N-1)$ with $r_{W,H}(n)$ surjective. Indeed since $d-f-x \geqslant 3$, we have $Z(d-f,\ x;\ N-1) \neq \emptyset$. If a general $W \in Z(d-f,\ x;\ N-1)$ spans H, the claim follows from the inductive assumption, (4) and the inequality $f-1 \geqslant j$ which holds by 3.3. If a general $W \in Z(d-f,\ x;\ N-1)$ does not span H, it spans a linear space of dimension $d-f-x \geqslant 3$ and we may use the inductive assumption and the inequality

$$n(d-f)+1-x \leqslant \binom{d-f-x+n}{n}$$

which is true if $n \geqslant 3$, $d-f \geqslant 6$.

We may assume that a curve W as in the claim contains $j+1$ general points of H because $d-f-x \geqslant j+1$. By the inductive assumption, the inequality $f-p \geqslant N+j+1$ and Corollary 2.2 we may find $X \in Z(f,\ p;\ N)$, X intersecting transversally H, with $\mathrm{card}(X \cap W) = j+1$ and $r_{W \cup (X \cap H),H}(n)$ surjective. Since W can be degenerate to a suitable union of lines, $X \cup W$ is a smooth point of Hilb $\mathbb{P}^N$ and $W \cup W \in Z(d,\ g;\ N)$.

Take $A \subset \mathbb{P}^N \setminus H$, $B \subset H$, with $\mathrm{card}(A) = k$, $\mathrm{card}(B) = e-k$, A and B general. It is sufficient to prove that $r_{X \cup W \cup A \cup B}(n)$ is injective, hence bijective. Take $f \in H^0(\mathbb{P}^N, \mathscr{I}_{X \cup W \cup A \cup B}(n))$. The restriction of h to H vanishes on $W \cup (X \cap H) \cup B$, hence vanishes identically. Thus h is divided by the equation z of H. Since h/z vanishes on $X \cup A$, we have $h = 0$.

Case (B): $k > e$, $p \geqslant k - e$. Assume $d - f \leqslant g - p + n - 2$. Since $d - f \geqslant 2(g - p) - N - 2$, we find $d - f \leqslant 2n - 2 + N$, contradicting 3.4. We take a general $E \in Z(f, p - k + e; N)$ with $r_E(n-1)$ surjective, hence $h^0(\mathbb{P}^N, \mathscr{I}_E(n-1)) = e$. Note that by 3.3 and a degeneration of E to a union of lines, we may assume that E contains $1 + k - e + j$ general points of a hyperplane H. We may take $W \in Z(d - f, x; N - 1)$, $W \subset H$, with $r_{W,H}(n)$ surjective and $\operatorname{card}(W \cap E) = 1 + k - e + j$ because $d - f - x \geqslant N - 1$ and $d - f - x \geqslant j + k - e + 1$ by 3.4; in particular W spans H. By 2.2 we may deform E to E', W to W' with $r_{E'}(n-1)$ surjective, $r_{W' \cap (E' \cap H), H}(n)$ surjective and $\operatorname{card}(E' \cap W') = 1 + k - e + j$. Note that $W' \cup E' \in Z(d, g; N)$. As in case A) we prove the surjectivity of $r_{E' \cup W'}(n)$.

Case (C): $k > e$, $p < k - e$. Note that we have $p = g = g'$ because by 2.3 we cannot have $f \leqslant 2p + N + 1 \leqslant 2n - 5 + N$; hence $f = d'$. By a particular case of the main result of [4] there exists $F \subset \mathbb{P}^N$, F disjoint union of a rational curve T of degree $f - (k - e - g)$ and $(k - e - g)$ lines with $r_F(n-1)$ surjective. By 3.4(c) we may find a curve W contained in a hyperplane H, W rational and connected, $\deg W = d''$, with $r_{W,H}(n)$ surjective, W intersecting every connected component of F and intersecting T exactly at $1 + g$ points. We conclude as in case (A).

Case (D): $k \leqslant e$, $d - f \leqslant g - p$. Since $d - f \geqslant 2(g - p) - N - 2$, we have $d - f \leqslant g - p \leqslant N + 2$. If $g'' \neq 0$, we have $d' \leqslant 2g'$ and $d - f \geqslant d'' \geqslant 2g'' - 1$, hence $g'' \leqslant (N + 3)/2$. First assume $g'' \geqslant 2$. We take $E \in Z(d', g'; N)$, E intersecting transversally a hyperplane H, and a connected elliptic curve $W \subset H$, with $\deg W = d''$ and $\operatorname{card}(E \cap W) = g''$. This is possible because $d'' \geqslant 2g'' - 1 \geqslant 3$. It is sufficient to prove that we may find E and W as above with $r_{\mathrm{W} \cup (\mathrm{E} \cap \mathrm{H}), \mathrm{H}}(n)$ surjective. Set $u = \min(N, g')$. By [1] (as used in 1.1) we may find $C \in Z(u + N, u; N)$ with $r_C(2)$ surjective. We may assue that C intersects transversally H. From the linear normality of C and the exact sequence

$$0 \to \mathscr{I}_C(1) \to \mathscr{I}_C(2) \to J_{C \cap H, H}(2) \to 0$$

we obtain that $r_{C \cap H, H}(2)$ is surjective. Now we take a hyperplane A of H containing exactly g'' points of $C \cap H$; this is possible because $g'' \leqslant N - 1$ for $N \geqslant 5$. In A we add an elliptic curve W, $\deg(W) = d''$, W containing g'' points of $C \cap H$. We may assume $r_{W,A}(3)$ surjective (even if $d'' \geqslant N$) by the inductive assumption. As in case A) we find that

$r_{W \cup (C \cap H), H}(3)$ is surjective. By 2.1 we may find $E \supset C$ with the properties we are looking for.

If $g'' \leqslant 1$ we take as A a hyperplane of H containing $1 + g''$ points of $C \cap H$ and we take in A a connected rational curve of degree d'' containing $1 + g''$ points of C.

Case (E): $d - f \leqslant 5$. By case (D) we may assume $d - f \geqslant g - p + 1$. We take a suitable $Y \in Z(f, p; N)$ and we add in a hyperplane H a connected, rational curve of degree $d - f$ containing $g - p + 1$ points of Y.

The proof of Lemma 3.1 is over.

PROOF OF LEMMA 3.2: Since 1.1 works even in the injective range we may assume $n \geqslant 4$. Let s, s' be the maximal integers with $0 \leqslant s \leqslant g$, $0 \leqslant s' \leqslant g$ and

$$(n-2)(2s+N) + 1 - s \leqslant \binom{N+n-2}{N} + n - 3$$

$$(n-2)(\max(s'+N, 2s'-1)) \leqslant \binom{N+n-2}{N} + n - 3$$

Let r, r' be the only integers with $r \geqslant 2s + N$, $r' \geqslant \max(s' + N, 2s' - 1)$ and satisfying

$$\binom{N+n-2}{N} \leqslant (n-2)r + 1 - s \leqslant \binom{N+n-2}{N} + n - 3 \tag{5}$$

$$\binom{N+n-2}{N} \leqslant (n-2)r' + 1 - s' \leqslant \binom{N+n-2}{N} + n - 3 \tag{6}$$

We have $s \leqslant s'$, $r \leqslant r' < d$ because (d, g) has critical value n. If $s < g$ we have $r \leqslant 2s + N + 1$ by the maximality of s. Hence $d - r \geqslant 2(g - s) - N - 2$. Set $x' = \min(g - s, [(d - r + 1)/2])$ and $j' = g - s - x$; we have $j' \leqslant (N + 3)/2$. From the definitions of h and i we find

$$(n-1)(d-r) + 1 - (g-s) + r - 1 + h - i = \binom{N+n-2}{N-1} \tag{7}$$

We need the following numerical lemmas.

SUBLEMMA 3.5: *If $N \geqslant 5$ and $n \geqslant 4$ we have $r \geqslant 2n + N - 5$.*

PROOF: We have

$$(n-2)(2n+N-5) \leqslant \binom{N+n-2}{N}. \quad \square$$

SUBLEMMA 3.6: *Fix* $N \geqslant 5$, $n \geqslant 4$. *We have* $d - r \geqslant g - s + n - 3$ *and* $d - r \geqslant 6$.

PROOF: Assume $d - r \leqslant g - s + n - 4$. Then (7) gives a contradiction if $(N, n) \neq (5, 4)$. If $N = 5$, $n = 4$, by definition we find $s \leqslant 3$ and $r \leqslant 12$. Hence (7) gives $11 \geqslant d - r \geqslant g - s$. We obtain $d > 2g - 1$, contradiction. The last part is similar. □

Let H be a hyperplane of $\mathbb{P}^N$. As in the proof of 3.1 we distinguish a few cases.

Case (A): $h \leqslant i$. We take $X \in Z(r, s; N)$ with $r_X(n-2)$ injective. As in the corresponding case of 3.1 we may find $W \in Z(d-r, x'; N-1)$, $W \subset H$, with $r_{W,H}(n)$ of maximal rank and $\text{card}(W \cap X) = 1 + j'$ (use 3.5, 3.6). Since $h \leqslant i$ we may deform $W \cup X$ to $W' \cup X'$ with $r_{W' \cup X'}(n - 1)$ injective.

Case (B): $h > i$, $s \geqslant n - 2 - n + i$. Set $m = r - 1$, $m' = s - (n - 2 - h + i)$. Take $Y \in Z(m, m'; N)$ with $r_Y(n-2)$ injective. By 3.6 we may find $W \in Z(d - m, x'; N - 1)$, $W \subset H$, with $r_{W,H}(n-1)$ of maximal rank. We may apply to $Y \cup W$ the smoothing theorems for k-secants, $k = 1, 2$, because $m - m' \geqslant N + 1 + (n - 2 - h + i)$.

Case (C): $h > i$, $s < n - 2 - h + i$. If $s < g$, then $r \leqslant 2s + N + 1$. By 3.5 we have $s = g$. By [4] we may find a curve Y, $\deg Y = r - 1$, Y disjoint union of a rational curve T, $\deg T = r - 1 - (n - 2 - s - h + i)$, and $n - 2 - s - h + i$ disjoint lines, with $r_Y(n-2)$ injective. By 3.6 we may find $W \in Z(d - r + 1, 0; N - 1)$, $W \subset H$, with $r_{W,H}(n-1)$ of maximal rank, W intersecting every connected component of Y and intersecting T in exactly $1 + g$ points.
The proofs of 3.2 and Theorem 1 are over.

§4. Proof of theorem 2

As a byproduct of the proof of Theorem 1, we will obtain a proof of Theorem 2. From this proof it would be possible to obtain an explicit bound for the functions e_N; however this bound is too weak in any explicit situation. Since if $d < 2g - 1$, $d \geqslant g + N$, the genus of a triple $(d, g; N)$ with critical value n goes to infinity as n goes to infinity, we may fix $(d, g; N)$ with $N \geqslant 5$, $d \leqslant 2g - 2$, $d \geqslant g + N$, $d \leqslant 2g - n + N + 1$ and critical value $n \geqslant g - N$; it is sufficient to prove that a general element in $Z(d, g; N)$ has maximal rank.

We use the notations of Section 3, but with these new bounds on d. First consider the surjective part as in 3.1. The definitions of f, p, d', g' make sense even now. Certainly we have $s \leqslant g' < g$ because $d < 2g - 1$.

Hence $f \leqslant 2p+n+1$. Again we define k, k', e, x, j, d'', g'' with the same formulas. Now we have $d-f \geqslant 2(g-p)-n$ and $2j \leqslant (n-3)$.

First assume $d-f \geqslant 4n+1$. In case (A) we do not need the assumptions "$d-f \geqslant 6$" and "$d-f \geqslant g-p+1$"; hence we do not need cases (D) and (E). Indeed by the assumptions on $d-f$ and n, we may take $W \in Z(d-f, x; N-1)$, W spanning a hyperplane H and containing $n+1 \geqslant 1+j$ general points of H. In case (B) we may take $W \in Z(d-f, x; N-1)$ intersecting Y at $1+(n-2-h+i)-s+g-x \leqslant 2n+1$ points, because $d-f-x \geqslant 2n$. Case (C) cannot occur now because $p<g$.

Now assume $d-f \leqslant 4n$. Set $D=f-4n-2$, $G=p-2n-1$. We need two numerical lemmas.

LEMMA 4.1: *Assume* $N \geqslant 5$ *and* $n \geqslant 11$. *We have* $p \geqslant 3n+2$, *hence* $D \geqslant 2n+2+N$.

PROOF. If $p \leqslant 3n$, we have $f \leqslant 6n+N+1$ and (2) gives a contradiction. □

LEMMA 4.2: *Assume* $N \geqslant 5$, $n \geqslant 11$ *and* $d-f \leqslant 4n$. *Then* $e \geqslant (4n+2)(n-1)+n-2$.

PROOF: Use (2) and (4). □

We repeat the construction of 3.1 substituting (f, g) with (D, G). By Lemma 4.2 we have $k+(4n+2)(n-1) \leqslant e$ if $n \geqslant 11$, hence it is sufficient to consider case (A). Now we show what to change in the proof of 3.2 to obtain the injectivity part of Theorem 2. We may define r and s using the same formulas. Now we have $s<g$ because $(2g-1, g)$ has critical value at least n; now we have $d-r \geqslant 2(g-s)-n$.

If $d-r \geqslant 4n+1$, we may copy the proof of 3.2 with the same modifications just given. We conclude using the following lemma.

LEMMA 4.3: *If* $N \geqslant 5$ *and* $n \geqslant 11$, *we have* $d-r \geqslant 4n+1$.

PROOF: Assuming $d-r \leqslant 4n$. Since

$$r \leqslant \binom{N+n-2}{N} \Big/ (n-3),$$

the lemma follows from (6). □

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