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NEARLY PRINCIPAL MINORS OF M-MATRICES

Gerard Sierksma and Evert Jan Bakker

Abstract

It is well-known that two-two minors $c_{ii}c_{kj} - c_{ij}c_{ki}$ of the adjoint $\{c_{ij}\}$ of an irreducible nonsingular M-matrix with at least one of its elements c_{ii} on the main diagonal are nonnegative. In this paper this result is generalized for nearly principal minors having all but one of its diagonal elements on the main diagonal of the original matrix. Nearly principal minors of both an M-matrix and its adjoint are studied. There is also given a new proof of the Metzler theorem about strict domination of the main diagonal of the adjoint, i.e. $c_{ii} > c_{ij}$, under very weak conditions. No irreducibility conditions are needed throughout the paper.

1. Notation and definitions

A nonsingular M-matrix is any matrix of the form $sI - A$ with $s > 0$, $A \geqslant 0$ and such that the Perron-Frobenius root of A is $< s$. Nonsingularity is equivalent to the Hawkins-Simon condition, i.e. all principal minors of $sI - A$ positive. For the definitions of irreducibility and adjoint we refer to Berman and Plemmons [2]. In [2] there is an extensive list of properties of M-matrices. We denote $\langle n \rangle = \{1, 2, 3, \dots, n\}$, $i_{\langle k \rangle} = \{i_1, \dots, i_k\} \subset \langle n \rangle$ with $i_1 < i_2 \dots < i_k$, and $i'_{\langle k \rangle} = \langle n \rangle \setminus i_{\langle k \rangle}$. By $A(i_{\langle k \rangle}, j_{\langle k \rangle})$ is denoted the matrix consisting only of the rows $i_1, \dots, i_k$ and the columns $j_1, \dots, j_k$ of the original (n, n)-matrix A. Similarly $A(i'_{\langle k \rangle}, j'_{\langle k \rangle})$ is the matrix A with the rows $i_1, \dots, i_k$ and the columns $j_j, \dots, j_k$ deleted, and will be called the complement of $A(i_{\langle k \rangle}, j_{\langle k \rangle})$. Let $s, t \in \langle n \rangle$.

An (s, t)-Nearly Principal Minor ((s, t)-NPM) of order k of an (n, n)-matrix A is a minor $C(i_{\langle k \rangle}, j_{\langle k \rangle}) = \det A(i_{\langle k \rangle}, j_{\langle k \rangle})$ of A of order k such that

$$i_{\langle k \rangle} \cap j'_{\langle k \rangle} = \{ i_s \} \quad \text{and} \quad j_{\langle k \rangle} \cap i'_{\langle k \rangle} = \{ j_t \}.$$

So an (s, t)-NPM of order k contains $k-1$ main diagonal elements of A. Note that if $C(i_{\langle k\rangle}, j_{\langle k\rangle})$ is an (s, t)-NPM, then $C(i'_{\langle k\rangle}, j'_{\langle k\rangle})$ is an (s', t')-NPM with $s' = |\langle j_t\rangle \cap i'_{\langle k\rangle}|$ and $t' = |\langle i_s\rangle \cap j'_{\langle k\rangle}|$.

2. Nearly principal minors

In the following theorem the relationship between an NPM and its complement is established.

THEOREM 1: *If $C(i_{\langle k\rangle}, j_{\langle k\rangle})$ is a (p, q)-NPM with $i_p < j_q$, then $C(i'_{\langle k\rangle}, j'_{\langle k\rangle})$ is an (s, t)-NPM such that*

$$s = j_q - q \quad \text{and } t = i_p - p + 1.$$

PROOF: As $C(i_{\langle k\rangle}, j_{\langle k\rangle})$ is a (p, q)-NPM, it follows from the definition that $C(i'_{\langle k\rangle}, j'_{\langle k\rangle})$ is an (s, t)-NPM with

$$i'_s = j_q \quad \text{and } j'_t = i_p.$$

Defining, $N(i_{\langle k\rangle}; \rho) = |\{i \in i_{\langle k\rangle} \mid i \leqslant \rho\}|$, we have

$$N(i_{\langle k\rangle}; \rho) + N(i'_{\langle k\rangle}; \rho) = \rho,$$

and

$$\begin{aligned} N(i_{\langle k\rangle}; \rho) &= N(j_{\langle k\rangle}; \rho) && \text{if } \rho < i_p \text{ or } \rho \geqslant j_q; \\ &= N(j_{\langle k\rangle}; \rho) + 1 && \text{if } i_p \leqslant \rho < j_p. \end{aligned}$$

Hence,

$$\begin{aligned} s &= N(i'_{\langle k\rangle}; i'_s) = i'_s - N(i_{\langle k\rangle}; i'_s) = j_q - N(i_{\langle k\rangle}; j_q) \\ &= j_q - N(j_{\langle k\rangle}; j_q) = j_q - q. \end{aligned}$$

Moreover,

$$\begin{aligned} t &= N(j'_{\langle k\rangle}; j'_t) = j'_t - N(j_{\langle k\rangle}; j'_t) = i_p - N(j_{\langle k\rangle}; i_p) \\ &= i_p - [N(i_{\langle k\rangle}; i_p) - 1] = i_p - p + 1. \end{aligned}$$

We now consider nonsingular M-matrices A. The sign of a minor of A, derived from A by deleting k rows and k columns with precisely one row- and one column-index different, is determined.

THEOREM 2: *Let $C(i_{\langle k\rangle}, j_{\langle k\rangle})$ be a (p, q)-NPM of a nonsingular M-matrix A. Then*

$$(-1)^{p+q+1}C(i_{\langle k\rangle}, j_{\langle k\rangle}) \geqslant 0, \quad \textit{and}$$

$$(-1)^{i_p+j_q+p+q}C(i'_{\langle k\rangle}, j'_{\langle k\rangle}) \geqslant 0.$$

PROOF: We may assume that $j_q > i_p$, so that $q \geqslant p$. The matrix $A(i_{\langle k\rangle}, j_{\langle k\rangle})$ can be extended to a principal matrix by adding a row and a column, namely as row the elements of A with coordinates $(j_q, j_1), \ldots, (j_q, i_p), \ldots, (j_q, j_k)$ and as column the elements of A with coordinates $(i_1, i_p), \ldots, (j_q, i_p), \ldots, (i_k, i_p)$. This extended matrix is a nonsingular M-matrix. It follows from [2] N38 that the elements of the inverse of this extended matrix are nonnegative. As the element with coordinates (j_q, i_p) according to the matrix A is an element with coordinates $(q+1, p)$ relative to the new extended matrix, it follows that $(-1)^{p+q+1}C(i_{\langle k\rangle}, j_{\langle k\rangle}) \geqslant 0$, which proves the first part of the theorem. The second part follows from Theorem 1 by observing that $C(i'_{\langle k\rangle}, j'_{\langle k\rangle})$ is an (s, t)-NPM with $s = j_q - q$ and $t = i_p - p + 1$. As $s + t + 1 = i_p + j_p - (p+q) + 2$, it follows that

$$0 \leqslant (-1)^{s+t+1}C(i'_{\langle k\rangle}, j'_{\langle k\rangle}) = (-1)^{i_p+j_q-(p+q)+2}C(i'_{\langle k\rangle}, j'_{\langle k\rangle})$$

$$= (-1)^{i_p+j_q+p+q}C(i'_{\langle k\rangle}, j'_{\langle k\rangle}).$$

THEOREM 3: *Let C_{st} be an (s, t)-NPM of the adjoint of a nonsingular M-matrix and C'_{st} its complement. Then*

$$(-1)^{s+t}C_{st} \geqslant 0, \quad \textit{and}$$

$$(-1)^{i_s+j_t+s+t+1}C'_{st} \geqslant 0.$$

PROOF: Concerning the first part of the theorem, see Murphy [3]. The second part follows directly from Theorems 1 and 2.

EXAMPLE: The following diagram represents the adjoint $C = \{c_{ji}\}$ of a nonsingular M-matrix of order 10. Take for example the (s, t)-NPM with main diagonal elements c_{33}, c_{44}, c_{77}, and c_{1010} and off-diagonal element $c_{i_s j_t}$. If (i_s, j_t) is in a '+'-block then $C_{st} \geqslant 0$, and if (i_s, j_t) is in a '−'-block then $C_{st} \leqslant 0$. So the sign of an (s, t)-NPM

	1	2	3	4	5	6	7	8	9	10
1	+	+			+	+		−	−	
2	+	+			+	+		−	−	
3			c_{33}							
4				c_{44}						
5	+	+			+	+		−	−	
6	+	+			+	+		−	−	
7							c_{77}			
8	−	−			−	−		+	+	
9	−	−			−	−		+	+	
10										c_{1010}

is precisely the sign of the term $c_{i_1 i_1}, \ldots, c_{i_s j_t}, \ldots, c_{i_k i_k}$ in the development of the determinant, i.e. the term with all the $k-1$ main diagonal elements. So if the number of successive c_{ii}'s is even in the NPM, e.g. $c_{11}c_{22}c_{55}c_{66}c_{77}c_{88}$, then such an (s, t)-NPM is nonnegative, independent on the choice of s and t.

The following corollary can also be found in Sierksma [6]. Theorem 3 provides a direct proof of this corollary.

COROLLARY 4: *Let* $C = \{c_{ji}\}$ *be the adjoint of a nonsingular M-matrix with order* $n \geqslant 3$. *Then*

$$c_{ii}c_{kj} - c_{ij}c_{ki} \geqslant 0$$

for each $i, j, k \in \langle n \rangle$.

PROOF: $c_{ii}c_{kj} - c_{ij}c_{ki}$ is an NPM except for the sign. If $k = j$ then $c_{ii}c_{kj} - c_{ij}c_{ki} > 0$ according to Jacobi's formula and the Hawkins-Simon condition. If $k < i$ and $i < j$ then $s = 1$ and $t = 2$, so $C_{st} = C_{12} = c_{ki}c_{ij} - c_{ii}c_{kj} \leqslant 0$. Similarly for $i < k$ and $j < i$. If $i < k$ and $i < j$ then $C_{st} = C_{11} = c_{ii}c_{kj} - c_{ij}c_{ki} \geqslant 0$. Similarly for $k < i$ and $i > j$.

Nearly principal minors of order 2 occur many times in the theory of nonnegative matrices and its applications. Without references we mention that nearly principal minors play a part in mathematical economics especially in the theory of the Leontief model, e.g. in the theory of relative changes of final demand and gross output, in the theory of taxes and subsidies, and in the theory of Stolper-Samuelson and Kemp-Wegge on international trade. In the following paragraph we study the well-known theorem of Metzler which is used e.g. in the Leontief model when considering absolute changes of gross production and final demand.

3. Metzler's theorem

Principal minors of order 1 are just the elements of the main diagonal, and off-diagonal elements are (1, 1)-NPM's of order 1. Metzler's theorem (see e.g. Seneta [5]) asserts that $c_{ii} > c_{ij}$, for each $i,\ j \in \langle n \rangle\ i \neq j$, where $C = \{c_{ji}\}$ is the adjoint of an M-matrix sI-T with $T \gg 0$ and all row sums of T strictly less than 1; moreover, T needs to be irreducible. It follows directly from Theorem 3 that $c_{ij} > 0$ for each $i,\ j \in \langle n \rangle$, $i \neq j$. In the following theorem we drop some of these conditions.

THEOREM 5: *Let T be a nonnegative matrix with all, except the k-th, row sums strictly less then s and let $C = \{c_{ji}\} = \operatorname{adj}(sI - T)$. Then for each $j \neq k$ the following holds*

$$c_{kk} > c_{kj}$$

PROOF: Without loss of generality we may assume that $k = 1$. Let $T = \{t_{ij}\}$. Take $j \neq 1$. Then

$$c_{11} - c_{1j} = \begin{vmatrix} s - t_{22} & \cdots & -t_{2n} \\ \vdots & & \vdots \\ -t_{n2} & \cdots & s - t_{nn} \end{vmatrix} - (-1)^{j+1} \begin{vmatrix} -t_{21} & \cdots & t_{2j-1} & -t_{2j+1} & \cdots & -t_{2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -t_{n1} & \cdots & t_{nj-1} & -t_{nj+1} & \cdots & s - t_{nn} \end{vmatrix}$$

$$
= \begin{vmatrix} s-t_{22} & \cdots & -t_{2n} \\ \vdots & & \vdots \\ -t_{n2} & \cdots & s-t_{nn} \end{vmatrix} - (-1)^{j+1}(-1)^{j-2}
$$

$$
\times \begin{vmatrix} s-t_{22} & \cdots & -t_{2j-1} & -t_{21} & -t_{2j+1} & \cdots & -t_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -t_{n2} & \cdots & -t_{nj-1} & -t_{n1} & -t_{nj+1} & \cdots & s-t_{nn} \end{vmatrix}
$$

$$
= \begin{vmatrix} s-t_{22} & \cdots & -t_{2j}-1 & -(t_{2j}+t_{21}) & -t_{2j+1} & \cdots & -t_{2n} \\ -t_{32} & \cdots & -t_{3j-1} & -(t_{3j}+t_{31}) & -t_{3j+1} & \cdots & -t_{3n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -t_{n2} & \cdots & -t_{nj-1} & -(t_{nj}+t_{n1}) & -t_{nj+1} & \cdots & s-t_{nn} \end{vmatrix}
$$

$$
= \det(sI - T^*).
$$

Clearly, $sI - T^*$ is an M-matrix with *all* row sums > 0. It then follows that the Perron-Frobenius root of T^* is $< s$ (see e.g. Takayama [4] Theorem 4.c.10 (Remark 3)). Note that the irreducibility is not needed. So $sI - T^*$ is nonsingular and therefore has all its principal minors positive (see e.g. [2] Theorem 8.2.3). Hence, the above determinant is also positive and therefore we have in fact that $c_{11} - c_{1j} > 0$.

COROLLARY 6: *Let T be a nonnegative matrix and $C = \{c_{ji}\}$ be the adjoint of $sI - T$ with all row sums strictly positive. Then for all i, $j \in \langle n \rangle$ with $i \neq j$ the following holds*:

$$
c_{ii} > c_{ij}.
$$

PROOF: A direct consequence of Theorem 5.

Acknowledgement

Theorem 1 and the proof of Theorem 2, as they are now, were suggested by the referee.

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