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THE INFINITESIMAL M. NOETHER THEOREM FOR SINGULARITIES

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Introduction

In 1882 M. Noether [25] has shown that for a general surface of degree $d \geqslant 4$ in $\mathbb{P}^3 = P^3_{\mathbb{C}}$ each curve in S is the intersection of S with some hypersurface S' in $\mathbb{P}^3$. Recently Carlson-Green-Griffiths-Harris [7] have given an infinitesimal version of this result: If S is a smooth hypersurface of degree $d \geqslant 4$ in $\mathbb{P}^3$ and C is a curve on S such that for each first order deformation $\tilde{S}$ of S the curve C can be lifted to a first order deformation $\tilde{C} \subseteq \tilde{S}$ then $C = S \cap S'$ with some hypersurface $S' \subseteq \mathbb{P}^3$.

The purpose of this paper is to derive a similar result for singularities. Moreover we obtain with our methods, that isolated Gorenstein-singularities $(X, 0)$ of dimension $d \geqslant 3$ with vanishing tangent functor $T^{d-2}_{X,0}$ are almost factorial, i.e. each divisor $D \subseteq X$ is set theoretically given by one equation, or – equivalently – the divisor class group $\mathrm{Cl}(\mathcal{O}_{X,0})$ is a torsion group, see [27], [10]. By a result of Huneke [20] and Buchweitz [6] the assumption on the vanishing of $T^{d-2}_{X,0}$ is always satisfied for isolated Gorenstein singularities which are linked to complete intersections.

As an application we generalize results of Griffiths-Harris [11] and Harris-Hulek [16] on the splitting of normal bundle sequences.

We remark that throughout this paper we work in characteristic 0.

§1. The Main Lemma

Let k be a field of characteristic 0 and $A = k[\![T]\!]_n/\mathfrak{a}$ a normal complete k-algebra with an isolated singularity of dimension $d \geqslant 3$. We set $X = \mathrm{Spec}(A)$, $U := X \setminus \{m_A\}$. By Ω^1_X resp. Ω^1_U we denote the sheaf associated to the module of differentials $\Omega^1_A = \coprod_{1 \leqslant i \leqslant n} A \cdot \mathrm{d}T_i / A \cdot d(\mathfrak{a})$.

The logarithmic derivativion $d\log : \mathcal{O}^x_U \to \Omega^1_U$ induces a map $\mathrm{Pic}(U) = H^1(U, \mathcal{O}^x_U) \to H^1(U, \Omega^1_U)$. Since A has isolated singularity we have $\mathrm{Cl}(A) \cong \mathrm{Pic}(U)$, see [10], (18.10) (b), and we obtain a map

$$\xi : \mathrm{Cl}(A) \to H^1(U, \Omega^1_U).$$

In this section we will show:

Main Lemma 1.1: *If* depth $A \geqslant 3$ *then* $\mathrm{Ker}(\xi)$ *is the torsion of* $\mathrm{Cl}(A)$. *In particular, if* $H^1(U, \Omega^1_U)$ *vanishes then* $\mathrm{Cl}(A)$ *is a torsion group and* A *is almost factorial.*

If $k \subseteq K$ is a subfield and if $A_K := A \widehat{\otimes}_k K$, $X_K := \mathrm{Spec}(A_K)$, $U_K := X_K \setminus \{m_{A_K}\}$, then $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(A_K)$ and $\mathrm{Cl}(A) = \mathrm{Cl}(A_K)$ by [24] if k and K are algebraically closed. Therefore by standard arguments we can easily reduce our assertion to the case $k = \mathbb{C}$, which we shall henceforth assume. Before proving (1.1) in this case we need three lemmata:

Lemma 1.2: *Let* E *be a complete algebraic* $\mathbb{C}$*-scheme. Then the canonical mapping induced by the logarithmic derivation*

$$(\mathrm{Pic}(E)/\mathrm{Pic}^{\tau}(E)) \otimes_{\mathbb{Z}} \mathbb{C} \to H^1(E, \Omega^1_E)$$

is injective.

Proof: If E is in addition smooth, then (1.2) is well known and follows from the Lefschetz-theorem on (1, 1) sections, see [12], p. 163. In the general case, let $f: E' \to E$ be a resolution of singularities of E and consider the following diagram:

$$\begin{array}{ccc} (\mathrm{Pic}(E)/\mathrm{Pic}^{\tau}(E)) \otimes_{\mathbb{Z}} \mathbb{C} & \to & H^1(E, \Omega^1_E) \\ \downarrow \varphi & & \downarrow \\ \mathrm{Pic}(E')/\mathrm{Pic}^{\tau}(E') \otimes_{\mathbb{Z}} \mathbb{C} & \to & H^1(E', \Omega^1_{E'}). \end{array}$$

By [13], Exp. XII, Théorème 1.1 the map $\mathrm{Pic}(E) \to \mathrm{Pic}(E')$ is of finite type. It follows that $f^*: \mathrm{Pic}(E)/\mathrm{Pic}^{\tau}(E) \to \mathrm{Pic}(E')/\mathrm{Pic}^{\tau}(E')$ is injective, since $\ker(f^*)$ is a torsion free discrete group scheme of finite type and so vanishes. Hence in the diagram φ is injective, from which the general case follows.

Lemma 1.3: *Let* $A = \mathbb{C}\{X\}_n/\mathfrak{A}$ *be a normal (convergent) analytic algebra of dimension* $d \geqslant 3$ *with isolated singularity and set* $X = \mathrm{Spec}(A)$, $U = X \setminus \{m_A\}$. *Let* $X' \xrightarrow{\pi} X$ *be a resolution of singularities of* X *such that* $E = \pi^{-1}(m_A) = E_1 \cup \ldots \cup E_k$ *is a divisor with normal crossings. Then* $H^1_E(X', \Omega^1_{X'})$ *is a* $\mathbb{C}$*-vectorspace of rank* k.

Proof: The groups $H^1_E(X', \Omega^1_{X'})$, $H^{d-1}(X', \Omega^{d-1}_{X'})$ are finite dimensional and dual to each other as the reasoning in the proof of prop. (2.2) in [18] shows. Let $\pi^{an}: (X'^{an}, E^{an}) \to (X^{an}, 0)$ be the corresponding analytic map. Then $H^{d-1}(X', \Omega^{d-1}_{X'}) \cong H^{d-1}(E^{an}, \Omega^{d-1}_{X'^{an}})$, since ($X'_{(n)}$ indicates the nth infinitesimal neighbourhood of E in X')

$$H^{d-1}\left(X'_{(n)}, \Omega^{d-1}_{X'_{(n)}}\right) \cong H^{d-1}\left(E^{an}, \Omega^{d-1}_{X'^{an}_{(n)}}\right)$$

by the GAGA-theorems and since in the algebraic as well as in the analytic situation the comparison theorem holds. By Oshawa [26]

$$H^{2d-2}(E^{an}, \mathbb{C}) \cong \coprod_{p+q=2d-2} H^{pq}$$

where $H^{pq} = H^q(E^{an}, \Omega^p_{X'^{an}})$ and $H^{pq} = \overline{H^{qp}}$. Since E^{an} is real $(2d-2)$-dimensional with components $E_1, \ldots, E_k$ the group $H^{2d-2}(E^{an}, \mathbb{C})$ is a k-dimensional $\mathbb{C}$-vectorspace. Since $\overline{H}^{d,d-2} \cong H^{d-2,d} \cong H^d(E^{an}, \Omega^{d-2}_{X'^{an}}) = 0$ we get $H^{d-1,d-1} \cong H^{2d-2}(E^{an}, \mathbb{C}) \cong \mathbb{C}^k$ as desired.

LEMMA 1.4: *Situation as in* (1.3). *Assume moreover that* $H^1(X', \mathcal{O}_{X'}) = 0$. *Then the canonical map induced by the logarithmic derivation*

$$\operatorname{Pic}(X') \otimes_{\mathbb{Z}} \mathbb{C} \to H^1(X', \Omega^1_{X'})$$

is injective.

PROOF: From $H^1(X', \mathcal{O}_{X'}) = 0$ we get that $\operatorname{Pic}(X') \to \operatorname{Pic}(E)/\operatorname{Pic}^0(E)$ is injective, see e.g. the arguments in [3], Appendix or in [24]. From this fact together with (1.2) the assertion easily follows.

We will now prove (1.1): As remarked above we may assume $k = \mathbb{C}$. By Artin [1] A is the completion of a convergent analytic $\mathbb{C}$-algebra and by Bingener [2] the divisor class group of a normal analytic algebra with isolated singularity does not change under completion. Hence we may as well assume that A is a convergent analytic $\mathbb{C}$-algebra. With the notation of (1.3) we consider the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & H^1_E(X', \mathcal{O}^\times_{X'}) \otimes_{\mathbb{Z}} \mathbb{C} & \to & \operatorname{Pic}(X') \otimes_{\mathbb{Z}} \mathbb{C} & \to & \operatorname{Pic}(U) \otimes_{\mathbb{Z}} \mathbb{C} & \to 0 \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\ \cdots & H^1_E(\Omega^1_{X'}) & \to & H^1(X', \Omega^1_{X'}) & \to & H^1(U, \Omega^1_{X'}) & \to \cdots \end{array}$$

Here α, β, γ are induced by the logarithmic derivation, and $H^1_E(X', \mathcal{O}^\times_{X'})$ is easily seen to be the free subgroup of $\operatorname{Pic}(X')$ generated by $E_1, \ldots, E_k$. By (1.4) β is injective, hence α is injective, and since by (1.3) $H^1_E(\Omega^1_{X'})$ is of rank k the map α is even bijective. Hence we obtain by a simple diagram chasing that γ is injective as desired.

REMARK 1.5: For a normal isolated singularity of dimension $d \geqslant 3$ $\operatorname{Cl}(A)$ has a natural structure of a Lie-group, see [4], [5]. More generally as in (1.1) the proof given above shows that

$$\operatorname{Cl}(A)/\operatorname{Cl}^\tau(A) \to H^1(U, \Omega^1_U)$$

is injective (without the assumption depth $A \geqslant 3$).

COROLLARY 1.6: *Let* $A = k[\![X]\!]_n/\mathfrak{a}$ *be a Cohen-Macaulay ring of dimension d such that A is regular in codimension* $\leqslant 2$ (*i.e. A satisfies* R_2). *Set* $X = \operatorname{Spec} A$, $U = \operatorname{Reg} X$ *and let* $\xi : \operatorname{Cl}(A) \to H^1(U, \Omega^1_U)$ *be the mapping induced by the logarithmic derivation. Then* Ker ξ *is a torsion group.*

PROOF: If $d = 3$ then (1.6) is contained in (1.1). If $d > 3$ let $t \in A$ be a nonzero divisor such that $B = A/tA$ has property R_2 too; set $V := V(t) \cap U \subseteq U$. In the diagram

$$\begin{array}{ccc} \operatorname{Pic}(U) & \overset{\xi}{\to} & H^1(U, \Omega^1_U) \\ \rho \downarrow & & \downarrow \\ \operatorname{Pic}(V) & \overset{\xi}{\to} & H^1(V, \Omega^1_V) \end{array}$$

the restriction map ρ is injective by [23] or [15], Exp. XI. Now the result follows by induction on d.

REMARK 1.7: If $A = \coprod_{i \geqslant 0} A_i$ is quasihomogeneous, $A_0 = \mathbb{C}$, then the results above can be shown under much weaker assumptions: Set $X := \operatorname{Spec}(A)$, $U := X \setminus \{m_A\}$, m_A denoting the maximal homogeneous ideal. By $\operatorname{Pic}_h(U)$ we denote the subgroup of $\operatorname{Pic}(U)$ generated by those invertible $\mathcal{O}_U$-modules $\mathcal{L}$ such that $\Gamma(U, \mathcal{L})$ has a grading. Then

$$\operatorname{Pic}_h(U)/\operatorname{Pic}_h^\tau(U) \overset{\xi}{\to} H^1(U, \Omega^1_U)$$

is injective, if depth $A \geqslant 3$. Here we do not assume that A has isolated singularity or even that A is reduced. If A is in addition normal then the same holds also for the completion of A since in this case $\operatorname{Pic}_h(U) = \operatorname{Pic}(U) = \operatorname{Pic}(\hat{U})$ by [9], (1.5) and its proof, where $\hat{U} := \operatorname{Spec}(\hat{A}) \setminus \{\mathrm{m}_{\hat{A}}\}$. We shortly sketch the proof in the homogeneous case: $H^1(U, \Omega^1_U)$ has a natural grading and $\xi(\operatorname{Pic}_h(U))$ is easily seen to be contained in $H^1(U, \Omega^1_U)_0$. If $Y = \operatorname{Proj}(A)$ the natural mapping $\operatorname{Pic}(Y)/\mathbb{Z} \cdot [\mathcal{O}_Y(1)] \to \operatorname{Pic}_h(U)$ given by $\mathcal{L} \mapsto \coprod_{i \geqslant 0} \mathrm{H}^0(Y, \mathcal{L}(i))$ is bijective. In the diagram

$$\begin{array}{cccc} & \operatorname{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{C} & \to & \operatorname{Pic}_h(U) \otimes_{\mathbb{Z}} \mathbb{C} \\ & \beta \downarrow & & \downarrow \xi \\ \mathbb{C} = H^0(Y, \mathcal{O}_Y) \overset{\alpha}{\to} & H^1(Y, \Omega^1_Y) & \to & H^1(U, \Omega^1_U)_0 \end{array}$$

where the last exact sequence is induced by the Euler-sequence, $\alpha(\mathbb{C}) = \mathbb{C} \cdot \beta([\mathcal{O}_Y(1)])$. Since β is injective by (1.2) this implies the injectivity of ξ. We remark that these arguments can be carried over to the quasihomogeneous case. One may ask if ξ is also injective under these weaker assumptions if A is not quasihomogeneous.

§2. Applications

Let k be always a field of characteristic 0. In the following we will formulate our results for complete local k-algebras $A = k[\![X]\!]_n/\mathfrak{a}$. We remark that they are also valid in the corresponding analytic or algebraic situation.

THEOREM 2.1: *Let $A = k[\![X]\!]_n/\mathfrak{a}$ be an isolated Gorenstein singularity of dimension $d \geqslant 3$ satisfying $T_A^{d-2}(A) = 0$. Then A is almost factorial.*

PROOF: It is well known and follows easily from the spectral sequence

$$E_2^{pq} = \operatorname{Ext}_A^p\big(T_p^A(A),\ A\big) \Rightarrow T_A^{p+q}(A),$$

that $T_A^{d-2}(A) = \operatorname{Ext}_A^{d-2}(\Omega_A^1, A)$ in this case. By Grothendieck-duality $\operatorname{Ext}_A^{d-2}(\Omega_A^1, A)$ is dual to $H_m^2(\Omega_A^1) \cong H^1(U, \Omega_U^1)$, where $U = \operatorname{Spec}(A) \setminus \{m_A\}$ as usual. By (1.1) our result follows.

In particular (2.1) implies, that a 3-dimensional rigid isolated Gorenstein singularity is almost factorial. We remark that the condition $T_A^{d-2}(A) = 0$ in (2.1) is necessary: If A is the completion of the local ring at the vertex of the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with respect to $\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(2)$, then A is an isolated Gorenstein singularity, which is even rigid, but $\operatorname{Cl}(A) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/(2)$.

From (2.1) it is easily to deduce a similar result for non isolated singularities.

COROLLARY 2.2: *Let $A = k[\![X]\!]_n/\mathfrak{a}$ be a d-dimensional Gorenstein singularity which is regular in codimension $\leqslant k$, where $3 \leqslant k < d$. Suppose $T_A^{d-2}(A) = \ldots = T_A^{k-1}(A) = 0$. Then A is almost factorial.*

PROOF: In the case $d = k+1$ this is just (2.1). In the case $d > k+1$ choose $t \in A$ such that $B = A/tA$ is also regular in codimension $\leqslant k$. By the Lefschetz theorem of [23] or [15], Exp. XI, $\operatorname{Cl}(A) \to \operatorname{Cl}(B)$ is injective. From the exact cohomology sequence of tangent functors

$$\cdots \to T_A^i(A) \xrightarrow{t} T_A^i(A) \to T_A^i(B) \to T_A^{i+1}(A) \to \cdots$$

$$\cdots \to T_{A/B}^i(B) \to T_B^i(B) \to T_A^i(B) \to T_{A/B}^{i+1}(B) \to \cdots$$

and the vanishing of $T_{A/B}^i(B)$, $i \geqslant 2$, we obtain $T_B^{d-3}(B) = \ldots = T_B^{k-1}(B) = 0$. Now the assertion follows by induction on d.

In the quasihomogeneous case (2.2) has been shown by Buchweitz (unpublished). By [6], [20], the assumptions on the vanishing of the tangent functors are satisfied, if A is linked to a complete intersection. For other results in this direction see also [21], [28].

In the case $d = 3$ we now show a refined version of (2.1), which is an analogue of the infinitesimal M. Noether theorem in [7]. Let $A = k[\![X]\!]_n/\mathfrak{a}$ be an isolated Gorenstein singularity of dimension 3 and $U := \operatorname{Spec}(A) \setminus \{m_A\}$. Suppose L is a reflexive A-module of rank 1 and denote by $\mathscr{L}$ the associated invertible sheaf on U. If $k[\epsilon] \to A'(\epsilon^2 = 0)$ is a first order deformation of A, we set $U' = \operatorname{Spec}(A') \setminus \{m_{A'}\}$.

THEOREM 2.3: *Suppose that for each first order deformation $k[\epsilon] \to A'$ of A $\mathscr{L}$ can be extended to a locally free sheaf $\mathscr{L}'$ on U'. Then L is a torsion element in* $\operatorname{Cl}(A)$.

PROOF: Let $\xi_L \in H^1(U, \Omega^1_U)$ be the class associated to L under the map $\operatorname{Cl}(A) \to H^1(U, \Omega^1_U)$. It is well known that the group $\operatorname{Ext}^1_A(\Omega^1_A, A)$ describes the first order deformations of A. Denote by $[A']$ the cohomology class in $\operatorname{Ext}^1_A(\Omega^1_A, A)$ associated to A'. Then it is not difficult to see that in the canonical pairing

$$\operatorname{Ext}^1_A(\Omega^1_A, A) \times H^1(U, \Omega^1_U) \xrightarrow{\langle\, ,\, \rangle} H^2(U, \mathcal{O}_U)$$

$\langle [A'], \xi_L \rangle$ is just the obstruction for extending $\mathscr{L}$ to a $\mathscr{L}'$. But by Grothendieck duality this pairing is nondegenerated, and so by our assumption $\xi_L = 0$, which implies by (1.1) that L is a torsion element in $\operatorname{Cl}(A)$.

For the case of complete intersections it is possible to strengthen (2.3):

PROPOSITION 2.4: *Let A be as in* (2.3) *and suppose moreover that A is a complete intersection. Then $A' = k[\![X]\!]_n/\mathfrak{a}^2$ is parafactorial.*

PROOF: Let $\mathscr{L}'$ be a locally free module on $U' := \operatorname{Spec}(A') \setminus \{m_{A'}\}$. If $A = k[\![X]\!]_n/(f_1, \dots, f_{n-3})$, denote by A_ι the first order deformation

$$A_\iota := k[\![X]\!]_n/\big(f_1, \dots, f_{\iota-1}, f_\iota^2, f_{\iota+1}, \dots, f_{n-3}\big), \quad \epsilon \mapsto \bar{f}_\iota,$$

of A and $U_\iota := \operatorname{Spec}(A_\iota) \setminus \{m_{A_\iota}\}$. Moreover let $\mathscr{L}$ resp. $\mathscr{L}_\iota$ be the sheaf on U resp. U_ι induced by $\mathscr{L}'$. By assumption $\mathscr{L}$ can be extended to the locally free sheaf $\mathscr{L}_\iota$ on U_ι, hence with the notations in the proof of the last result

$$\langle [A_\iota], \xi_{\mathscr{L}} \rangle = 0, \quad i = 1, \dots, n-3.$$

But the $[A_t]$ generate $\mathrm{Ext}^1_A(\Omega^1_A, A)$ as an A-module and so $\xi_{\mathscr{L}} = 0$, and $\mathscr{L} \in \mathrm{Pic}(U)$ is a torsion element. Since $\mathrm{Pic}(U) = \mathrm{Cl}(A)$ is known to have no torsion, see [4] (3.2), we get $\mathscr{L} \cong \mathcal{O}_U$ and hence $\mathscr{L}' \cong \mathcal{O}_{U'}$ as desired.

We will now apply these results to normal bundles of Gorenstein singularities.

THEOREM 2.5: *Let* $A = k[\![X]\!]_n/\mathfrak{a}$ *be a d-dimensional isolated Gorenstein singularity,* $d \geqslant 3$, $W := \mathrm{Spec}(k[\![X]\!]_n) \setminus \{m\}$, $X := \mathrm{Spec}(A)$, $U := X \setminus \{m_A\}$, $Y \subseteq X$ *a divisor and* $V := Y \setminus \{m_A\}$. *If the sequence of normal bundles*

$$0 \to \mathcal{N}_{V/U} \to \mathcal{N}_{V/W} \to \mathcal{N}_{U/W} \otimes \mathcal{O}_V \to 0$$

splits on V *then* Y *represents a torsion element in* $\mathrm{Cl}(A)$, *i.e.* Y *is given set-theoretically by one equation.*

PROOF: First we will assume $d = 3$. Let R denote the ring $k[\![X]\!]_n$ and $\tilde{B} := H^0(V, \mathcal{O}_V)$, which by Grothendieck's finiteness theorem is finite over $B := H^0(Y, \mathcal{O}_Y)$. Then $H^0(V, \mathcal{N}_{V/U}) = T^1_{A/B}(\tilde{B})$, $H^0(V, \mathcal{N}_{V/W}) = T^1_{R/B}(\tilde{B})$, $H^0(V, \mathcal{N}_{U/W} \otimes \mathcal{O}_V) = T^1_{R/A}(\tilde{B})$, and our assumption implies that

$$0 \to T^1_{A/B}(\tilde{B}) \to T^1_{R/B}(\tilde{B}) \xrightarrow{\gamma} T^1_{R/A}(\tilde{B}) \to 0$$

is exact. In particular in the diagram

$$\begin{array}{ccc} T^1_{R/B}(\tilde{B}) & \xrightarrow{\gamma} & T^1_{R/A}(\tilde{B}) \\ \downarrow \alpha & & \downarrow \beta \\ T^1_B(\tilde{B}) & \xrightarrow{\delta} & T^1_A(\tilde{B}) \end{array}$$

γ is onto, and since R is regular, α, β are surjective too, from which we obtain the surjectivity of δ. Consider the diagram

$$\begin{array}{ccc} & & T^1_A(A) \\ & & \downarrow \\ T^1_B(\tilde{B}) & \xrightarrow{\delta} & T^1_A(\tilde{B}) \end{array}$$

That δ is surjective means: If $k[\epsilon] \to A'$ is a first order deformation of A then there exists an extension $[B'] \in T^1_B(\tilde{B})$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & A' & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \tilde{B} & \to & B' & \to & B & \to & 0 \end{array}$$

see [8], §1. In particular $V \subseteq U$ can be extended to a first order deformation $V' := \mathrm{Spec}(B') \setminus \{m_{B'}\} \subseteq U' := \mathrm{Spec}(A') \setminus \{m_{A'}\}$ or, equivalently, $\mathscr{L} = \mathcal{O}_U(V)$ can be extended to a locally free sheaf $\mathscr{L}' = \mathcal{O}_{U'}(V')$. By (2.3) $\mathscr{L}$ is a torsion element in $\mathrm{Cl}(A)$ and so Y can be described set-theoretically by one equation.

Now suppose $d > 3$. Let $t \in R$ be a generic linear combination of $X_1, \ldots, X_n$ with coefficients in k. Set $\overline{W} := V(t) \subseteq W$, $\overline{U} := V(t) \cap U$, $\overline{V} := V(t) \cap V$. Then $\overline{W}$, $\overline{U}$ are smooth, and $\overline{A} := A/tA$ is an isolated Gorenstein singularity of dimension $d-1$. Once more by [15,23] $\mathrm{Cl}(A) \to \mathrm{Cl}(\overline{A})$ is injective, and moreover the normal bundle sequence

$$0 \to \mathcal{N}_{\overline{V}/\overline{U}} \to \mathcal{N}_{\overline{V}/\overline{W}} \to \mathcal{N}_{\overline{U}/\overline{W}} \otimes \mathcal{O}_{\overline{V}} \to 0$$

splits, since it is the restriction of our original normal bundle sequence to $\overline{V}$. Now the assertion follows by induction on d.

Applying this result to the cone over a projective variety we immediately obtain a generalization of the results [11] Chap. IV, (f), and [16] mentioned in the introduction.

COROLLARY 2.6: *Suppose $X \subseteq \mathbb{P}^n = \mathbb{P}^n_k$ is an arithmetically Cohen-Macaulay submanifold of dimension $d \geqslant 2$ such that $\omega_X = \mathcal{O}_X(\ell)$ for some ℓ. If $Y \subseteq X$ is a 1-codimensional Cartier-divisor and if the sequence of normal bundles*

$$0 \to \mathcal{N}_{Y/X} \to \mathcal{N}_{Y/\mathbb{P}^n} \to \mathcal{N}_{X/\mathbb{P}^n} \otimes \mathcal{O}_Y \to 0$$

splits, then there is a hypersurface $H \subseteq \mathbb{P}^n$ such that $Y = H \cap X$ set-theoretically.

REMARKS: (1) In the case $d = 3$ in (2.5) it is obviously sufficient to require that $H^0(V, \mathcal{N}_{V/W}) \to H^0(V, \mathcal{N}_{U/W} \otimes \mathcal{O}_V)$ is surjective. Similarly in (2.6) it suffices that $H^0(\mathcal{N}_{Y/\mathbb{P}^n}(\ell)) \to H^0(\mathcal{N}_{X/\mathbb{P}^n} \otimes \mathcal{O}_Y(\ell))$ is surjective if $d = 2$.

(2) If in (2.5) resp. (2.6) $\mathrm{Cl}(A)$ resp. $\mathrm{Pic}(X)$ has no torsion then Y is even scheme-theoretically given by one equation. This is e.g. satisfied if A resp. X is a complete intersection, see [4], (3.2).

(3) I do not known whether these results continue to be true without the assumption $\mathrm{char}(A) = 0$. At least the proofs given here do not apply since we have heavily used the Hodge-decomposition theorem of Oshawa.

(4) If the problem mentioned at the end of section 1 would be true, then (2.3) would be valid in the case of any 3-dimensional Gorenstein singularity (not necessarily isolated) if $\mathscr{L}$ is assumed to be locally free. In order to show this let $\mathrm{Ext}^1_A(\Omega^1_A, A) \xrightarrow{\alpha} T^1_A(A)$ be the map induced by

the canonical projection $L_A^{\bullet} \to \Omega_A^1$, where $L_A^{\bullet}$ is the cotangent complex of A. Then in the diagram

$$\begin{array}{lllll} \mathrm{Ext}_A^1(\Omega_A^1, A) \times & H^1(U, \Omega_U^1) & \searrow \langle\ ,\ \rangle & \\ \downarrow \alpha & \uparrow \xi & & H^2(U, \Omega_U^1) \\ T_A^1(A) \times & \mathrm{Pic}(U) & \nearrow \{\ ,\ \} & \end{array}$$

is commutative in the sense, that $\{\alpha(x), [\mathscr{L}]\} = \langle x, \xi_{\mathscr{L}} \rangle$. Here $\{[A'], \mathscr{L}\}$ denotes the obstruction of extending $\mathscr{L}$ to A'. Now the proof of (2.3) applies. In a similar way, then it would be possible to generalize (2.4), (2.5). In (2.5) we could replace the condition "isolated singularity" by "U is locally a complete intersection in W". By the last remark in section 1 this is at least true for quasihomogeneous singularities, and so we obtain:

COROLLARY 2.7: (2.6) *remains true if the condition "submanifold" is replaced by "locally a complete intersection".*

References

[1] M. ARTIN: Algebraic approximation of structures over complete local rings. *Pub. Math. I.H.E.S.* 36 (1969) 23–58.

[2] J. BINGENER: Über die Divisorenklassengruppen lokaler Ringe. *Math. Ann.* 229 (1977) 173–179.

[3] J. BINGENER, H. FLENNER: Variation of the divisor class group. *J.f.d.r.u.a. Math.* 351 (1984) 20–41.

[4] J. BINGENER, U. STORCH: Zur Berechnung der Divisorenklassengruppen kompletter lokaler Ringe. *Nova acta Leopoldina NF* 52, Nr. 240 (1981) 7–63.

[5] J.F. BOUTOT: *Schéma de Picard Local. Lecture Notes in Math.* 632. Springer Berlin-Heidelberg-New York (1978).

[6] R.-O. BUCHWEITZ: Contributions à la théorie des singularités. Thèses à l'Université Paris VII (1981).

[7] J. CARLSON, M. GREEN, P. GRIFFITHS, J. HARRIS: Infinitesimal variations of Hodge structure (I). *Comp. Math.* 50 (1983) 109–205.

[8] H. Flenner: Über Deformationen holomorpher Abbildungen. *Habilitationsschrift* Osnabrück (1978).

[9] H. FLENNER: Divisorenklassengruppen quasihomogener Singularitäten, *J.f.d.r.u.a. Math.* 328 (1981) 128–159.

[10] R.M. FOSSUM: The divisor class group of a Krull domain. *Erg. d. Math. Bd.* 74, Springer Berlin-Heidelberg-New York (1973).

[11] P. GRIFFITHS, J. HARRIS: Infinitesimal variations of Hodge structures (II): An infinitesimal invariant of Hodge classes. *Comp. Math.* 50 (1983) 207–265.

[12] P. GRIFFITHS, J. HARRIS: *Principles of Algebraic Geometry*. John Wiley (1978).

[13] A. GROTHENDIECK et al.: *Théorie des Intersections et Théorème de Riemann-Roch (SGA 6). Lecture Notes in Math.* 225, Springer Berlin-Heidelberg-New York (1971).

[14] A. GROTHENDIECK: *Local Cohomology. Lecture Notes in Math.* 41, Springer Berlin-Heidelberg-New York (1967).

[15] A. GROTHENDIECK: *Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux (SGA 2).* North Holland Publ. Company Amsterdam (1968).

[16] J. HARRIS, K. HULEK: On the normal bundle of curves on complete intersection surfaces. *Math. Ann.* 264 (1983) 129–135.

[17] R. HARTSHORNE: *Algebraic Geometry. Graduate Texts in Math.* 52. Springer Berlin-Heidelberg-New York (1966).

[18] R. HARTSHORNE, A. OGUS: On the factoriality of local rings of small embedding codimension. *Communications in Algebra* 1 (1974) 415–437.

[19] K. HULEK: Complete intersection curve, the splitting of the normal bundle and the Veronese surface. Preprint Erlangen.

[20] C. HUNEKE: Numerical invariants of liaison classes. *Inv. Math.* 75 (1984) 301–325.

[21] C. HUNEKE, B. ULRICH: Divisor class group and deformations. Am. J. Math. 107 (1985) 1265–1304.

[22] A. KUSTIN, M. MILLER: Deformation and linkage of Gorenstein algebras. *Trans. AMS.* 284 (1984) 501–534.

[23] J. LIPMAN: Rings with a discrete divisor class group: Theorem of Danilov-Samuel. *Amer. J. Math.* 101 (1979) 203–211.

[24] J. LIPMAN: Picard schemes of formal schemes; application to rings with discrete divisor class of group. In: *Classification of algebraic Varieties and compact complex Manifolds. Lec. Notes in Math.* 412, Springer Berlin-Heidelberg-New York (1974).

[25] M. NOETHER: Zur Grundlegung der Theorie der algebraischen Raumkurven. *J.f.d.r.u.a. Math.* 93 (1882) 271–318.

[26] T. OSHAWA: A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds. *Inv. Math.* 63 (1981) 335–354.

[27] U. STORCH: Fastfaktorielle Ringe. *Schriftenreihe des Math. Inst. der Univ. Münster*, Heft 36 (1967) 1–42.

[28] B. ULRICH: Gorenstein rings as specializations of unique factorization domains. *J. Alg.* 86 (1984) 129–140.

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Note added in proof: By using different arguments, G. Ellingsrud, L. Grusiu, C. Peskine, S.A. Strømme: On the normal bundle of curves on smooth projective surfaces. Inv. math. 80 (1985) 181–184, could also give a generalization of the theorem of Griffiths and Harris on the splitting of normal bundles.