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THE GENERALIZED RADON–HURWITZ NUMBERS

Marc A. Berger and Shmuel Friedland

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Abstract

The generalized Radon–Hurwitz number, $\rho(m, n)$, designed to characterize the dimensions for which normed bilinear maps exist, is discussed. The values of $\rho(m, n)$ are computed when (i) $n - m \leqslant 3$; (ii) $n - m = 4$ and m is odd; (iii) $m \leqslant 9$. Tables are provided and many structure results for the Radon–Hurwitz matrices are developed.

Notation

$\mathbb{R}^m$ denotes the real m-dimensional Euclidean space with norm

$$|u|^2 = \sum_{i=1}^{m} u_i^2$$

and inner product

$$\langle u, v\rangle = \sum_{i=1}^{m} u_i v_i$$

S^{m-1} denotes the unit ball (i.e. the set of unit vectors) in $\mathbb{R}^m$. $\mathbb{R}P^{m-1}$

denotes the real projective space obtained from S^{m-1} by identifying antipodal points. x always denotes a vector in $\mathbb{R}^m$, y' always denotes a vector in $\mathbb{R}^k$ and y always denotes a vector in $\mathbb{R}^{k-1}$. Since we have much more occasion to work in $\mathbb{R}^{k-1}$ than in $\mathbb{R}^k$ we have chosen to let y (without the prime) denote a vector in $\mathbb{R}^{k-1}$, and the prime indicates an extra dimension. Thus

$$y' = (y, y_k).$$

The dimension of a subspace $\mathscr{U}$ is denoted $\dim(\mathscr{U})$. For two spaces $\mathscr{U}$, $\mathscr{V}$ we let $\mathrm{lin}(\mathscr{U}, \mathscr{V})$ denote the set of linear mappings from $\mathscr{U}$ to $\mathscr{V}$ (with domain $\mathscr{U}$); and we let $\mathscr{C}(\mathscr{U}, \mathscr{V})$ denote the set of continuous mappings from $\mathscr{U}$ to $\mathscr{V}$ (with domain $\mathscr{U}$). For a matrix A we use the following symbols:

a_{pq} – the (p, q) entry of A
A^t – the transpose of A
$\mathrm{tr}(A)$ – the trace of A
$\ker(A)$ – the kernel (i.e. the null space) of A
$\mathrm{null}(A)$ – the nullity of A (i.e. the dimension of its kernel)

§1. Introduction

Let M_{mn} denote the vector space of all real $m \times n$ matrices, and let Q_{mn} denote the subset of those matrices $A \in M_{mn}$ which satisfy

$$AA^t = \alpha(A) I \tag{1}$$

for some scalar $\alpha(A)$. (Of course $\alpha(A)$ must be nonnegative and it is zero if and only if $A = 0$. It can be written explicitly as $\alpha(A) = \dfrac{1}{m}\mathrm{tr}(AA^t)$.) If $m > n$ then Q_{mn} is trivial, so we will always take $m \leqslant n$. When $m = n$ we denote these sets by M_n and Q_m, respectively.

DEFINITION I: The *generalized Radon–Hurwitz number* $\rho(m, n)$ is the maximal dimension of a subspace contained in Q_{mn}.

The number $\rho(m) := \rho(m, m)$ is the classical Radon–Hurwitz number. It was computed independently by Radon [23] and Hurwitz [18]. If we factor m as

$$m = (2a+1)2^{b+4c},\ 0 \leqslant b \leqslant 3, \tag{2}$$

then $\rho(m)$ is given by

$$\rho(m) = 2^b + 8c. \tag{3}$$

This number $\rho(m)$ is of central importance in many mathematical problems. $\rho(m)-1$ is the dimension of the maximal Clifford algebra in M_m. Thus this number can be traced as far back as Clifford [10]. The famous result of Adams [1] in algebraic topology (K-theory) asserts that $\rho(m)-1$ is the maximal number of independent vector fields on S^{m-1}. This number appears in the study of nonsingular bilinear maps (Lam [20]), imbeddings of real projective spaces (Adem [4], Berrick [9]), orthogonal designs (Geramita and Seberry [14]) and strictly hyperbolic partial differential equations (Friedland, Robbin and Sylvester [13]).

The problem of determining the generalized Radon-Hurwitz number $\rho(m, n)$ was first formulated by Hurwitz [17], as we shall describe shortly. (See the remarks following Definition 2.V.) In the classical case $m = n$ Eckmann [11] used a special group structure in M_m to prove that $\rho(m)$ is given by (2), (3) above. When $m \neq n$ we do not have this nice group structure. The proofs in §3 and §5 show, however, that some of this structure can be recovered, and that there is a very intricate algebraic structure involved here. Cf. Adem [6], [7].

In §2 we discuss several alternate definitions of $\rho(m, n)$. In §3 we analyze the structure of matrices which satisfy the Radon–Hurwitz condition and present some elementary calculations for $\rho(m, n)$. Here we compute the values of $\rho(m, n)$ for $m \leqslant 9$. In §4 we describe the Adams result in the perspective of a nonlinear Radon–Hurwitz number. In §5 we compute the values of $\rho(m, n)$ for (i) $n - m \leqslant 3$, and (ii) $n - m = 4$ with m odd. The vaues of $\rho(m, m+1)$, $\rho(m, m+2)$ can be obtained directly from Lam [19], and some values of $\rho(m, m+3)$, $\rho(m, m+4)$ can be obtained by applying the topological techniques in Adams [2]. Our techniques here are entirely different from those of Adams or Lam.

§2. Alternate definitions for $\rho(m, n)$

For any subspace $\mathcal{U} \subset Q_{mn}$ we can choose a basis $E_1, \dots, E_k \in \mathcal{U}$ such that

$$A(y')A^t(y') = |y'|^2 I, \quad y' \in \mathbb{R}^k, \tag{1}$$

where

$$A(y') = \sum_{i=1}^{k} y_i E_i. \tag{2}$$

Equivalent to (1)

$$E_i E_j^t + E_j E_i^t = 2\delta_{ij} I \quad (i, j = 1, \dots, k). \tag{3}$$

Thus we arrive at the following alternate definition for $\rho(m, n)$.

DEFINITION I: (Alternate): $\rho(m, n)$ is the maximal k for which there exists a map $A \in \text{lin}(\mathbb{R}^k, Q_{mn})$ satisfying (1). Equivalently it is the maximal number, k, of real $m \times n$ matrices $E_i = (e_{pq}^{(i)})$, $i = 1, \ldots, k$, which can satisfy (3).

For $A \in M_{mn}$ let us write

$$A = [B, C] \tag{4}$$

in block partitioned form, where $B \in M_m$ and $C \in M_{ml}$ $(l = n - m)$. Let Q_{mn}^* be the subset

$$Q_{mn}^* = \{A = [B, C] \in Q_{mn} : B^t = -B\}. \tag{5}$$

If $\mathscr{U} \subset Q_{mn}$ is any subspace and Φ is any $n \times n$ orthogonal matrix, then $\mathscr{U}\Phi$ is a subspace of Q_{mn} of the same dimension as $\mathscr{U}$. In particular by such a transformation we may assume without loss of generality in (3) that

$$E_k = [I, 0]. \tag{6}$$

It then follows from (3) that $E_1, \ldots, E_{k-1} \in Q_{mn}^*$, and the subspace $\mathscr{V}$ spanned by $\{E_1, \ldots, E_{k-1}\}$ is a $(k-1)$-dimensional subspace of Q_{mn}^*. Conversely, if $\mathscr{V}$ is any $(k-1)$-dimensional subspace of Q_{mn}^* then the subspace $\mathscr{U}$ spanned by $\mathscr{V}$ and $[I, 0]$ is a k-dimensional subspace of Q_{mn}. Thus we also have the following definition for $\rho(m, n)$.

DEFINITION II: (Alternate): $\rho(m, n) - 1$ is the maximal dimension of a subspace of Q_{mn}^*.

The condition on $A(y) = [B(y), C(y)]$ that

$$A(y) \in Q_{mn}^*, \quad A(y)A^t(y) = |y|^2 I, \quad y \in \mathbb{R}^{k-1} \tag{7}$$

is

$$B^t(y) = -B(y), \quad y \in \mathbb{R}^{k-1}, \tag{8}$$

$$-B^2(y) = |y|^2 I - C(y)C^t(y), \quad y \in \mathbb{R}^{k-1}. \tag{9}$$

If

$$A(y) = \sum_{i=1}^{k-1} y_i E_i \tag{10}$$

then equivalent to (8), (9) in terms of the basis elements $E_i = [B_i, C_i]$

$$B_i^t = -B_i \tag{11}$$

$$-B_i^2 = I - C_i C_i^t \tag{12}$$

$$B_i B_j + B_j B_i = C_i C_j^t + C_j C_i^t \tag{13}$$

for $i, j = 1, \ldots, k-1$ with $i \neq j$. This then leads us to the following definition.

DEFINITION III: (Alternate): $\rho(m, n) - 1$ is the maximal k' for which there exists a map $A \in \mathrm{lin}(\mathbb{R}^{k'}, Q^*_{mn})$ satisfying (7). Equivalently it is the maximal number, k' of real $m \times m$ and $m \times l$ matrices B_i and C_i, $i = 1, \ldots, k'$, respectively, which can satisfy (11)–(13).

The following result is immediate.

PROPOSITION IV: *The following are equivalent.*
(*a*) *Condition* (3) *holds.*
(*b*) $E_1^t x, \ldots, E_k^t x$ *are orthonormal whenever* $x \in S^{m-1}$.
(*c*) $|A^t(y')x| = |x||y'|$
(*d*) $\sum_{q=1}^{n} (\langle D_q x, y' \rangle)^2 = |x|^2 |y'|^2$ *where* $D_q = (d_{ip}^{(q)}) = (e_{pq}^{(i)})$.

We note that (d) amounts to the bilinear map $D: \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n$ defined by

$$\mathscr{D}(x, y') = (\langle D_1 x, y' \rangle, \ldots, \langle D_n x, y' \rangle) \tag{14}$$

satisfying

$$|\mathscr{D}(x, y')| = |x||y'|. \tag{15}$$

Such a map (bilinear, satisfying (15)) is called a *normed* map. Thus we arrive now at the next alternate definition.

DEFINITION V: (Alternate): $\rho(m, n)$ is the maximal k for which there exists a normed map $\mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n$.

Hurwitz [17] first formulated the problem of finding the minimal n for which there exists a normed map $\mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n$, given k and m. Of course this is equivalent to the determination of $\rho(m, n)$, since what is really under investigation is the set of triples (m, k, n) for which such a normed map exists. Some results about normed maps are given in Adem

[3]–[7]. Lam [22] contains recent new results. For a general survey of results and techniques see Lam [21] and Shapiro [24].

The $k \times m$ matrices $D_1, \ldots, D_n$ in Proposition IV(d) are said to be dual to the $m \times n$ matrices $E_1, \ldots, E_k$. This duality, $d_{ip}^{(q)} = e_{pq}^{(i)}$, is best illustrated by the three-dimensional box in Figure 1 in the Appendix. Namely, the D's are the "horizontally stacked" $k \times m$ matrices, and the E's are the "vertically stacked" $m \times n$ matrices. This figure also illustrates the symmetry in the k, m variables, corresponding to Proposition 3.I(a) below.

§3. Preliminary results

PROPOSITION I:

(*a*) $\rho(m, n) \geqslant k \Leftrightarrow \rho(k, n) \geqslant m$

(*b*) $\rho(m, n)$ *is increasing in n and decreasing in m*

(*c*) $\max_{m \leqslant k \leqslant n} \rho(k) \leqslant \rho(m, n) \leqslant n$

(*d*) $\sum_{i=1}^{t} \rho(m, n_i) \leqslant \rho(m, \sum_{i=1}^{t} n_i)$

(*e*) $\min_{1 \leqslant i \leqslant t} \rho(m_i, n_i) \leqslant \rho(\sum_{i=1}^{t} m_i, \sum_{i=1}^{t} n_i)$

(*f*) *If* $\rho(m, n) \geqslant k$ *then* $\binom{n}{r}$ *is even for* $n - k < r < m$.

PROOF:

(a) It follows at once from Definition 2.V.

(b) It is a direct consequence of (a) above that $\rho(m, n)$ decreases in m. To see that it increases in n note that $[A, 0]$ satisfies (1.1) whenever A does.

(c) The lower bound follows from (b) above, and the upper bound follows from Proposition 2.IV(b).

(d) Let $\mathcal{U}_i$ be a subspace of Q_{mn_i}, $i = 1, \ldots, t$. Then

$$\mathcal{U} = \{[A_1, \ldots, A_t] : A_i \in \mathcal{U}_i;\ i = 1, \ldots, t\}$$

is a subspace of Q_{mn}, where $n = \sum_{i=1}^{t} n_i$. Since $\mathcal{U} \approx \mathcal{U}_1 x \ldots x \mathcal{U}_t$ the dimension of $\mathcal{U}$ is

$$\dim(\mathcal{U}) = \sum_{i=1}^{t} \dim(\mathcal{U}_i).$$

(e) The direct sum $\bigoplus_{i=1}^{t} A_i(y')$ satisfies (2.1) whenever each $A_i(y')$ does.

(f) According to Hopf [16] a necessary condition for the existence of a normed map $\mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n$ is that $\binom{n}{r}$ be even for $n - k < r < m$. (Stiefel [25] first arrived at this condition for bilinear maps. Behrend [8] extended it to odd homogeneous polynomials and, finally, Hopf extended it to biskew maps. This work pioneered the development of the theory of cohomology rings.) □

Using Proposition I, along with the formula (1.3) for $\rho(m)$, we can now fill in Table 1 of the Appendix. The Supplement which follows that table contains steps which can be used as instructions.

By carefully going over the steps in that Supplement one discovers two interesting periodicities

PROPOSITION II:

(*a*) *If* $m \leqslant \min(9, 2^t)$ *then*

$$\rho(m, n + 2^t) = \rho(m, n) + 2^t$$

(*b*) *Let* $t \leqslant 4$. *If* $m = 2^t a + b$, $n = 2^t a + c$ *where* $1 \leqslant b \leqslant c \leqslant \min(b + 7, 2^t - 1)$ *then*

$$\rho(m + 2^t, n + 2^t) = \rho(m, n).$$

PROPOSITION III: *Let* $A = [B, C] \in Q^*_{mn}$.

(*a*) *If* m *is odd then the spectral radius of* CC^t *and* C^tC *is* α, *where* $\alpha = \alpha(A)$ *is the scalar appearing in* (1.1), *and any other nonzero eigenvalues come in pairs.*

(*b*) *If* m *is even then any nonzero eigenvalues of* CC^t *and* C^tC *come in pairs.*

(*c*) *If* $A \neq 0$ *then* $m - \operatorname{rank}(C)$ *is even.*

PROOF:

The Proposition follows from the following sequence of observations.

(i) Since A satisfies (1.1) and B is skew symmetric

$$\alpha I - CC^t = -B^2. \tag{1}$$

(ii) Since B is skew symmetric $-B^2$ is positive semi-definite. Thus, by (1) $r(CC^t) \leqslant \alpha$. Furthermore, if m is odd then B must be singular, and so $r(CC^t) = \alpha$.

(iii) Since B is skew symmetric the nonzero eigenvalues of B^2 come in pairs.

(iv) The nonzero eigenvalues and their respective multiplicities coincide for CC^t and C^tC.

(v) α is an eigenvalue of $\alpha I - CC^t$ of multiplicity $m - \operatorname{rank}(CC^t)$, and $\operatorname{rank}(C) = \operatorname{rank}(CC^t)$. □

PROPOSITION IV: *Let* $B(y)=\sum_{i=1}^{k-1} y_i B_i$ *and* $C(y)=\sum_{i=1}^{k-1} y_i C_i$ *satisfy* (2.8), (2.9).

(*a*)

$$\bigcap_{i=1}^{k-1} \ker\left(C_i^t C_i\right) = \bigcap_{y\in\mathbb{R}^{k-1}} \ker\left(C^t(y)C(y)\right)$$

$$\bigcap_{i=1}^{k-1} \ker\left(I - C_i^t C_i\right) = \bigcap_{y\in\mathbb{R}^{k-1}} \ker\left(|y|^2 I - C^t(y)C(y)\right)$$

(*b*) *Set*

$$p=\dim\left[\bigcap_{i=1}^{k-1} \ker\left(C_i^t C_i\right)\right]$$

$$q=\dim\left[\bigcap_{i=1}^{k-1} \ker\left(I - C_i^t C_i\right)\right]$$

Then $k \leqslant \rho(m+q,\, n-p)$.

PROOF:

(a) Since $\ker(C^tC)=\ker(C)$ the first equality is immediate. To establish the second equality let $v \in \bigcap_{i=1}^{k-1} \ker(I - C_i^tC_i)$. Then it follows from (2.11), (2.12) that

$$B_iC_iv = 0\ (i=1,\dots,k-1). \tag{2}$$

Next we compute, using (2.11), (2.12),

$$\begin{aligned}|B_iC_jv|^2 &= -\langle C_j^tB_i^2C_jv,\, v\rangle = \langle C_j^t\left(I - C_iC_i^t\right)C_jv,\, v\rangle \\ &= |v|^2 - |C_i^tC_jv|^2;\end{aligned} \tag{3}$$

and for $i \neq j$, using (2.11), (2.13) and (2) above,

$$\begin{aligned}\langle B_iC_jv,\, B_jC_iv\rangle &= -\langle C_i^tB_jB_iC_jv,\, v\rangle \\ &= \langle C_i^t\left(B_iB_j - C_iC_j^t - C_jC_i^t\right)C_jv,\, v\rangle \\ &= -|v|^2 - \langle C_i^tC_jv,\, C_j^tC_iv\rangle.\end{aligned} \tag{4}$$

Thus

$$\left|\left(B_iC_j + B_jC_i\right)v\right|^2 = -\left|\left(C_i^tC_j + C_j^tC_i\right)v\right|^2, \tag{5}$$

and so

$$\left(C_i^t C_j + C_j^t C_i\right) v = 0. \tag{6}$$

(b) (Part I). If $p > 0$ choose an orthonormal basis $\{v_1, \ldots, v_p\}$ for $\bigcap_{y \cap \mathbb{R}^{k-1}} \ker(C^t(y)C(y))$, and then complete this to an orthonormal basis $\{u_1, \ldots, u_{l-p}, v_1, \ldots, v_p\}$ for all of $\mathbb{R}^l$, where $l = n - m$. Let Φ be the orthogonal $l \times l$ matrix

$$\Phi = [u_1, \ldots, u_{l-p}, v_1, \ldots, v_p].$$

If $\mathscr{V} \subset Q_{mn}^*$ is the subspace

$$\mathscr{V} = \{[B(y), C(y)] : y \in \mathbb{R}^{k-1}\}$$

and if we set

$$\tilde{C}(y) = C(y)\Phi, \ y \in \mathbb{R}^{k-1},$$

then

$$\tilde{\mathscr{V}} = \{[B(y), \tilde{C}(y)] : y \in \mathbb{R}^{k-1}\}$$

is a subspace of Q_{mn}^* of the same dimension (namely, $k-1$). On the other hand, since

$$C(y)v_j = 0 \ (y \in \mathbb{R}^{k-1};\ j = 1, \ldots, p)$$

it is clear that the last p columns of each matrix in $\tilde{\mathscr{V}}$ are zero. Thus $k \leqslant \rho(m, n-p)$. Furthermore, since

$$|y|^2 I - \tilde{C}^t(y)\tilde{C}(y) = \Phi^t\left[|y|^2 I - C^t(y)C(y)\right]\Phi$$

it is also clear that

$$\dim\left[\bigcap_{y \in \mathbb{R}^{k-1}} \ker\left(|y|^2 I - \tilde{C}^t(y)\tilde{C}(y)\right)\right]$$
$$= \dim\left[\bigcap_{y \in \mathbb{R}^{k-1}} \ker\left(|y|^2 I - C^t(y)C(y)\right)\right],$$

and thus q remains unchanged under the transition from $C(y)$ to $\tilde{C}(y)$.

(Part II). If $q > 0$ choose an orthonormal basis $\{w_1, \ldots, w_q\}$ for

$\bigcap_{y \in \mathbb{R}^{k-1}} \ker(|y|^2 I - C^t(y)C(y))$. Let Ψ be the orthogonal $l \times q$ matrix

$$\Psi = [w_1, \ldots, w_q].$$

Since

$$C^t(y)C(y)\Psi = |y|^2\Psi, \; y \in \mathbb{R}^{k-1}, \tag{7}$$

it follows from (2.8), (2.9) that

$$B(y)C(y)\Psi = 0, \; y \in \mathbb{R}^{k-1}. \tag{8}$$

Set

$$A(y') = \begin{bmatrix} B(y) + y_k I & C(y) \\ \Psi^t C^t(y) & -y_k \Psi^t \end{bmatrix}, \; y' \in \mathbb{R}^k. \tag{9}$$

Then it follows from (2.8), (2.9) and (7), (8) above that $A(y')$ satisfies (2.1). □

PROPOSITION V: *Let* $A(y) = [B(y), C(y)]$ *satisfy* (2.8)–(2.10). *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be the subspaces of* $\mathbb{R}^{k-1}$

$$\mathcal{M} = \{ y \in \mathbb{R}^{k-1} : C^t(y)C(y) = |y|^2 I \},$$

$$\mathcal{N} = \{ y \in \mathbb{R}^{k-1} : C(y) = 0 \}.$$

(*a*) *For any* $\xi \in \mathcal{M}$, $\eta \in \mathcal{M}^\perp$

$$C^t(\eta)C(\xi) + C^t(\xi)C(\eta) = 0.$$

(*b*) *For any* $\xi \in \mathcal{N}$, $\eta \in \mathcal{N}^\perp$

$$B(\eta)B(\xi) + B(\xi)B(\eta) = 0.$$

(*c*) *If* $C(\eta)$ *is not injective then* $\eta \in \mathcal{M}^\perp$. *In particular*, $\mathcal{M} \perp \mathcal{N}$.
(*d*) *If* $B(\eta)$ *is singular then* $\eta \in \mathcal{N}^\perp$.
N.B. $\mathcal{M}$ *is a subspace since*

$$y \in \mathcal{M} \Leftrightarrow \operatorname{tr}(|y|^2 I - C^t(y)C(y)) = 0$$

PROOF:
(a) For any $\xi, \eta \in \mathbb{R}^{k-1}$

$$B(\xi)B(\eta) + B(\eta)B(\xi) = C(\xi)C^t(\eta) + C(\eta)C^t(\xi) - 2\langle \xi, \eta \rangle I. \tag{10}$$

It follows from (2.8), (2.9) that for any $\xi \in \mathscr{M}$

$$B(\xi)C(\xi) = 0$$

Thus by multiplying (10) on the left by $C^t(\xi)$ and on the right by $C(\xi)$, and using the definition of $\mathscr{M}$ we have (for any η)

$$C^t(\eta)C(\xi) + C^t(\xi)C(\eta) = 2\langle \xi, \eta \rangle I. \tag{11}$$

(b) This follows at once from (10).

(c) If $C(\eta)$ is not injective choose $w \neq 0$ such that $C(\eta)w = 0$. It follows from (11) that for any $\xi \in \mathscr{M}$

$$C^t(\eta)C(\xi)w = 2\langle \xi, \eta \rangle w.$$

By taking the inner product of both sides with respect to w we arrive at

$$0 = 2\langle \xi, \eta \rangle |w|^2.$$

Hence $\eta \perp \xi$.

(d) If $B(\eta)$ is singular choose $w \neq 0$ such that $B(\eta)w = 0$. It follows from (10) that for any $\xi \in \mathscr{N}$

$$B(\eta)B(\xi)w = -2\langle \xi, \eta \rangle w.$$

By taking the inner product of both sides with respect to w and using the skew symmetry of $B(\eta)$ we arrive at

$$0 = -2\langle \xi, \eta \rangle |w|^2.$$

hence $\eta \perp \xi$. □

COROLLARY VI: *Let $A(y) = [B(y), C(y)]$ satisfy* (2.8)–(2.10). *Let $\eta \in \mathbb{R}^{k-1}$ be such that $B(\eta)$ is singular, and set*

$$\nu = \mathrm{null}(B(\eta)).$$

Then

$$\dim(\mathscr{N}) \leqslant \rho(\nu) - 1$$

where $\mathscr{N}$ is the subspace in Proposition V.

PROOF: It follows from Proposition V(b) that $\mathscr{S} = \ker(B(\eta))$ is invariant under $B(\xi)$, for any $\xi \in \mathscr{N}$. By restricting $B(\xi)$ to $\mathscr{S}$, then, we obtain a family of skew symmetric matrices which satisfy

$$-\left[B(\xi)\,|_{\mathscr{S}}\right]^2 = |\xi|^2 I,\ \xi \in \mathscr{N}.$$

Thus, by Definition 2.IV, the desired inequality follows. □

The next Proposition is from Friedland [12, Thm. 8.11].

PROPOSITION VII: *Let $A(y)$ be a real $m \times n$ matrix polynomial in the variable $y \in \mathbb{R}^{k-1}$. Assume that $A(y)$ is rank one, so that it is not identically zero and all 2×2 minors vanish. Then there exist vector polynomials $u(y) \in \mathbb{R}^m$, $v(y) \in \mathbb{R}^n$ with relatively prime coordinates, and a (scalar) polynomial $a(y)$ such that*

$$A(y) = a(y)u(y)v^t(y), \quad y \in \mathbb{R}^{k-1}.$$

$u(y)$ and $v(y)$ are unique up to scalar multiplication. Furthermore if $A(y)$ is symmetric then we can choose $u(y) = v(y)$.

PROOF: Choose $y \in \mathbb{R}^{k-1}$, $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$ such that

$$A(y)\alpha \neq 0, \quad A^t(y)\beta \neq 0.$$

Since the polynomial ring $\mathbb{R}[y]$ is a unique factorization domain we can set

$$A(y)\alpha = b(y)u(y), \quad A^t(y)\beta = c(y)v(y)$$

where $b(y)$, $c(y)$ are (scalar) polynomials, and $u(y)$, $v(y)$ have relatively prime coordinates. Thus in a neighbourhood of y we have an equality

$$A(y) = a(y)u(y)v^t(y).$$

Since $A(y)$, $u(y)$, $v^t(y)$ are polynomials, $a(y)$ must be a rational function of y. Say, then,

$$a(y) = p(y)/q(y),$$

where $p(y)$ and $q(y)$ are relatively prime polynomials. We claim that $q(y)$ must be constant. Otherwise there would be an irreducible factor $r(y)$ in $q(y)$. Since the coordinates of $u(y)$ are relatively prime there must be a coordinate $u_i(y)$ which is relatively prime to $r(y)$. Since the (i, j)-entry of $A(y)$ is a polynomial it must be that $r(y)$ divides each coordinate $v_j(y)$ – but this contradicts our choice of $v(y)$.

Note that once y, α, β are chosen then $u(y)$ and $v(y)$ are unique up to scalar multiplication. If $A(y)$ is symmetric we can take $\alpha=\beta$, and thus $u(y)$ and $v(y)$ can be chosen the same. □

PROPOSITION VIII: *Let* $A(y)=[B(y), C(y)]$ *satisfy* (2.8)–(2.10).

(*a*) *If*

$$\operatorname{rank}(C(y))=\operatorname{rank}\big(C^t(y)C(y)\big)\leqslant 1, \quad y\in\mathbb{R}^{k-1},$$

then

$$k\leqslant \max\big(\rho(m, m+1), n-m+1\big).$$

(*b*) *If*

$$\operatorname{rank}\big(|y|^2 I - C^t(y)C(y)\big)\leqslant 1, \quad y\in\mathbb{R}^{k-1}$$

then

$$k\leqslant \max\big(\rho(n-1, n), n-m+1\big).$$

PROOF:

(a) If m is even it follows from Proposition III(c) that $C(y)\equiv 0$, and thus $k\leqslant\rho(m)$. Suppose, then, that m is odd. According to Proposition III(a) the eigenvalues of $C^t(y)C(y)$ must be $(|y|^2, 0,\dots,0)$. According to Proposition VII we write

$$C^t(y)C(y)=a(y)u(y)u^t(y), \; y\in\mathbb{R}^{k-1}$$

where $u(y)$ is a vector polynomial in $\mathbb{R}^{n-m}$ and $a(y)$ is a (scalar) polynomial. Since $C(y)$ is linear only one of a, u can vary with y. If u is constant then

$$\dim\left[\bigcap_{y\in\mathbb{R}^{k-1}}\ker\big(C^t(y)C(y)\big)\right]=n-m-1,$$

and thus, by Proposition IV, $k\leqslant\rho(m, m+1)=\rho(m+1)$. Otherwise, if a is constant we can write

$$C^t(y)C(y)=u(y)u^t(y)$$

where

$$u(y)=\sum_{i=1}^{k-1} y_i u_i. \tag{12}$$

Then

$$|y|^2=\operatorname{tr}\big(C^t(y)C(y)\big)=|u(y)|^2, \tag{13}$$

and $\{u_1, \ldots, u_{k-1}\}$ must be an orthonormal system. Hence $k-1 \leqslant n-m$.
(b) If n is even it follows from Proposition III that $C^t(y)C(y) = |y|^2 I$, and thus $k \leqslant \rho(n)$. Suppose, then, that n is odd. According to Proposition III the eigenvalues of $C^t(y)C(y)$ must be $(|y|^2, |y|^2, \cdots, |y|^2, 0)$. According to Proposition VII we write

$$|y|^2 I - C^t(y)C(y) = a(y)u(y)u^t(y), \quad y \in \mathbb{R}^{k-1}.$$

Exactly as above, if u is constant then

$$\dim\left[\bigcap_{y \in \mathbb{R}^{k-1}} \ker\left(|y|^2 I - C^t(y)C(y)\right)\right] = n - m - 1$$

and (by Proposition IV) $k \leqslant \rho(n-1, n) = \rho(n)$. Otherwise, if a is constant then $k - 1 \leqslant n - m$. □

§4. Nonlinear theory

Let R_{mn} denote the subset of those matrices $A \in M_{mn}$ with rank m. Conventionally when $m = n$ this set is denoted GL_m. Let R^0_{mn}, GL^0_m denote the sets $R_{mn} \cup \{0\}$, $GL_m \cup \{0\}$, respectively, with the zero matrix appended.

$\rho(m, n)$ is the maximal k for which lin($\mathbb{R}^k$, Q_{mn}) contains an injective map. It is this perpective that reveals how striking the theorem of Adams [1] is, for a consequence of it is the following. (See Friedland, Robbin and Sylvester [13]).

THEOREM I: *If $\mathscr{C}(S^{k-1}, GL_m)$ contains an odd map then $k \leqslant \rho(m)$. In particular $\rho(m)$ is the maximal dimension of a subspace of GL^0_m.*

This theorem tells us that there exists an injective linear map $\mathbb{R}^k \to Q_m$ if and only if there exists a continuous odd map $S^{k-1} \to GL_m$.

COROLLARY II: *If $\mathscr{C}(S^{k-2}, GL_m)$ contains an odd map which is skew symmetric then $k \leqslant \rho(m)$.*

PROOF: If $\psi : S^{k-2} \to GL_m$ is an odd map which is skew symmetric then the map

$$\tilde{\psi}(y') = |y|\psi\left(\frac{y}{|y|}\right) + y_k I$$

is an odd map in $\mathscr{C}(S^{k-1}, GL_m)$. □

COROLLARY III: *Let $A(y) = [B(y), C(y)]$ be an odd map in $\mathscr{C}(S^{k-2}, Q^*_{mn})$.*

(a) If $B(y)$ is nonsingular for all $y \in S^{k-2}$ then $k \leqslant \rho(m)$.

(b) If $C(y)$ is of rank *$n - m$ for all $y \in S^{k-2}$ then $k \leqslant \rho(n)$.*

PROOF:

(a) If $B(y)$ is nonsingular then it must be an odd map in $\mathscr{C}(S^{k-2}, GL_m)$ which is skew symmetric. Hence by Corollary II $k \leqslant \rho(m)$.

(b) For a single matrix $A = [B, C] \in Q^*_{mn}$ suppose C^tC is invertible. Let

$$D = -(C^tC)^{-1}C^tBC. \tag{1}$$

Since

$$-B^2 = \alpha I - CC^t \tag{2}$$

where $\alpha = \alpha(A)$ is the scalar in (1.1) it follows that C^tC and C^tBC commute. Thus since B is skew symmetric so is D. Furthermore by using the commutativity of C^tC and C^tBC, the commutativity of CC^t and B, and (2) it follows that

$$-D^2 + C^tC = \alpha I, \tag{3}$$

$$BC + CD = 0. \tag{4}$$

Thus

$$\tilde{A} = \begin{bmatrix} B & C \\ -C^t & D \end{bmatrix} \tag{5}$$

lies in Q^*_n, and satisfies

$$\tilde{A}\tilde{A}^t = \alpha I. \tag{6}$$

In terms of $y \in S^{k-2}$ if we now set

$$D(y) = -[C^t(y)C(y)]^{-1}C^t(y)B(y)C(y), \tag{7}$$

$$\tilde{A}(y) = \begin{bmatrix} B(y) & C(y) \\ -C^t(y) & D(y) \end{bmatrix}, \tag{8}$$

then $\tilde{A}(y)$ will be an odd map in $\mathscr{C}(S^{k-1}, Q^*_n)$. Since $\tilde{A}(y)$ is never zero ($C(y)$ is of maximal rank), it follows as above that $k \leqslant \rho(n)$. □

Gitler and Lam [15] have demonstrated that $\mathscr{C}(S^{27}, R_{1332})$ contains an odd map, yet lin($\mathbb{R}^{28}$, R^0_{1332}) does not contain any injective maps. Thus we cannot hope to generalize Theorem I in its strongest form. We can obtain the following result, though.

PROPOSITION IV: *If* $\mathscr{C}(S^{k-1}, R_{m\,m+1})$ *contains an odd map then* $k \leqslant \max(\rho(m), \rho(m+1))$. *In particular*

(*a*) $\rho(m, m+1) = \max(\rho(m), \rho(m+1))$;

(*b*) $\rho(m, m+1)$ *is the maximal dimension of a subspace of* $R^0_{m\,m+1}$.

PROOF: Define $\psi: R_{m\,m+1} \to GL_{m+1}$ by setting $\psi_{m+1\,q}(A)$ equal to $(-1)^q$ multiplied by the determinant of the $m \times m$ submatrix obtained from A by striking the q^{th} column, and $\psi_{pq}(A) = a_{pq}$ for $p \leqslant m$. Any odd map $\varphi \in \mathscr{C}(S^{k-1}, R_{m\,m+1})$ can be lifted to $\hat{\varphi} = \psi \circ \varphi \in \mathscr{C}(S^{k-1}, GL_{m+1})$. If m is odd then $\hat{\varphi}$ is odd and thus, by Theorem I, $k \leqslant \rho(m+1)$. If m is even, then for any $y \in S^{k-1}$ the rows of $\hat{\varphi}(y')$ form a basis for $\mathbb{R}^{m+1}$, the last one an even function of y' and the rest odd functions. Hence, according to Friedland, Robbin and Sylvester [13, Thm. A] $k \leqslant \rho(m)$. □

The result $\rho(m, m+1) = \max(\rho(m), \rho(m+1))$ can be proved by elementary means, without resort to Theorem I, as will be apparent in §5 below.

§5. Basic results

In this section we compute $\rho(m, m+l)$ for $l \leqslant 3$, and $\rho(2m'+1, 2m'+5)$.

THEOREM I: *For odd m*

$$\rho(m, m+2) = \max(\rho(m+1), 3).$$

PROOF: It follows from Table 1 and Proposition 3.II(a) that $\rho(3, m) \geqslant m-2$ (for all m). Hence, from Proposition 3.I(a) follows $\rho(m, m+2) \geqslant 3$ (for all m). It follows from Proposition I(c) that $\rho(m, m+2) \geqslant \rho(m+1)$ (for all m). This establishes the lower bound $\rho(m, m+2) \geqslant \max(\rho(m+1), 3)$ (for all m).

Let $A(y) = [B(y), C(y)]$ satisfy (2.8)–(2.10). According to Proposition 3.III(c) $\operatorname{rank}(C(y)) = 1$ for $y \neq 0$. Thus, according to Proposition 3.VIII(a), $k \leqslant \max(\rho(m, m+1), 3)$. Since m is odd $\rho(m, m+1) = \rho(m+1)$. This establishes the upper bound $\rho(m, m+2) \leqslant \max(\rho(m+1), 3)$. □

THEOREM II: *For even m*

$$\rho(m, m+2) = \max(\rho(m), \rho(m+2)).$$

PROOF: From Proposition 3.I(b) follows the lower bound $\rho(m, m+2) \geqslant \max(\rho(m), \rho(m+2))$. Let $A(y) = [B(y), C(y)]$ satisfy

(2.8)–(2.10). According to Proposition 3.III(b) the eigenvalues of $C^t(y)C(y)$ are $(\gamma(y), \gamma(y))$, where $\gamma(y)$ is a quadratic form satisfying

$$0 \leqslant \gamma(y) \leqslant |y|^2, \quad y \in \mathbb{R}^{k-1},$$

Thus

$$C^t(y)C(y) = \gamma(y)I, \quad y \in \mathbb{R}^{k-1}.$$

Let $\mathcal{N}$ be the subspace

$$\mathcal{N} = \{y \in \mathbb{R}^{k-1} : C(y) = 0\} = \{y \in \mathbb{R}^{k-1} : \gamma(y) = 0\}.$$

For $y \notin \mathcal{N}$ we can define

$$D(y) = -\gamma^{-1}(y)C^t(y)B(y)C(y), \tag{1}$$

as in (4.7), and

$$\tilde{A}(y) = \begin{bmatrix} B(y) & C(y) \\ -C^t(y) & D(y) \end{bmatrix} \tag{2}$$

will be an odd continuous function of y satisfying (2.7). Since

$$D^2(y) = -[|y|^2 - \gamma(y)]I$$

it follows that

$$D(y) = \omega(y)T, \tag{3}$$

where

$$\omega^2(y) = |y|^2 - \gamma(y),$$

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Suppose now that $k > \rho(m)$. (Otherwise we are through.) According to Corollary 4.III there must be a nonzero η for which $B(\eta)$ is singular. Since $\text{null}(B(\eta)) = 2$ it follows then from Corollary 3.VI that $\dim(\mathcal{N}) \leqslant 1$. Thus $\omega(y)$ in (3) is defined on all of S^{k-2} except for at most two antipodal points, where $\gamma(y) = 0$. Since $\omega(y)$ can only approach ± 1 at these points and since S^{k-1} is simply connected for $k \geqslant 4$, it follows that $\omega(y)$, and hence $\tilde{A}(y)$, has an odd continuous extension to all of S^{k-2}, as long as $k \geqslant 4$. Furthermore we may suppose $k \geqslant 4$ since $\max(\rho(m), \rho(m+2)) \geqslant 4$. With this extension, then, $\tilde{A}(y)$ is an odd map in $\mathscr{C}(S^{k-2}, GL_{m+2})$ which is skew symmetric. Thus by Corollary 4.II $k \leqslant \rho(m+2)$. This establishes the upper bound $\rho(m, m+2) \leqslant \max(\rho(m), \rho(m+2))$. □

THEOREM III: *For even m*

$$\rho(m, m+3) = \rho(m, m+2).$$

PROOF: From Proposition 3.I(b) follows the lower bound $\rho(m, m+3) \geqslant \rho(m, m+2)$ (for all m). Let $A(y) = [B(y), C(y)]$ satisfy (2.8)–(2.10). According to Proposition 3.III(b) the eigenvalues of $C^t(y)C(y)$ are $(\gamma(y), \gamma(y), 0)$, where $\gamma(y)$ is a quadratic form satisfying

$$0 \leqslant \gamma(y) \leqslant |y|^2, \quad y \in \mathbb{R}^{k-1}.$$

Accordingly

$$F(y) = \gamma(y)I - C^t(y)C(y) \tag{4}$$

is a symmetric rank one polynomial matrix. If $F(y)$ is identically zero then $\gamma(y)$, and hence $C(y)$, is also identically zero, in which case $k \leqslant \rho(m)$. Otherwise, if $F(y)$ is not identically zero, we can use Proposition 3.VII to write

$$F(y) = a(y)u(y)u^t(y), \tag{5}$$

for a vector polynomial $u(y) \in \mathbb{R}^3$ and a (scalar) polynomial $a(y)$. If u is a constant vector we can let $|u| = 1$, in which case

$$F(y) = \gamma(y)uu^t.$$

Then

$$C^t(y)C(y)u = 0, \quad y \in \mathbb{R}^{k-1},$$

and it follows from Proposition 3.IV(b) that $k \leqslant \rho(m, m+2)$.

Otherwise, if $u(y)$ varies with y it must be linear in y, and $a(y)$ must be constant. We can then write

$$F(y) = u(y)u^t(y), \tag{6}$$

$$u(y) = \sum_{i=1}^{k-1} y_i u_i. \tag{7}$$

Furthermore,

$$\gamma(y) = \mathrm{tr}(F(y)) = |u(y)|^2. \tag{8}$$

Condition (2.7) remains valid under an orthogonal transformation of the y variable, $y = \Phi\bar{y}$, and thus it may be assumed without loss of generality that

$$\gamma(y) = \sum_{i=1}^{p} \gamma_i y_i^2, \quad y \in \mathbb{R}^{k-1}, \tag{9}$$

where $0 \leqslant p \leqslant k-1$ and each γ_i is positive. From (7), (8) follows that in fact $p \leqslant 3$. Let $\mathcal{N}$ be the subspace

$$\mathcal{N} = \{ y \in \mathbb{R}^{k-1} : C(y) = 0 \} = \{ y \in \mathbb{R}^{k-1} : \gamma(y) = 0 \}.$$

Exactly as in the proof of Theorem II above, we conclude that if $k > \rho(m)$ then $\dim(\mathcal{N}) \leqslant 1$. Since

$$k - 1 = p + \dim(\mathcal{N})$$

and since $\rho(m, m+2) \geqslant 4$, it suffices now, in order to conclude the proof, to demonstrate that if $k > \rho(m)$ and if $\mathcal{N}$ is non-trivial then $p \leqslant 2$.

Suppose then that $p = 3$, $C_4 = 0$. We show how this leads to a contradiction. Observe that from (4), (6), (8) follows

$$C(y)u(y) = 0, \quad y \in \mathbb{R}^{k-1}. \tag{10}$$

Expanding this we obtain

$$C_i u_i = 0 \quad (i = 1, 2, 3), \tag{11}$$

$$C_i u_j = -C_j u_i \quad (i, j = 1, 2, 3). \tag{12}$$

From (7), (8), (9) follows

$$\langle u_i, u_j \rangle = \delta_{ij} \gamma_i \quad (i, j = 1, 2, 3). \tag{13}$$

From (4), (6), (7) follows

$$\gamma_i I - C_i^t C_i = u_i u_i^t \quad (i = 1, 2, 3),$$

and thus using (13) we further conclude that

$$C_i^t C_i u_j = \gamma_i u_j \quad (i, j = 1, 2, 3;\ i \neq j). \tag{14}$$

From (12), (13), (14) follows that $C_1 u_2$, $C_1 u_3$, $C_2 u_3$ are mutually orthogonal nonzero vectors, and hence, using (11), (12), (13) we conclude that

$$\bigcup_{\eta\in\mathcal{N}^{\perp}} \text{range}(C(\eta)) = \text{span}(C_1u_2,\ C_1u_3,\ C_2u_3). \tag{15}$$

From (6), (10) follows that $C(y)F(y)=0$, and so from (2.9), (4) we conclude that

$$-B^2(y)C(y) = \left[\,|y|^2-\gamma(y)\right]C(y), \quad y\in\mathbb{R}^{k-1}. \tag{16}$$

In particular, for $\eta\in\mathcal{N}^{\perp}$, $\eta\neq 0$ it follows that the two-dimensional eigenspace of $-B^2(\eta)$ corresponding to the eigenvalue $|\eta|^2-\gamma(\eta)$ is precisely range($C(\eta)$). On the other hand, according to Proposition 3.IV(b),

$$B(\eta)B_4 + B_4B(\eta) = 0,\ \eta\in\mathcal{N}^{\perp} \tag{17}$$

Thus this eigenspace, range($C(\eta)$), is invariant under B_4. Their union $\bigcup_{\eta\in\mathcal{N}^{\perp}}$ range($C(\eta)$) is then also invariant under B_4. According to (15), though, this union is a three-dimensional subspace of $\mathbb{R}^m$. Since B_4 is both skew symmetric and nonsingular (in fact orthogonal) it cannot have an odd dimensional invariant subspace. This is our desired contradiction. □

THEOREM IV: *For odd m*

$$\rho(m,\ m+3) = \rho(m+1,\ m+3).$$

PROOF: From Proposition 3.I(b) follows the lower bound $\rho(m,\ m+3)\geqslant\rho(m+1,\ m+3)$ (for all m). Let $A(y)=[B(y),\ C(y)]$ satisfy (2.8)–(2.10). According to Proposition 3.III(a) the eigenvalues of $C^t(y)C(y)$ are $(|y|^2, \gamma(y), \gamma(y))$, where $\gamma(y)$ is a quadratic form satisfying

$$0\leqslant\gamma(y)\leqslant|y|^2, \quad y\in\mathbb{R}^{k-1}.$$

Accordingly

$$F(y) = C^t(y)C(y) - \gamma(y)I \tag{18}$$

is a symmetric rank one polynomial matrix. If $F(y)$ is identically zero then rank($C(y)$) = 3, $y\neq 0$, and thus $k\leqslant\rho(m+3)$ (Corollary 4.III(b)). Otherwise, if $F(y)$ is not identically zero, we can use Proposition 3.VII to write

$$F(y) = a(y)u(y)u^t(y), \tag{19}$$

for a vector polynomial $u(y) \in \mathbb{R}^3$ and a (scalar) polynomial $a(y)$. If u is a constant vector we can let $|u| = 1$, in which case

$$F(y) = \left[|y|^2 - \gamma(y)\right] uu^t.$$

Then

$$C^t(y)C(y)u = |y|^2 u, \quad y \in \mathbb{R}^{k-1},$$

and it follows from Proposition 3.IV(b) that $k \leqslant \rho(m+1, m+3)$.

Otherwise, if $u(y)$ varies with y it must be linear in y, and $a(y)$ must be constant. We can then write

$$F(y) = u(y)u^t(y), \tag{20}$$

$$u(y) = \sum_{i=1}^{k-1} y_i u_i. \tag{21}$$

Furthermore,

$$|y|^2 - \gamma(y) = \operatorname{tr}(F(y)) = |u(y)|^2. \tag{22}$$

Whenever $\gamma(y) \neq 0$ we can define

$$D(y) = -\left[C^t(y)C(y)\right]^{-1} C^t(y)B(y)C(y), \tag{23}$$

as in (2.7), and then

$$\tilde{A}(y) = \begin{bmatrix} B(y) & C(y) \\ -C^t(y) & D(y) \end{bmatrix}$$

will satisfy (2.7). If $\gamma(y)$ is identically zero then $\operatorname{rank}(C(y)) \leqslant 1$, $y \in \mathbb{R}^{k-1}$ and we conclude from Proposition 3.VIII that

$$k \leqslant \max(\rho(m+1), 4) \leqslant \rho(m+1, m+3).$$

Otherwise $D(y)$ defined by (23) is a rational function of y. We show now that in fact it is locally linear in y, hence globally linear – in which case $\tilde{A}$ above actually lies in $\operatorname{lin}(\mathbb{R}^{k-1}, Q^*_{m+3})$. Then

$$k \leqslant \rho(m+3) \leqslant \rho(m+1, m+3).$$

Accordingly, we devote the remainder of this proof to establishing that $D(y)$ is locally linear in y. From (18), (20), (22) follows

$$C^t(y)C(y) = |y|^2 u(y), \quad y \in \mathbb{R}^{k-1},$$

and thus from (2.8), (2.9) follows

$$B(y)C(y)u(y)=0, \quad y \in \mathbb{R}^{k-1}.$$

Using (23) we see then that

$$D(y)u(y)=0, \quad y \in \mathbb{R}^{k-1}, \quad \gamma(y) \neq 0. \tag{24}$$

From (4.3) follows

$$\operatorname{tr}(-D^2(y)) = 2[\,|y|^2 - \gamma(y)], \quad y \in \mathbb{R}^{k-1}, \quad \gamma(y) \neq 0. \tag{25}$$

Choose η such that $0 < \gamma(\eta) < |\eta|^2$. This can be done since we are assuming that $\gamma(y)$ is neither identically 0 or $|y|^2$. Then $u(\eta) \neq 0$. Since $D(\eta)$ is a skew-symmetric 3×3 matrix, the conditions (24), (25) require that

$$D(\eta)v = \pm u(\eta) \times v, \quad v \in \mathbb{R}^3, \tag{26}$$

where $\times$ is the vector cross product. Since $D(y)$ is continuous the choice of $\pm$ must be fixed in a neighbourhood of η, and we are through. □

THEOREM V: *For odd m*

$$\rho(m, m+4) = \max(\rho(m, m+3), 5).$$

PROOF: It follows from Table 1 and Proposition 3.II(a) that $\rho(5, m) \geq m-4$ (for all m). Hence, from Proposition 3.I(a) follows $\rho(m, m+4) \geq 5$ (for all m). It follows from Proposition 3.I(b) that $\rho(m, m+4) \geq \rho(m, m+3)$ (for all m). This establishes the lower bound $\rho(m, m+4) \geq \max(\rho(m, m+3), 5)$ (for all m).

Let $A(y) = [B(y), C(y)]$ satisfy (2.8)–(2.10). According to Proposition 3.III(a) the eigenvalues of $C^t(y)C(y)$ are $(|y|^2, \gamma(y), \gamma(y), 0)$, where $\gamma(y)$ is a quadratic form satisfying

$$0 \leq \gamma(y) \leq |y|^2, \quad y \in \mathbb{R}^{k-1}.$$

Accordingly

$$F(y) = C^t(y)C(y)[C^t(y)C(y) - \gamma(y)I] \tag{27}$$

is a symmetric rank one polynomial matrix. If $F(y)$ is identically zero then $\gamma(y) \equiv |y|^2$ and it follows from Proposition 3.VIII(b) that $k \leq \max(\rho(m+3), 5)$. Otherwise we use Proposition 3.VII to write

$$F(y) = a(y)u(y)u^t(y), \quad y \in \mathbb{R}^{k-1}, \tag{28}$$

where $u(y)$ is a vector polynomial in $\mathbb{R}^4$ and $a(y)$ is a (scalar) polynomial. Similarly,

$$G(y) = \left[C^t(y)C(y) - |y|^2 I\right]\left[C^t(y)C(y) - \gamma(y)I\right] \tag{29}$$

is also a symmetric rank one polynomial matrix. If $G(y)$ is identically zero then $\gamma(y) \equiv 0$ and it follows from Proposition 3.VIII(a) that $k \leqslant \max(\rho(m+1), 5)$. Otherwise we again use Proposition 3.VII to write

$$G(y) = b(y)v(y)v^t(y), \quad y \in \mathbb{R}^{k-1}, \tag{30}$$

where $v(y)$ is a vector polynomial in $\mathbb{R}^4$ and $b(y)$ is a (scalar) polynomial. Observe from (27)–(30) that

$$|y|^2\left[|y|^2 - \gamma(y)\right] = a(y)|u(y)|^2, \tag{31}$$

$$|y|^2\gamma(y) = b(y)|v(y)|^2. \tag{32}$$

Combining (27)–(30) we obtain

$$\begin{aligned} C^t(y)C(y) - \gamma(y)I &= \frac{F(y) - G(y)}{|y|^2} \\ &= \frac{|y|^2 - \gamma(y)}{|u(y)|^2} u(y)u^t(y) \\ &\quad - \frac{\gamma(y)}{|v(y)|^2} v(y)v^t(y), \quad y \in \mathbb{R}^{k-1}. \end{aligned} \tag{33}$$

Furthermore $u(y)$ and $v(y)$ must be orthogonal since $F(y)G(y) = 0$. Thus we find from (33) that

$$C^t(y)C(y)u(y) = |y|^2 u(y), \quad C^t(y)C(y)v(y) = 0. \tag{34}$$

Since $F(y)$ and $G(y)$ are fourth order in y, there are several cases to consider regarding the degrees of a, b, u, v in (28), (30). All but one of the cases, however, are immediate. Indeed, if $u(y)$ is constant (i.e. independent of y) then we use (34) to conclude from Proposition 3.IV that $k \leqslant \rho(m+1, m+4)$. Similarly if $v(y)$ is constant then $k \leqslant \rho(m, m+3)$. If $a(y)$ is constant then we may choose it to be one, and (33) becomes

$$C^t(y)C(y) - \gamma(y)I = \frac{u(y)u^t(y)}{|y|^2} - \frac{\gamma(y)}{|v(y)|^2} v(y)v^t(y).$$

Thus

$$|y|^2 \,\Big|\, u(y)u^t(y)|v(y)|^2.$$

Since (for $k-1 \geqslant 2$) $|y|^2$ is irreducible and the coordinates of $u(y)$ are relatively prime (recall Proposition 3.VII) it follows that

$$|y|^2 \Big| |v(y)|^2.$$

This means that for $v(y)$ to be linear in y we must have $k-1 \leqslant 4$. (Cf. the proof of Proposition 3.VIII(a) – in particular the remark following (12), (13).) Similarly if $b(y)$ is constant and $u(y)$ is linear then, again, we must have $k-1 \leqslant 4$. Suppose that $u(y)$, $v(y)$ are both linear in y. Then using the irreducibility of $|y|^2$, it follows from (31), (32) that one of the alternatives

$$|y|^2 \Big| |u(y)^2, \quad |y|^2 \Big| |v(y)|^2, \quad |y|^2 \Big| |u(y)|^2 + |v(y)|^2$$

holds. If the first or second alternative holds then, as just mentioned above, $k-1 \leqslant 4$. For the third alternative observe that

$$|u(y)|^2 + |v(y)|^2 = |u(y) + v(y)|^2,$$

since $u(y)$ and $v(y)$ are orthogonal. Thus, under this alternative as well, $k-1 \leqslant 4$. The only remaining case to consider then is where $u(y)$, $v(y)$ are both quadratic.

Accordingly, we devote the remainder of this proof to the case $u(y)$, $v(y)$ quadratic. We choose $a(y)$, $b(y)$ to be one. Observe that from (31), (32) and the orthogonality of $u(y)$, $v(y)$ follows

$$|u(y) \pm v(y)| = |y|^2, \quad y \in \mathbb{R}^{k-1}. \tag{35}$$

Since (from (33))

$$|y|^2 \Big| u(y)u'(y) - v(y)v'(y)$$

and since the coordinates of $u(y)$ are assumed to be relatively prime, it follows that one of the alternatives

$$u(y) + v(y) = |y|^2 w, \quad u(y) - v(y) = |y|^2 w \tag{36}$$

holds, where $w \in \mathbb{R}^4$ is a fixed unit vector. (That it is a unit vector follows from (35).) Set

$$\omega(y) = C(y)w.$$

It then follows from (31), (34), (36) and the orthogonality of $u(y)$, $v(y)$ that

$$|\omega(y)|^2 = |y|^2 - \gamma(y), \tag{37}$$

$$\left[\,|y|^2 I - C(y)C'(y)\right]\omega(y) = 0. \tag{38}$$

Thus from (2.8), (2.9) we deduce that

$$B(y)\omega(y)=0. \tag{39}$$

By rotating if necessary let us assume that

$$w=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

This corresponds to $\omega(y)$ being the fourth column of $C(y)$, and we write

$$C(y)=[E(y),\ \omega(y)]$$

where the dimensions of $E(y)$ are $m\times 3$. Then

$$E(y)E^t(y)=C(y)C^t(y)-\omega(y)\omega^t(y) \tag{40}$$

and since, by (38), $\omega(y)$ is an eigenvector of $C(y)C^t(y)$ with eigenvalue $|y|^2$, it follows from (37) that the eigenvalues of $E(y)E^t(y)$ are $(\gamma(y), \gamma(y), \gamma(y), 0, \dots, 0)$. Thus

$$E^t(y)E(y)=\gamma(y)I. \tag{41}$$

Observe that since, by (2.9), $C(y)C^t(y)$ and $B(y)$ commute, it follows from (39), (40) that $E(y)E^t(y)$ and $B(y)$ likewise commute.

Wherever $\gamma(y)>0$ set

$$D(y)=-\frac{E^t(y)B(y)E(y)}{\gamma(y)}.$$

This matrix $D(y)$ is skew symmetric, since $B(y)$ is, and since $E(y)E^t(y)$ and $B(y)$ commute it satisfies

$$B(y)E(y)+E(y)D(y)=0.$$

Thus the matrix

$$\tilde{A}(y)=\begin{bmatrix}B(y) & E(y)\\ -E^t(y) & D(y)\end{bmatrix}$$

satisfies

$$\tilde{A}(y)\tilde{A}^t(y)=\begin{bmatrix}|y|^2I-\omega(y)\omega^t(y) & 0\\ 0 & \gamma(y)I+D(y)D^t(y)\end{bmatrix}.$$

Clearly this is nonsingular for $\gamma(y) > 0$. Thus if $\gamma(y) > 0$ for all $y \neq 0$ then $\tilde{A}$ constitutes an odd map in $\mathscr{C}(S^{k-2}, GL_{m+3})$ which is everywhere skew symmetric. According to Corollary 4.II, then, $k \leqslant \rho(m+3)$.

Thus we must concentrate on the case where

$$\mathscr{N} = \{ y \in \mathbb{R}^{k-1} : E(y) = 0 \} = \{ y \in \mathbb{R}^{k-1} : \gamma(y) = 0 \}$$

is a nontrivial subspace, for the remainder of this proof. Since $\operatorname{rank}(C(y)) \leqslant 3$ it follows that

$$\omega(y) \in \operatorname{range}(E(y)), \quad y \notin \mathscr{N}.$$

Then, using the linearity of $\omega(y)$ and $E(y)$ in y, we obtain that

$$\omega(\mathscr{N}) \subset \bigcap_{\substack{\eta \in \mathscr{N}^{\perp} \\ \eta \neq 0}} \operatorname{range}(E(\eta)). \tag{42}$$

Since, by (37), $\omega|_{\mathscr{N}}$ is an isometry it follows that

$$\dim(\mathscr{N}) \leqslant \dim\left(\bigcap_{\substack{\eta \in \mathscr{N}^{\perp} \\ \eta \neq 0}} \operatorname{range}(E(\eta)). \right). \tag{43}$$

Furthermore, we know on account of (41) that for $y \notin \mathscr{N}$, $\operatorname{range}(E(y))$ is precisely the eigenspace of $E(y)E^t(y)$ corresponding to the eigenvalue $\gamma(y)$. Thus we conclude from (42) that

$$[\gamma(\eta)I - E(\eta)E^t(\eta)]\,\omega(\mathscr{N}) = 0, \quad \eta \in \mathscr{N}^{\perp}.$$

This shows that if $\mathscr{N} \neq 0$ then $\gamma(\eta)$ can be written as the sum of three terms, each a product of two linear forms in η. Since $\gamma(\eta)$ is positive definite we must have

$$\dim(\mathscr{N}^{\perp}) \leqslant 3. \tag{44}$$

Since

$$k - 1 = \dim(\mathscr{N}) + \dim(\mathscr{N}^{\perp})$$

we conclude the proof by establishing

$$\dim(\mathscr{N}^{\perp}) \geqslant 2 \Rightarrow \dim\left(\bigcap_{\substack{\eta \in \mathscr{N}^{\perp} \\ \eta \neq 0}} \operatorname{range}(E(\eta)) \right) \leqslant 2, \tag{45}$$

$$\dim(\mathscr{N}^{\perp}) \geqslant 3 \Rightarrow \dim\left(\bigcap_{\substack{\eta \in \mathscr{N}^{\perp} \\ \eta \neq 0}} \operatorname{range}(E(\eta)) \right) \leqslant 1, \tag{46}$$

and invoking (43). To simplify the rest of this discussion we assume without loss of generality, as in the proof of Theorem III (see the remark preceding (9)), that

$$\gamma(y)=\sum_{i=1}^{p}\gamma_i y_i^2,$$

where $p=\dim(\mathcal{N}^{\perp})$ and each γ_i is positive. Then, analogous to (2.3), the condition (41) is

$$E_i^t E_j + E_j^t E_i = 2\delta_{ij}\gamma_i I \quad (i,\ j=1,\dots,p), \tag{47}$$

where $E(y)=\sum_{i=1}^{p} y_i E_i$. To establish (45) suppose $p\geqslant 2$ and that in fact range(E_1) = range(E_2). Then there is a 3×3 orthogonal matrix Q such that $E_2=E_1Q$. From (47), though, it follows that Q must also be skew symmetric, which is impossible. Thus range(E_1) ≠ range(E_2) and, since each range is three-dimensional (45) follows.

To establish (46) suppose $p\geqslant 3$ and let $\omega_1=E_1\alpha$, $\omega_2=E_1\beta$ be any two linearly independent vectors. Choose η_2, η_3 not both zero such that

$$\eta_2\langle E_2\alpha,\ E_1\beta\rangle+\eta_3\langle E_3\alpha,\ E_1\beta\rangle=0.$$

Then it follows from (47) that

$$(\eta_2E_2+\eta_2E_3)\mathcal{M}\perp E_1\mathcal{M}$$

where $\mathcal{M}=\text{span}(\alpha,\ \beta)$. Since range($\eta_2E_2+\eta_3E_3$) is three-dimensional it is impossible that it should contain $E_1\mathcal{M}$, and thus (46) follows. □

Regarding the next result in line we conclude this Section with the following

CONJECTURE VI: *For m even*

$$\rho(m,\ m+4)=\max(\rho(m),\ \rho(m+2),\ \rho(m+4),\ 6).$$

For a summary of the preceding results see Tables 2 and 3 in the Appendix.

Appendix: Figures and tables

TABLE 1. $\rho(m, n)$

m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2		2	2	4	4	6	6	8	8	10	10	12	12	14	14
3			1	4	4	4	5	8	8	8	9	12	12	12	13
4				4	4	4	4	8	8	8	8	12	12	12	12
5					1	2	3	8	8	8	8	8	9	10	11
6						2	2	8	8	8	8	8	8	10	10
7							1	8	8	8	8	8	8	8	9
8								8	8	8	8	8	8	8	8
9									1	2	3	4	5	6	7
10										2	2	4	4	6	6
11											1	4	4	4	5
12												4	4	4	4
13													1	2	3
14														2	2
15															1

TABLE 2. Values of $\rho(m, n)$

$\rho(m, m+1)$	m	odd	$= \rho(m+1)$	{(Thm. 4.IV)
	m	even	$= \rho(m)$	
$\rho(m, m+2)$	m	odd	$= \max(\rho(m+1), 3)$	(Thm. 5.I)
	m	even	$= \max(\rho(m), \rho(m+2))$	(Thm. 5.II)
$\rho(m, m+3)$	m	odd	$= \max(\rho(m+1), \rho(m+3))$	(Thm. 5.IV)
	m	even	$= \max(\rho(m), \rho(m+2))$	(Thm. 5.III)
$\rho(m, m+4)$	m	odd	$= \max(\rho(m+1), \rho(m+3), 5)$	(Thm. 5.V)

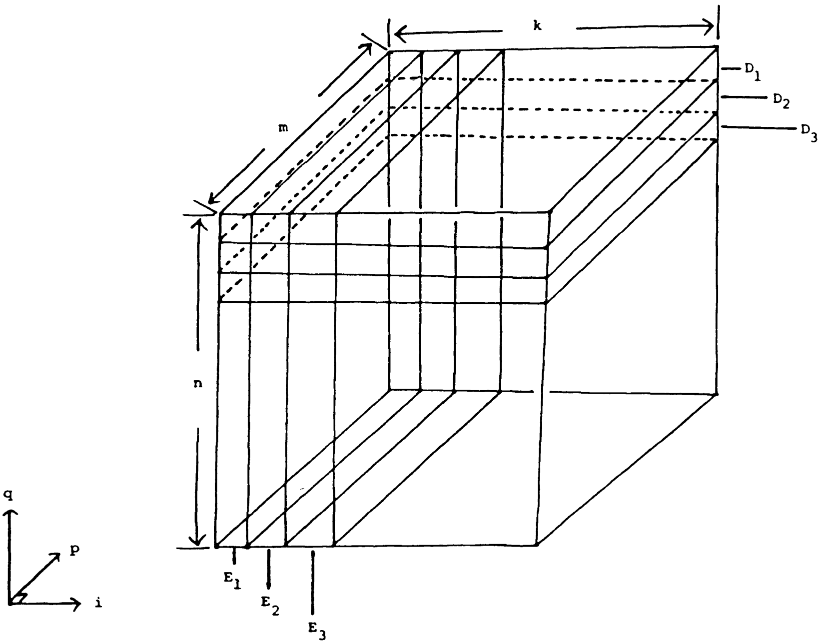

Figure 1. Duality condition. The horizontally stacked $k \times m$ matrices are dual to the vertically stacked $m \times n$ matrices. $d_{ip}^{(q)} = e_{pq}^{(i)}$

TABLE 3 $\rho(m, n)$

m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	49	40
2		2	2	4	4	6	6	8	8	10	10	12	12	14	14	16	16	18	18	20	20	22	22	24	24	26	26	28	28	30	30	32	32	34	34	36	36	38	38	40
3			1	4	4	4	5	8	8	8	9	12	12	12	13	16	16	16	17	20	20	20	21	24	24	24	25	28	28	28	29	32	32	32	33	36	36	36	37	40
4				4	4	4	4	8	8	8	8	12	12	12	12	16	16	16	16	20	20	20	20	24	24	24	24	28	28	28	28	32	32	32	32	36	36	36	36	40
5					1	2	3	8	8	8	8	8	9	10	11	16	16	16	16	16	17	18	19	24	24	24	24	24	25	26	27	32	32	32	32	32	33	34	35	40
6						2	2	8	8	8	8	8	8	10	10	16	16	16	16	16	16	18	18	24	24	24	24	24	24	26	26	32	32	32	32	32	32	34	34	40
7							1	8	8	8	8	8	8	8	9	16	16	16	16	16	16	16	17	24	24	24	24	24	24	24	25	32	32	32	32	32	32	32	33	40
8								8	8	8	8	8	8	8	8	16	16	16	16	16	16	16	16	24	24	24	24	24	24	24	24	32	32	32	32	32	32	32	32	40
9									1	2	3	4	5	6	7	16	16	16	16	16	16	16	16	16	17	18	19	20	21	22	23	32	32	32	32	32	32	32	32	32
10										2	2	4	4	6	6	10																32	32	32	32	32	32	32	32	32
11											1	4	4	4	5																									
12												4	4	4	4																									
13													1	2	3	9	9																							
14														2	2	9	9																							
15															1	9	9	9	9																					
16																9	9	9	9																					
17																	1	2	3	4	5	6	7	8	9															
18																		2	2	4	4	6	6	8	8															
19																			1	4	4	4	5	8	8	8	9													
20																				4	4	4	4	8	8	8	8													
21																					1	2	3	8	8	8	8	8	9											
22																						2	2	8	8	8	8	8	8											
23																							1	8	8	8	8	8	8	8	9									
24																								8	8	8	8	8	8	8	8									
25																									1	2	3	4	5	6	7									
26																										2	2	4	4	6	6									
27																											1	4	4	4	5									
28																												4	4	4	4									
29																													1	2	3	10	10							
30																														2	2	10	10							
31																															1	10	10	10	10					
32																																10	10	10	10					
33																																	1	2	3	4	5	6	7	8
34																																		2	2	4	4	6	6	8
35																																			1	4	4	4	5	8
36																																				4	4	4	4	8
37																																					1	2	3	8
38																																						2	2	8
39																																							1	8
40																																								8

Supplement

Steps for filling in Table 1

(i) Use (1.3) to fill in the 15 entries

$$\{(1,1),(2,2),(3,3),\dots,(15,15)\}$$

(ii) Use the fact that $\rho(m,n) \geqslant \frac{n}{m}\rho(m)$ whenever $m \mid n$ (Proposition 3.I(d)) and the lower bound $\rho(m,n) \leqslant n$ from Proposition 3.I(c) to conclude that

$$\rho(m,n) = n \tag{1}$$

for the 23 entries

$$\{(1,2),(1,3),(1,4),\dots,(1,15),(2,4),(2,6),(2,8),\\ \dots,(2,14),(4,8),(4,12)\}$$

Then use the monotonicity condition (Proposition 3.I(b)) to conclude that (1) holds for the 6 additional entries.

$$\{(3,4),(3,8),(3,12),(5,8),(6,8),(7,8)\}$$

(iii) Use the fact that $\rho(m,n) \leqslant n-m+1$ whenever $\binom{n}{m-1}$ is odd (Proposition 3.I(f)) and the monotonicity condition (Proposition 3.I(b)) to conclude that

$$\rho(m,n) = n-m+1 \tag{2}$$

for the 27 entries

$$\{(2,3),(2,5),(2,7),\dots,(2,15),(3,6),(3,10),(3,14),\\ (4,7),(4,11),(4,15),(5,6),(5,12),(6,7),(6,13),\\ (7,14),(8,15),(9,10),(9,12),(10,11),(10,13),\\ (11,14),(12,15),(13,14),(14,15)\}$$

Then use the monotonicity condition (Proposition 3.I(b)) to conclude that

$$\rho(m,n) = \rho(m-1,n) \tag{3}$$

for the 32 entries

$$\{(3,5),(3,9),(3,13),(4,5),(4,6),(4,9),(4,10),(4,13),$$
$$(4,14),(5,9),(5,10),(5,11),(6,9),(6,10),(6,11),$$
$$(6,12),(7,9),(7,10),(7,11),(7,12),(7,13),(8,9),$$
$$(8,10),(8,11),(8,12),(8,13),(8,14),(10,12),$$
$$(11,12),(11,13),(12,13),(12,14)\}$$

(iv) Use the fact that $\rho(m, n) \leqslant n - m + 1$ whenever $\binom{n}{m-1}$ is odd (Proposition 3.I(f)) and the lower bound in Proposition 3.I(e) (with $t = 2$ and $(m, n) = (4, 4)$ or $(8, 8)$ or $(12, 12)$) to conclude that (2) holds for the 7 entries

$$\{(5,7),(9,11),(9,13),(9,14),(9,15),(10,15),(13,15)\}.$$

Then use the lower bound from the symmetry condition (Proposition 3.I(d)) to conclude that (1) holds for the 6 additional entries

$$\{(3,7),(3,11),(3,15),(5,13),(6,15),(7,15)\}.$$

Repeat this step once more to add the 2 additional entries

$$\{(5,15),(11,15)\}$$

to the list. (The calculation for $\rho(11, 15)$ depends on prior knowledge that $\rho(3, 7) = 5$, which has just been established above.) Finally, by Proposition 3.I(e) $\rho(10, 14) \geqslant 6$ and thus by monotonicity (Proposition 3.I(b))

$$\rho(10,14)=6.$$

By symmetry (Proposition 3.I(a)) $\rho(6, 14) \geqslant 10$ and thus by monotonicity

$$\rho(6,14)=10.$$

Finally, since $\rho(10, 14) \geqslant 5$ we have $\rho(5, 14) \geqslant 10$, by symmetry. Since $\binom{14}{4}$ is odd it follows from Proposition 3.I(f) that

$$\rho(5,14)=10.$$

REMARK: A useful test for divisibility of binomial coefficients by a prime p is the following. Write the p-adic expansions

$$m = \sum_{i=0}^{t} \beta_i p^i, \quad n = \sum_{i=0}^{t} \gamma_i p^i.$$

Then

$$p \Big| \binom{n}{m} \Leftrightarrow \beta_i \leqslant \gamma_i \ (i = 0, \dots, t).$$

In other words if in subtracting m from n in base p we have to "borrow" then p divides $\binom{n}{m}$, and vice versa. (This is not true for composite p – e.g. $4 \Big| \binom{6}{2}$.) For $p = 2$ this affords a very simple test for the parity of $\binom{n}{m-1}$, which can be used to help fill in Table 1. (Cf. Behrend [8]).

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