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THE PLANCHEREL FORMULA FOR THE PSEUDO-RIEMANNIAN SPACE $SL(n, \mathbb{R})/GL(n-1, \mathbb{R})$

G. van Dijk and M. Poel

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0. Introduction

The main result of this paper is a Plancherel formula for the rank one symmetric space $X = SL(n, \mathbb{R})/GL(n-1, \mathbb{R})$, $n \geq 3$. This means a desintegration of the left regular representation of G on $L^2(X)$ into irreducible unitary representations. One can also formulate it in terms of spherical distributions (cf. [7]). Then we are determining a desintegration of the δ -distribution at the origin of X into extremal positive-definite spherical distributions.

Section 1 and 2 are concerned with a precise definition. We also ask for uniqueness of the desintegration and introduce once more the notion of a generalized Gelfand pair. Special attention is paid to the relative discrete series. Section 3 contains the abstract theory, while Section 4 is devoted to an explicit determination of a parametrization of the relative discrete series for the space under consideration. The results we obtain are applied in Section 5 where the Plancherel formula is determined by a method previously used by Faraut [5]. This paper is a continuation of earlier work [7] and depends heavily on it. Recently Molčanov [9] has obtained the Plancherel formula for the case $n = 3$ by a quite different method. Our analysis of the relative discrete series seems to have some analogy with work of Kengmana [6].

1. Invariant Hilbert subspaces of $D'(G/H)$

Let G be a real Lie group and H a closed subgroup of G . Throughout this paper we assume both G and H to be unimodular. Let us fix Haar

measures dg on G , dh on H and a G -invariant measure dx on G/H in such a way that $dg = dx dh$.

We shall take all scalar products anti-linear in the first and linear in the second factor.

Let π be a continuous unitary representation of G on a Hilbert space \mathcal{H} . A vector $v \in \mathcal{H}$ is said to be a C^∞ -vector if the map $g \rightarrow \pi(g)v$ is in $C^\infty(G, \mathcal{H})$. The subspace \mathcal{H}_∞ of C^∞ -vectors in \mathcal{H} can be endowed with a natural Sobolev-type topology (cf. [2], §1). Let us recall the definition. Let \mathfrak{g} be the Lie algebra of G . For any $X \in \mathfrak{g}$ and $v \in \mathcal{H}_\infty$, put

$$\pi(X)v = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp tX)v.$$

Then $\pi(X)$ leaves \mathcal{H}_∞ stable. The topology is defined by means of the set of norms $\|\cdot\|_m$ given by the following formula. Let X_1, \dots, X_n be a basis of \mathfrak{g} . Then

$$\|v\|_m^2 = \sum_{|\alpha| \leq m} \|\pi(X_1)^{\alpha_1} \dots \pi(X_n)^{\alpha_n} v\|^2$$

with $|\alpha| = \alpha_1 + \dots + \alpha_n$, α_i non-negative integers ($v \in \mathcal{H}_\infty$). \mathcal{H}_∞ becomes a Frechet space in this manner.

The topology does not depend on the choice of the basis of \mathfrak{g} . The space \mathcal{H}_∞ is G -invariant. The corresponding representation of G on \mathcal{H}_∞ is called π_∞ ; the map $(g, v) \rightarrow \pi_\infty(g)v$ is continuous $G \times \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$.

Denote $\mathcal{H}_{-\infty}$ the anti-dual of \mathcal{H}_∞ , endowed with the strong topology. The inclusion $\mathcal{H}_\infty \subset \mathcal{H}$ and the isomorphism of the Hilbert space \mathcal{H} with its anti-dual yield an inclusion $\mathcal{H} \subset \mathcal{H}_{-\infty}$, so $\mathcal{H}_\infty \subset \mathcal{H} \subset \mathcal{H}_{-\infty}$. The injections are continuous. G acts on $\mathcal{H}_{-\infty}$ and the corresponding representation is called $\pi_{-\infty}$. Denote by $D(G)$, $D(G/H)$ the space of C^∞ -functions with compact support on G and G/H respectively, endowed with the usual topology. Let $D'(G)$, $D'(G/H)$ be the topological anti-dual of $D(G)$ and $D(G/H)$ respectively, provided with the strong topology.

For $v \in \mathcal{H}_\infty$, $a \in \mathcal{H}_{-\infty}$ we put $\langle v, a \rangle = a(v)$ and we write $\langle a, v \rangle$ instead of $\overline{\langle v, a \rangle}$. Similarly we put $\langle \phi, T \rangle = \overline{\langle T, \phi \rangle} = T(\phi)$ for $\phi \in D(G/H)$, $T \in D'(G/H)$. Denote $\phi_0 \rightarrow \phi$ the canonical projection map $D(G) \rightarrow D(G/H)$ given by

$$\phi(x) = \int_H \phi_0(gh) dh \quad (x \in G/H, x = gH).$$

For any $a \in \mathcal{H}_{-\infty}$ and $\phi_0 \in D(G)$, put

$$\pi_{-\infty}(\phi_0)a = \int_G \phi_0(g)(\pi_\infty(g)a) dg.$$

Then $\pi_{-\infty}(\phi_0)a \in \mathcal{H}_{\infty}$. A vector $a \in \mathcal{H}_{-\infty}$ is called *cyclic* if $\{\pi_{-\infty}(\phi_0)a: \phi_0 \in D(G)\}$ is a dense subspace of \mathcal{H} . Define

$$\mathcal{H}_{-\infty}^H = \{ a \in \mathcal{H}_{-\infty}: \pi_{-\infty}(h)a = a \text{ for all } h \in H \}.$$

We say that π can be realized on a Hilbert subspace of $D'(G/H)$ if there is a continuous linear injection $j: \mathcal{H} \rightarrow D'(G/H)$ such that

$$j\pi(g) = L_g j$$

for all $g \in G$ (L_g denotes the left translation by g). The space $j(\mathcal{H})$ is said to be an invariant Hilbert subspace of $D'(G/H)$.

THEOREM 1.1: *π can be realized on a Hilbert subspace of $D'(G/H)$ if and only if $\mathcal{H}_{-\infty}^H$ contains non-zero cyclic elements. There is an one-to-one correspondence between the non-zero cyclic elements of $\mathcal{H}_{-\infty}^H$ and the continuous linear injections $j: \mathcal{H} \rightarrow D'(G/H)$ satisfying $j\pi(g) = L_g j$ ($g \in G$). To a cyclic vector $a \neq 0$ in $\mathcal{H}_{-\infty}^H$ corresponds j , such that $j^*: D(G/H) \rightarrow \mathcal{H}$ is given by $j^*(\phi) = \pi_{-\infty}(\phi_0)a$.*

The proof is quite similar to [2, Théorème 1.4].

Let π be a representation realized on $D'(G/H)$ and $j: \mathcal{H} \rightarrow D'(G/H)$ the corresponding injection. Denote by ξ_{π} the cyclic vector in $\mathcal{H}_{-\infty}^H$ defined by Theorem 1.1. Then we put

$$\langle T, \phi_0 \rangle = \langle \xi_{\pi}, \pi_{-\infty}(\phi_0)\xi_{\pi} \rangle \quad (\phi_0 \in D(G)).$$

T is a distribution on G which is left and right H -invariant. We call T the reproducing distribution of π (or \mathcal{H}). T is positive-definite, bi- H -invariant and

$$\| j^*\phi \|^2 = \langle T, \tilde{\phi}_0 * \phi_0 \rangle$$

for all $\phi_0 \in D(G)$. Here $\tilde{\phi}_0$ is given by $\tilde{\phi}_0(g) = \overline{\phi_0(g^{-1})}$ ($g \in G$). Given a positive-definite bi- H -invariant distribution T on G , the latter formula shows the way to define a G -invariant Hilbert subspace of $D'(G/H)$ with T as reproducing distribution. Indeed, let V be the space $D(G/H)$ provided with the inner product

$$(\phi, \psi) = \langle T, \tilde{\phi}_0 * \psi_0 \rangle.$$

Let V_0 be the subspace of V consisting of the elements of length zero and define \mathcal{H} to be the completion of V/V_0 and j^* the natural projection $D(G/H) \rightarrow \mathcal{H}$. Then clearly

$$\| j^*\phi \|^2 = \langle T, \tilde{\phi}_0 * \phi_0 \rangle$$

for all $\phi_0 \in D(G)$.

Furthermore, an easy calculation shows that jv is a C^∞ -function for all $v \in \mathcal{H}_\infty$. Actually

$$jv(x) = \langle \xi_\pi, \pi(g^{-1})v \rangle \quad (x = gH \in G/H).$$

Note that j can be defined on $\mathcal{H}_{-\infty}$ (as anti-dual of $j^*: D(G/H) \rightarrow \mathcal{H}_\infty$). Then $j(\xi_\pi)$ is precisely the reproducing distribution T , considered as an H -invariant element of $D'(G/H)$. One has

$$(jj^*)(\phi) = \phi_0 * T$$

for all $\phi_0 \in D(G)$.

Summarizing we have

PROPOSITION 1.2: *The correspondence $\mathcal{H} \rightarrow T$ which associates with each invariant Hilbert subspace of $D'(G/H)$ its reproducing distribution is a bijection between the set of G -invariant Hilbert subspaces of $D'(G/H)$ and the set of bi- H -invariant positive-definite distributions on G .*

Denote Γ_G the set of bi- H -invariant positive-definite distributions and $\text{ext}(\Gamma_G)$ the subset of those distributions which correspond to minimal G -invariant Hilbert subspaces of $D'(G/H)$ (or: to irreducible unitary representations π realized on a Hilbert subspace of $D'(G/H)$). Choose an admissible parametrization $s \rightarrow T_s$ of $\text{ext}(\Gamma_G)$ as in [12]. Here S is a topological Hausdorff space. Then one has

PROPOSITION 1.3 [12, Proposition 9]: *For every $T \in \Gamma_G$ there exists a (non-necessarily unique) Radon measure m on S such that*

$$\langle T, \phi_0 \rangle = \int_S \langle T_s, \phi_0 \rangle \, dm(s)$$

for all $\phi_0 \in D(G)$.

This result, except for the fixed parametrization independent of T , has been obtained by L. Schwartz and K. Maurin. See [12] for references. The proof of Proposition 1.3 is obtained by diagonalising a maximal commutative C^* -algebra commuting with the action of G in the Hilbert subspace, associated with T . The fixed parametrization can then be obtained by the techniques of [12]. Clearly we are mainly interested in the decomposition of the distribution $T \in \Gamma_G$ given by

$$\langle T, \phi_0 \rangle = \int_H \phi_0(h) \, dh \quad (\phi_0 \in D(G))$$

which corresponds to the δ -function at the origin of G/H .

This could be called a Plancherel formula for G/H .

Let G be a connected, non-compact, real semisimple Lie group with finite center and σ an involutive automorphism of G . Let H be an open subgroup of the group of fixed points of σ . The pair (G, H) is called a semisimple symmetric pair.

Let $\mathbb{D}(G/H)$ denote the algebra of G -invariant differential operators on G/H . It is known that $\mathbb{D}(G/H)$ is a commutative, finitely generated algebra. For any $D \in \mathbb{D}(G/H)$, define $'D$ by

$$\int_{G/H} \overline{D\phi(x)}\psi(x) \, dx = \int_{G/H} \overline{\phi(x)'}D\psi(x) \, dx$$

for all $\phi, \psi \in \mathbb{D}(G/H)$. Then $'D \in \mathbb{D}(G/H)$. So $\mathbb{D}(G/H)$ is generated by "self adjoint" elements. Let $D \in \mathbb{D}(G/H)$ be such that $D = 'D$. Then, regarding D as a density defined linear operator on $L^2(G/H)$, D is *essentially self-adjoint*. The proof of this fact in [11, Lemma 9] is incomplete. E.P. van den Ban [13] has recently shown the following non-trivial fact: any $D \in \mathbb{D}(G/H)$ maps $L^2(X)_\infty$ into itself and

$$\int_{G/H} \overline{D\phi(x)}\psi(x) \, dx = \int_{G/H} \overline{\phi(x)'}D\psi(x) \, dx$$

for all $\phi, \psi \in L^2(X)_\infty$. Now the reasoning of the proof of [11, Lemma 9] goes through, observing that $\phi_0 * \psi \in L^2(X)_\infty$ for any $\phi_0 \in D(G)$ and $\psi \in L^2(X)$. Let \mathcal{A} be the closed $*$ -algebra (C^* -algebra) generated by the spectral projections of the closures of all self-adjoint $D \in \mathbb{D}(G/H)$. By a result of Nelson [10, p. 603] any two of such closures strongly commute, so this algebra \mathcal{A} is abelian. As mentioned before, the main part of the proof of Proposition 1.3 is obtained by diagonalising a maximal commutative C^* -algebra commuting with the action of G in \mathcal{H} , \mathcal{H} being the Hilbert subspace associated with T .

So, in our situation, with $\mathcal{H} = L^2(G/H)$ we only have to extend \mathcal{A} to a maximal commutative C^* -algebra. The result is a desintegration of $L^2(G/H)$ into irreducible Hilbert subspaces, even a formula of the form

$$\phi(eH) = \int_S \langle T_s, \phi \rangle \, dm(s) \quad (\phi \in D(G/H))$$

such that T_s is a *common eigendistribution* for all $D \in \mathbb{D}(G/H)$, for m -almost all $s \in S$. Here we regard T_s as an element of $D'(G/H)$. For details of the (abstract) theory, we refer to [8], [12].

PROPOSITION 1.4: *Let (G, H) be a semisimple symmetric pair. There exists a (non-necessarily unique) Radon measure m on S such that*

$$(i) \quad \phi(eH) = \int_S \langle T_s, \phi \rangle \, dm(s) \quad (\phi \in D(G/H))$$

- (ii) for m -almost all $s \in S$, T_s is a common eigendistribution for all $D \in \mathbb{D}(G/H)$.

Would Proposition 1.4 answer a problem raised by Faraut [5, p. 371]?

2. $(\mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n-1, \mathbb{R}))$ is a generalized Gelfand pair for $n \geq 3$

We keep to the notation of Section 1. Generalizing the classical notion of a Gelfand pair, we define

DEFINITION 2.1: *The pair (G, H) is called a generalized Gelfand pair if for each irreducible unitary representation π on a Hilbert space \mathcal{H} , one has $\dim \mathcal{H}_{-\infty}^H \leq 1$.*

The following result is proved in [12].

PROPOSITION 2.2. *The following statements are equivalent:*

- (i) (G, H) is a generalized Gelfand pair
- (ii) For any unitary representation π which can be realized on a Hilbert subspace of $D'(G/H)$, the commutant of $\pi(G) \subset \mathcal{L}(\mathcal{H})$ is abelian. [$\mathcal{L}(\mathcal{H})$: the algebra of the continuous linear operators of \mathcal{H} into itself]
- (iii) For every $T \in \Gamma_G$ there exists a **UNIQUE** Radon measure m on S such that

$$\langle T, \phi_0 \rangle = \int_S \langle T_s, \phi_0 \rangle \, dm(s)$$

for all $\phi_0 \in D(G)$.

For a more detailed discussion of generalized Gelfand pairs, including examples, we refer to [14]. Most examples are connected with symmetric spaces.

Let G be a connected semisimple Lie group with finite center, σ an involutive automorphism of G and H an open subgroup of the group of fixed points of σ . Then it was recently shown by E.P. van den Ban [13] that for every irreducible unitary representation π of G on \mathcal{H} , $\dim \mathcal{H}_{-\infty}^H < \infty$. He actually shows the following. Choose a Cartan involution θ of G commuting with σ and let K be the group of fixed points of θ . Given a finite-dimensional irreducible representation δ of K and an infinitesimal character χ , we write $A(G/H; \chi)$ for the space of right H -invariant real analytic functions $\phi: G \rightarrow \mathbb{C}$ satisfying $z \cdot \phi = \chi(z)\phi$ for all $z \in Z(\mathfrak{g})$ [center of the universal enveloping algebra of the Lie algebra \mathfrak{g} of G], and $A_\delta(G/H; \chi)$ for the subspace of K -finite elements of type δ . Then $\dim_{\mathbb{C}} A_\delta(G/H; \chi) \leq |W(\phi)| \dim(\delta)^2$, where $W(\Phi)$ is the Weyl group of

the complexification \mathfrak{g}_c of \mathfrak{g} with respect to a Cartan subalgebra. This result clearly implies that $\mathcal{H}_{-\infty}^H$ is of finite dimension. Indeed, let v be a non-zero K -finite element in \mathcal{H} of type δ and let χ be the infinitesimal character of π . For any subset (ξ'_π) of linearly independent elements in $\mathcal{H}_{-\infty}^H$, the set of functions $\phi_t(g) = \langle \xi'_t, \pi(g^{-1})v \rangle$ ($g \in G$) is also linearly independent. Clearly $\phi_t \in A_\delta(G/H; \chi)$. The proof of van den Ban's result is completely in the spirit of Harish-Chandra's work. We now come to the pair $(\mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n-1, \mathbb{R}))$.

THEOREM 2.3: *The semisimple symmetric pair $(\mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n-1, \mathbb{R}))$ is a generalized Gelfand pair for $n \geq 3$.*

To prove this theorem, we apply a very useful criterion, due to Thomas (see [12, Theorem E]).

PROPOSITION 2.4: *Let $J: D'(G/H) \rightarrow D'(G/H)$ be an anti-automorphism. If $J\mathcal{H} = \mathcal{H}$ (i.e. $J|_{\mathcal{H}}$ anti-unitary) for all G -invariant or minimal G -invariant Hilbert subspaces of $D'(G/H)$ then (G, H) is a generalized Gelfand pair.*

The proof is rather easy and consists of showing (ii) of Proposition 2.2.

In our situation we take $JT = \bar{T}$. To show that J satisfies the conditions of Proposition 2.4, it suffices to show the following: any positive-definite bi- H -invariant distribution T on G satisfies $\check{T} = T$. Here $\langle T, \phi_0 \rangle = \langle T, \check{\phi}_0 \rangle$, $\check{\phi}_0(g) = \phi_0(g^{-1})$ ($g \in G, \phi_0 \in D(G)$). By desintegration (Proposition 1.3) one sees that T may even be assumed to be spherical. We shall use the notation of [7] from now on. There is a right H -invariant function Q on $G = \mathrm{SL}(n, \mathbb{R})$, defined by

$$Q(g) = [gx^0, x^0] = \text{trace } gx^0g^{-1}x^0$$

where

$$x^0 = \begin{pmatrix} 1 & & & \\ & 0 & & \emptyset \\ & & \ddots & \\ & \emptyset & & 0 \end{pmatrix}$$

with the following property. Put $X = G/H$. For $\phi \in D(X)$ define $M\phi$ on \mathbb{R} by

$$\int_X F(Q(x))\phi(x) dx = \int_{-\infty}^{\infty} F(t)M\phi(t) dt$$

for all $F \in C_c(\mathbb{R})$ (see [7, Section 7]). Call $\mathcal{H} = M(D(X))$ and put the

usual topology on \mathcal{X} . (See also Section 4). Then by [7, section 8] any spherical T is of the form $T = M'S$ for some $S \in \mathcal{X}'$. So $\langle T, \phi \rangle = \langle S, M\phi \rangle$ for all $\phi \in D(X)$. Here T has to be regarded as an H -invariant distribution on X . More precisely, putting

$$f^\#(x) = \int_H f(gh) \, dh \quad (x = gH, f \in D(G))$$

we have the equality $\langle T, f \rangle = \langle T, f^\# \rangle$, where on the right-hand-side T has to be regarded as an H -invariant distribution on X . The problem to be solved amounts to the relation $M(\check{f})^\# = Mf^\#$ for all $f \in D(G)$. For all $F \in C_c(\mathbb{R})$ one has

$$\begin{aligned} \int_{-\infty}^{\infty} F(t) M(\check{f})^\#(t) \, dt &= \int_X F(Q(x)) (\check{f})^\#(x) \, dx \\ &= \int_G F(Q(g)) \check{f}(g) \, dg \\ &= \int_G F(Q(g^{-1})) f(g) \, dg. \end{aligned}$$

Since $Q(g) = Q(g^{-1})$ we get the result and the proof of Theorem 2.3 is complete.

REMARK: $(\mathrm{SL}(2, \mathbb{R}), \mathrm{GL}(1, \mathbb{R}))$ is not a generalized Gelfand pair.

3. Invariant Hilbert subspaces of $L^2(G/H)$

We keep to the notations of Section 1. Let π be a unitary representation of G on a Hilbert space \mathcal{H} , which can be realized on a Hilbert subspace of $D'(G/H)$. Let $j: \mathcal{H} \rightarrow D'(G/H)$ be the corresponding injection. Define ξ_π, T, j^* as usual.

PROPOSITION 3.1: *The following conditions on π are equivalent:*

- (i) $j(\mathcal{H}) \subset L^2(G/H)$
- (ii) *There exists a constant $c > 0$ such that for all $\phi_0 \in D(G)$,*
 $|\langle T, \tilde{\phi}_0 * \phi \rangle| \leq c \|\phi\|_2^2$.

PROOF: (i) \Rightarrow (ii): The map $j: \mathcal{H} \rightarrow L^2(G/H)$ is closed and everywhere defined on \mathcal{H} , hence continuous by the closed graph theorem. This implies that $j^*: D(G/H) \rightarrow \mathcal{H}$ is continuous in the L^2 -topology of $D(G/H)$, so (ii) follows.

(ii) \Rightarrow (i): Clearly (ii) implies that j^* is continuous with respect to the L^2 -topology. Extend j^* to $L^2(G/H)$. Then clearly $j(\mathcal{H}) \subset L^2(G/H)$.

We shall say that π belongs to the *relative discrete series* of G (with respect to H) if π is irreducible and satisfies one of the conditions of Proposition 3.1. We shall occasionally use the terminology: π is *square-integrable mod H* .

PROPOSITION 3.2: *Let π be an irreducible unitary representation of G on \mathcal{H} , which can be realized on a Hilbert subspace of $D'(G/H)$. Let $j: \mathcal{H} \rightarrow D'(G/H)$ be the corresponding injection. The following statements are equivalent:*

- (i) π is square-integrable mod H
- (ii) $j(\mathcal{H})$ is a closed linear subspace of $L^2(G/H)$
- (iii) $j(v) \in L^2(G/H)$ for a non-zero element $v \in \mathcal{H}$.

PROOF: It suffices to prove the implication (iii) \rightarrow (ii). Let $P = \{w \in \mathcal{H}: j(w) \in L^2(G/H)\}$. Clearly P is a G -stable and non-zero linear subspace of \mathcal{H} , hence dense in \mathcal{H} . Now observe that $j: P \rightarrow L^2(G/H)$ is a closed linear operator: if $w_k \rightarrow w (w_k \in P, w \in \mathcal{H})$ and $jw_k \rightarrow f$ in $L^2(G/H)$, then obviously $j(w) \in D'(G/H)$ is equal to f as a distribution. Polar decomposition of j and applying Schur's Lemma yields: j can be extended to a continuous linear operator $\mathcal{H} \rightarrow L^2(G/H)$ with closed image (cf. [1, p. 48]).

REMARK: It also follows (see [1, p. 48]) that there is a constant $c > 0$ such that $\|jv\|_2 = c\|v\|$ for all $v \in \mathcal{H}$.

One has the following orthogonality relations.

PROPOSITION 3.3: *Let π, π' be irreducible unitary representations on $\mathcal{H}, \mathcal{H}'$, both belonging to the relative discrete series. Define T, T' and $\xi_\pi, \xi_{\pi'}$ as usual. Then one has:*

- (i) $\int_{G/H} \langle \pi(x^{-1})v, \xi_\pi \rangle \overline{\langle \pi'(x^{-1})v', \xi_{\pi'} \rangle} dx = 0$ for all $v \in \mathcal{H}_\infty, v' \in \mathcal{H}'_\infty$ if π is not equivalent to π' .
- (ii) There exists a constant $d_\pi > 0$, only depending on T , such that

$$\int_{G/H} \langle \pi(x^{-1})v, \xi_\pi \rangle \overline{\langle \pi(x^{-1})v', \xi_\pi \rangle} dx = d_\pi^{-1} \langle v, v' \rangle$$

for all $v, v' \in \mathcal{H}_\infty$.

To prove this proposition, one follows the well-known receipt to introduce the invariant hermitian form

$$(v, v') = \int_{G/H} \langle \pi(x^{-1})v, \xi_\pi \rangle \overline{\langle \pi'(x^{-1})v', \xi_{\pi'} \rangle} dx$$

on $\mathcal{H}_\infty \times \mathcal{H}'_\infty$. This form is continuous with respect to the topology on $\mathcal{H} \times \mathcal{H}'$. Schur's lemma now easily implies the result. The "only" dependency on T follows from the formula: $\|jj^*\phi_0\|_2^2 = d_\pi^{-1} \|j^*\phi_0\|^2$, so $\|\phi_0 * T\|_2^2 = d_\pi^{-1} \langle T, \tilde{\phi}_0 * \phi \rangle$ for all $\phi_0 \in D(G)$.

REMARK: Observe that $\|jv\|_2 = d_\pi^{-1/2} \|v\|$ for all $v \in \mathcal{H}_\infty$. So c , introduced before, is equal to $d_\pi^{-1/2}$.

The constant d_π is called the *formal degree* of π . It depends on the choice of j (or T). Once a canonical choice j (or T) is possible, d_π has a more realistic meaning.

EXAMPLE: Let G_1 be as usual. Let $G = G_1 \times G_1$ and $H = \text{diag}(G)$. Let π be an irreducible unitary representation of G . π can be realized on a Hilbert subspace of $D'(G/H) \simeq D'(G_1)$ if π is of the form $\bar{\pi}_1 \otimes \pi_1$, where π_1 is an irreducible unitary representation of G_1 on \mathcal{H}_1 whose (distribution-) character θ_1 exists [12]. Actually, the reproducing distribution T associated with π , can be taken equal to θ_1 . This is a canonical choice. The injection $j: \mathcal{H}_1 \hat{\otimes}_2 \mathcal{H}_1 \rightarrow D'(G_1)$ has the form

$$j(v \otimes w)(x) = \langle \pi(x^{-1})v, w \rangle \quad (x \in G_1, v, w \in \mathcal{H}_1).$$

In this case Propositions 3.2 and 3.3 yield the well-known properties of square-integrable representations of G_1 (cf. [1, 5.13–5.15]). Note that any square-integrable representation π_1 of G_1 has a distribution-character.

Let us assume that (G, H) is a generalized Gelfand pair. Denote by $E_2(G/H)$ the set of equivalence-classes of irreducible square-integrable representations mod H . Fix a representative π in each class, together with the realisation j_π on a Hilbert subspace of $L^2(G/H)$ and call this set of representatives S . Denote by T_π the reproducing distribution and by d_π the formal degree of π . Let \mathcal{H}_π be the space of π . Define $\mathcal{H}_d = \oplus_{j_\pi}(\mathcal{H}_\pi)$ and let E be the orthogonal projection of $L^2(G/H)$ onto \mathcal{H}_d . Then one has the following (partial) Plancherel formula for the relative discrete series.

PROPOSITION 3.4: For all $\phi_0 \in D(G)$,

$$\|E\phi\|_2^2 = \sum_{\pi \in S} d_\pi \langle T_\pi, \tilde{\phi}_0 * \phi \rangle.$$

Notice that $E\phi \in C^\infty(G/H)$ for all $\phi_0 \in D(G/H)$. So the formula in Proposition 3.4 is equivalent to

$$(E\phi)(eH) = \sum_{\pi \in S} d_\pi \langle T_\pi, \phi_0 \rangle \quad (\phi_0 \in D(G)).$$

The above formulae do not depend on the choice of the set S : $d_\pi T_\pi$ is independent of the choice of π in its equivalence class and the choice of j_π . In fact, $d_\pi \langle T_\pi, \tilde{\phi}_0 * \phi \rangle = \|E_\pi \phi\|_2^2$, where E_π is the orthogonal projection of $L^2(G/H)$ onto $j_\pi(\mathcal{H}_\pi)$. Indeed, choose an orthogonal basis (e_i) in \mathcal{H}_π . Then $d_\pi^{1/2} j(e_i)$ is an orthogonal basis for $j_\pi(\mathcal{H}_\pi)$ and $\|E_\pi \phi\|_2^2 = \sum_i d_\pi |(j e_i, \phi)|^2 = \sum_i d_\pi |(e_i, j^* \phi)|^2 = d_\pi \|j^* \phi\|^2 = d_\pi \langle T_\pi, \tilde{\phi}_0 * \phi \rangle$.

REMARK: The above proposition is easily extended to the case of finite multiplicity: $m_\pi = \dim \mathcal{H}_\infty^H < \infty$ for all $\pi \in E_2(G/H)$. Indeed, we can choose for each π , $T_\pi^1, \dots, T_\pi^{m(\pi)}$ such that the corresponding Hilbert subspaces are orthogonal (regarded as subspaces of $L^2(G/H)$) and the G -action is equivalent to π . Then the above formula reads;

$$\|E\phi\|_2^2 = \sum_{\pi \in S} \sum_{i=1}^{m(\pi)} d'_i \langle T'_i, \tilde{\phi}_0 * \phi \rangle \quad (\phi_0 \in D(G)).$$

Again $\sum_{i=1}^{m(\pi)} d'_i \langle T'_i, \tilde{\phi}_0 * \phi \rangle = \|E_\pi \phi\|_2^2$, where E_π is the orthogonal projection of $L^2(G/H)$ onto $\bigoplus_{i=1}^{m(\pi)} j'_i(\mathcal{H}'_i) = \text{Cl}(\sum_{\pi' \sim \pi} j_{\pi'}(\mathcal{H}_{\pi'}))$. If (G, H) is a semisimple symmetric pair, then E.P. van den Ban [13] has recently shown that the T'_i can be chosen in such a way that they are common eigendistributions of $\mathbb{D}(G/H)$, the algebra of G -invariant differential operators on G/H . However, different eigenvalues may occur.

4. The relative discrete series of $\text{SL}(n, \mathbb{R})/\text{GL}(n-1, \mathbb{R})$

We recall briefly some facts from [7].

Let $G = \text{SL}(n, \mathbb{R})$, $H = \text{S}(\text{GL}(1, \mathbb{R}) \times \text{GL}(n-1, \mathbb{R}))$, $n \geq 3$. (G, H) is a semisimple symmetric pair. Write $X = G/H$. Let x^0 be the $n \times n$ matrix given by $\begin{pmatrix} 1 & & \\ & \emptyset & \\ & & \emptyset \end{pmatrix}$. G acts on the space of real $n \times n$ matrices $M_n(\mathbb{R})$ by $g \cdot x = gxg^{-1}$ ($g \in G, x \in M_n(\mathbb{R})$). X is naturally isomorphic to $G \cdot x^0 = \{x \in M_n(\mathbb{R}) : \text{rank } x = \text{trace } x = 1\}$. We defined a function $Q: X \rightarrow \mathbb{R}$ by $Q(x) = [x, x^0]$, where $[x, y] = \text{trace } xy$ ($x, y \in M_n(\mathbb{R})$). Q has the following properties:

- a. Q is H -invariant.
- b. x^0 is a non-degenerate critical point for Q . The Hessian of Q in this point has signature $(n-1, n-1)$.
- c. Besides x^0 , the set $\mathcal{S} \subset X$, consisting of elements of the form

$$x = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & T & \\ 0 & & \end{pmatrix}$$

with $T \in M_{n-1}(\mathbb{R})$, $\text{rank } T = \text{trace } T = 1$, is a critical set of Q . For each $x \in \mathcal{S}$ one can choose coordinates x_1, \dots, x_{2n-2} near x such that $Q = x_1 x_2$ and \mathcal{S} is given by $x_1 = x_2 = 0$.

- d. If $x \neq x^0$ and $x \notin \mathcal{S}$ then x is not a critical point of Q .
- e. Q is real analytic.
- f. Q assumes all real values.
- g. For $\lambda \neq 0, 1$, $Q(x) = \lambda$ is an H -orbit.
- h. $Q(x) = 1$ consists of 4 H -orbits.
- i. $Q(x) = 0$ consists of 3 H -orbits.

Define for $f \in D(X)$ the function Mf on \mathbb{R} by the property

$$\int_X F(Q(x))f(x) \, dx = \int_{-\infty}^{\infty} Mf(t)F(t) \, dt$$

for all $F \in C_c^\infty(\mathbb{R})$. Put $\mathcal{X} = M(D(X))$. \mathcal{X} consists of all functions of the form

$$\phi(t) = \phi_0(t) + \phi_1(t) \log |t| + \phi_2(t) \eta(t)$$

where

$$\phi_t \in D(\mathbb{R}) \quad \text{and} \quad \eta(t) = \begin{cases} Y(t-1)(t-1)^{n-2} & \text{if } n \text{ is odd,} \\ (t-1)^{n-2} \log |t-1| & \text{if } n \text{ is even,} \end{cases}$$

Y being the Heaviside function: $Y(t) = 1$ if $t \geq 0$, $Y(t) = 0$ if $t < 0$.

Let \square_0 be the Laplace-Beltrami operator on X and put $\square = 2\square_0$. One can topologize \mathcal{X} in such a way that

- a. $M: D(X) \rightarrow \mathcal{X}$ is continuous.
- b. Any H -invariant eigendistribution T of \square is of the form $T = M'S$ for some $S \in \mathcal{X}'$.
- c. $\square \cdot M' = M' \cdot L$, where L is the second order differential operator on \mathbb{R} , given by $L = a(t)d^2/dt^2 + b(t)d/dt$, with $a(t) = 4t(t-1)$, $b(t) = 4(nt-1)$.

Denote $D'_{\lambda,H}(X)$ the space of H -invariant eigendistributions T of \square on X with eigenvalue λ .

PROPOSITION 4.1: $\dim D'_{\lambda,H}(X) = 2$ for all $\lambda \in \mathbb{C}$.

This is shown in [7, Proposition 7.10]. Let $T \in D'_{\lambda,H}(X)$. Then there is a continuous linear form S on $\mathcal{X} = M(D(X))$ satisfying $LS = \lambda S$ such that

$$T(f) = S(Mf) \quad (f \in D(X)).$$

Since $x \rightarrow Q(x)$ is submersive on $X - \{x^0\} - \mathcal{S}$, S is actually a distribution on $\mathbb{R} \setminus \{0, 1\}$. Since L is elliptic there, we see that S is an analytic function $\mathbb{R} \setminus \{0, 1\}$. By abuse of notation we shall also call it S . Now notice that $Lu = \lambda u$ is a hypergeometric differential equation. Let $\lambda = s^2 - \rho^2$ ($s \in \mathbb{C}$) with $\rho = n - 1$. In [7, section 8] we have given a basis $M'S_0$ and $M'S_2$ if n is odd, $M'S_0$ and $M'S_1$ if n is even, of $D'_{\lambda, H}(X)$, for all s satisfying $\text{Im } s \neq 0$. It is not difficult to extend the results in a natural way to all s with $\text{Re}(s) \geq 0$. If $s \in \mathbb{N}$, analytic continuation does not work, one has to construct the distributions S_0, S_1 and S_2 by the method of [7, Appendix 2 and Section 8]. Instead of giving full details we shall describe the asymptotics of S_0, S_1 and S_2 as $t \rightarrow \pm \infty$.

I. $s \notin \mathbb{Z}, \text{Re}(s) > 0$. ([7, Lemma 8.1])

$$S_i(t) \sim d_{i,+}(s)t^{\frac{1}{2}(s-\rho)} \quad (t \rightarrow \infty)$$

$$S_i(t) \sim d_{i,-}(s)(-t)^{\frac{1}{2}(s-\rho)} \quad (t \rightarrow -\infty) \quad (i = 0, 1, 2)$$

where

$$d_{0,+}(s) = \frac{\Gamma(\rho)\Gamma(s)}{\Gamma(\frac{1}{2}(s+\rho))^2},$$

$$d_{1,+}(s) = (-1)^{\rho-1} \frac{\Gamma(s)\Gamma(\frac{1}{2}(s-\rho))}{\Gamma(\rho-1)\Gamma(\frac{1}{2}(s-\rho+2))} \cdot \cos \pi(\frac{1}{2}(s-\rho)),$$

$$d_{2,+}(s) = 0$$

$$d_{0,-}(s) = \frac{\Gamma(\rho)\Gamma(s)}{\Gamma(\frac{1}{2}(s+\rho))^2} \cos \pi(\frac{1}{2}(s-\rho)),$$

$$d_{1,-}(s) = (-1)^{\rho-1} \frac{\Gamma(s)\Gamma(\frac{1}{2}(\rho-s))}{\Gamma(\rho-1)\Gamma(\frac{1}{2}(s-\rho+2))},$$

$$d_{2,-}(s) =$$

$$- \pi \frac{\Gamma(\frac{1}{2}(\rho-s))\Gamma(s) \cos \pi(\frac{1}{2}(s-\rho))}{\Gamma(\frac{1}{2}(\rho+s))\Gamma(\rho-1)\Gamma(\frac{1}{2}(s-\rho+2))\Gamma(\frac{1}{2}(-s-\rho+2))}.$$

II. $\text{Re}(s) = 0, s \neq 0$

$$S_i(t) \sim d_{i,+}(s)t^{\frac{1}{2}(s-\rho)} + d_{i,+}(-s)t^{\frac{1}{2}(-s-\rho)} \quad (t \rightarrow \infty)$$

$$S_i(t) \sim d_{i,-}(s)(-t)^{\frac{1}{2}(s-\rho)} + d_{i,-}(-s)(-t)^{\frac{1}{2}(-s-\rho)} \quad (t \rightarrow -\infty)$$

($i = 0, 1, 2$), where $d_{i,\pm}(s)$ are as before.

Here we apply [3, formula (36) on page 107].
We now split up the cases n odd and n even.

III. (odd) $s = \rho + 2k + 1$, $s > 0$ ($k \in \mathbb{Z}$)

Put $v = \rho + k + \frac{1}{2}$ then for $t \rightarrow \infty$

$$S_0(t) \sim \frac{\Gamma(\rho)\Gamma(2k+1)}{\Gamma(v)^2} t^{k+\frac{1}{2}}$$

$$S_0(-t) \sim \frac{\Gamma(\rho)\Gamma(k+\frac{3}{2}) \sin v\pi}{\Gamma(\rho+2k)\Gamma(-k-\frac{1}{2})} t^{-v}$$

The latter formula follows from [3, p. 109, formula (7)].

$$S_2(t) = 0$$

$$S_2(-t) \sim \frac{-2\pi^2}{\Gamma(\rho-1)\Gamma(1-v)\Gamma(\rho+2k)} t^{-v}$$

IV (odd) s even, $s = \rho + 2l$.

a. $l \geq 0$

S_0 is a polynomial of degree l with leading coefficient

$$\frac{\Gamma(\rho+2l)\Gamma(\rho)}{\Gamma(l+\rho)^2}$$

$$S_2(t) = 0 \quad (t \rightarrow \infty)$$

$$S_2(t) = \pi\alpha S_0(t) \quad (t \rightarrow -\infty),$$

where

$$\alpha = -\frac{\Gamma(\rho+l)^2}{\Gamma(l+1)^2\Gamma(\rho-1)\Gamma(\rho)}$$

b. $s = \rho + 2l$, $0 < s < \rho$ (so $l < 0$)

$$S_0(t) \sim \frac{\Gamma(\rho+2l)\Gamma(\rho)}{\Gamma(l+\rho)^2} t^l \quad (|t| \rightarrow \infty)$$

$$S_2(t) = 0 \quad (|t| \rightarrow \infty)$$

c. $s = \rho + 2l = 0$.

By [3, p. 110, formula (11)] we get

$$S_0(t) \sim \frac{(-1)^{t+1} \Gamma(\rho)}{\Gamma(\frac{1}{2}\rho)^2} t^{-\frac{1}{2}\rho} \log |t| + O(|t|^{-\frac{1}{2}\rho}) \quad (|t| \rightarrow \infty)$$

$$S_2(t) = 0 \quad (|t| \rightarrow \infty).$$

III (even) a. $s = \rho + 2k + 1, s > 0 (k \in \mathbb{Z})$.

Put $v = \rho + k + \frac{1}{2}$ then for $t \rightarrow \infty$

$$s_0(t) \sim \frac{\Gamma(\rho)\Gamma(\rho + 2k + 1)}{\Gamma(v)^2} t^{k+\frac{1}{2}}$$

$$S_0(-t) \sim (-1)^{k+1} 2 \frac{\Gamma(\rho)\Gamma(k + \frac{3}{2}) \sin v\pi}{\Gamma(\rho + 2k)\Gamma(-k - \frac{1}{2})} t^{-v}$$

$$S_1(t) \sim \frac{(-1)^{k+1} \pi \Gamma(v)^2}{\Gamma(\rho - 1)\Gamma(-k - \frac{1}{2})\Gamma(v + k + \frac{1}{2})\Gamma(k + \frac{3}{2})} t^{-v}$$

(apply [3, p. 109, formula (7)])

$$S_1(-t) \sim \frac{\Gamma(\rho + 2k + 1)\Gamma(-k - \frac{1}{2})}{\Gamma(\rho - 1)\Gamma(k + \frac{3}{2})} t^{k+\frac{1}{2}}$$

b. $s = 0$

$$S_0(t) \sim \frac{\Gamma(\rho)}{\Gamma(\frac{1}{2}\rho)^2} t^{-\frac{1}{2}\rho} \log |t| + O(t^{-\frac{1}{2}\rho}) \quad (t \rightarrow \infty)$$

$$S_0(t) \sim 2\pi \sin(\pi \frac{1}{2}\rho) \frac{\Gamma(\rho)}{\Gamma(\frac{1}{2}\rho)^2} (-t)^{-\frac{1}{2}\rho} \quad (t \rightarrow -\infty)$$

$$S_1(t) \sim \frac{2\pi \sin \pi \frac{1}{2}\rho}{\Gamma(1 - \frac{1}{2}\rho)\Gamma(\rho - 1)} \cdot \Gamma(\frac{1}{2}\rho) t^{-\frac{1}{2}\rho} \quad (t \rightarrow \infty)$$

$$S_1(t) \sim \frac{\Gamma(\frac{1}{2}\rho)}{\Gamma(1 - \frac{1}{2}\rho)\Gamma(\rho - 1)} (-t)^{-\frac{1}{2}\rho} \log |t| + O(|t|^{-\frac{1}{2}\rho})$$

$$(t \rightarrow -\infty)$$

(apply [3, p. 109, formula (7)]).

IV (even) $s = \rho + 2l$.

a. $l \geq 0$.

S_0 is a polynomial of degree l with leading coefficient

$$\frac{\Gamma(\rho + 2l)\Gamma(\rho)}{\Gamma(l + \rho)^2}$$

$$S_1(t) \sim \frac{\Gamma(l + \rho)^2}{\Gamma(\rho - 1)\Gamma(2l + \rho + 1)\Gamma(l + 1)} |t|^{-l-\rho} \quad (|t| \rightarrow \infty).$$

Here we apply [4, p. 170, formula (18)].

b. $s = \rho + 2l, 0 < s < \rho$ (so $l < 0$).

$$S_0(t) \sim \frac{\Gamma(\rho + 2l)\Gamma(\rho)}{\Gamma(l + \rho)^2} t^l \quad (|t| \rightarrow \infty).$$

By [3, p. 110, formula (14)] we get

$$S_1(t) \sim \frac{\Gamma(\rho + l)^2}{\Gamma(\rho + 2l + 1)\Gamma(\rho - 1)} t^{-\rho-l} \quad (|t| \rightarrow \infty).$$

Let $K = \text{SO}(n, \mathbb{R})$ and

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & \sinh t & & & & \\ & \sinh t & \cosh t & & & \\ & & & 1 & & \\ & & \emptyset & & \ddots & \\ & & & & & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

We quote the following lemma [7, Lemma 6.1].

LEMMA 4.2: *Every element $x \in X$ can be written as $x = ka_t x^0$ with $t \geq 0$ and $k \in K$. Then t is uniquely determined, and if $t > 0$ then k is determined uniquely modulo $M \cap K$, M being the centralizer of A in H . We can normalize the invariant measure dx on X in such a way that*

$$\int_X \phi(x) dx = \int_K \int_0^\infty \phi(ka_t x^0) A(t) dk dt$$

for all $\phi \in D(X)$. Here $A(t) = \sinh^{n-2}(2t) \cosh(2t)$.

Let π be a non-necessarily irreducible unitary representation of G which can be realized on a Hilbert subspace of $L^2(X)$. Let T be the reproducing distribution on X and ξ_π the cyclic distribution vector associated with π . So

$$\langle T, \phi \rangle = \langle \xi_\pi, \pi_{-\infty}(\phi_0)\xi_\pi \rangle \quad (\phi_0 \in D(G)).$$

For $g \in G, \phi \in D(X)$ define

$$\phi_g(x) = \phi(g \cdot x) \quad (x \in X).$$

Then $\langle T, \phi_g \rangle = \langle \xi_\pi, \pi(g^{-1})\pi_{-\infty}(\phi_0)\xi_\pi \rangle$, and $g \rightarrow \langle T, \phi_g \rangle$ actually is a C^∞ -function on X , which belongs to $L^2(X)$. Now apply Lemma 4.2 to conclude: for almost all $k \in K$ the function

$$\tau \rightarrow |\langle T, (\phi_k)_{a_\tau} \rangle|^2 A(\tau)$$

is in $L^1(0, \infty)$.

Since $A(\tau) \sim 2^{-\rho} e^{2\rho\tau}$ ($\tau \rightarrow \infty$) we get

LEMMA 4.3: *The function*

$$\tau \rightarrow e^{\rho\tau} \langle T, (\phi_k)_{a_\tau} \rangle$$

is in $L^2(0, \infty)$ for almost all $k \in K$.

We now assume, in addition, that $T \in D'_{\lambda, H}(X)$ for certain $\lambda \in \mathbb{C}$. (This is clearly so if π is irreducible).

Then we can write

$$T = \alpha M'S_0 + \beta M'S$$

for some $\alpha, \beta \in \mathbb{C}$, where $S = S_2$ if n is odd, $S = S_1$ if n is even. For any $\psi \in D(X)$ with $\text{Supp } \psi \subset \{x: [x, x^0] > 1\}$ one has

$$\langle M'S_0, \psi \rangle = \int_0^\infty S_0(t) M\psi(t) dt = \int_X S_0([x, x^0]) \psi(x) dx.$$

Put, as usual, $\xi^0 = \begin{pmatrix} 1 & & & -1 \\ & & & \emptyset \\ & & -1 & \\ & & & \emptyset \end{pmatrix} \in M_n(\mathbb{R})$ and $P_0(x) = [x, \xi^0]$ ($x \in X$). Because of $P_0(x) = 4 \lim_{\tau \rightarrow \infty} e^{-2\tau} [x, a_\tau x^0]$ for all $x \in X$, we have $\text{Supp } \psi_{a_\tau} \subset \{x \in X: [x, x^0] > 1\}$ for $\tau \rightarrow \infty$, provided $\text{Supp } \psi \subset$

$\{x: P_0(x) > 0\}$. Hence $\langle M'S_0, \psi_{a_\tau} \rangle = \int_X S([x, a_\tau x^0]) \psi(x) dx \ (\tau \rightarrow \infty)$.

Put $\lambda = s^2 - \rho^2, \operatorname{Re}(s) \geq 0$.

Applying the results on the asymptotic behaviour of S_0 , derived before, we obtain

(i) $\operatorname{Re}(s) > 0$

$$\langle M'S_0, \psi_{a_\tau} \rangle \sim \text{const.} \int_X P_0(x)^{\frac{1}{2}(s-\rho)} \psi(x) dx \cdot e^{(s-\rho)\tau}$$

(ii) $\operatorname{Re}(s) = 0, s \neq 0$

$$\begin{aligned} \langle M'S_0, \psi_{a_\tau} \rangle &\sim c_1 \cdot \int_X P_0(x)^{\frac{1}{2}(s-\rho)} \psi(x) dx \cdot e^{(s-\rho)\tau} \\ &\quad + c_2 \cdot \int_X P_0(x)^{\frac{1}{2}(-s-\rho)} \psi(x) dx \cdot e^{(-s-\rho)\tau} \end{aligned}$$

(iii) $s = 0$

$$\langle M'S_0, \psi_{a_\tau} \rangle \sim \text{const.} \int_X P_0(x)^{-\frac{1}{2}\rho} \psi(x) dx \cdot \tau e^{-\rho\tau}$$

where the constants are non-zero and $\tau \rightarrow \infty$.

We can clearly find a $\psi \in D(X)$ as above of the form ϕ_k , so that $\tau \rightarrow e^{\rho\tau} \langle T, (\phi_k)_{a_\tau} \rangle$ is in $L^2(0, \infty)$ (Lemma 4.3)

If n is odd, it follows now easily from (i), (ii) and (iii) that $\alpha = 0$, so $T = \beta M'S_2$.

If n is even then $\alpha = 0$ if $s = \rho + 2l$ ($l \in \mathbb{Z}, s > 0$) or $s = \rho + 2l + 1$ ($l \in \mathbb{Z}, s \geq 0$). In the other cases we get a linear relation between α and β .

A similar analysis for $\psi \in D(X)$ with $\operatorname{Supp} \psi \subset \{x: P_0(x) < 0\}$, using the asymptotic behaviour of S_0, S_1, S_2 for $\tau \rightarrow -\infty$, yields extra conditions on s .

The results are as follows.

THEOREM 4.4: *Let π be a unitary representation of G , realized on a Hilbert subspace of $L^2(X)$. Let T be the reproducing distribution of π and assume $T \in D'_{\lambda, H}(X)$. Then T is uniquely determined up to scalar multiplication. More precisely, if $\lambda = s^2 - \rho^2$ with $\operatorname{Re}(s) \geq 0$, then we have:*

n odd:

s is in the set

$$\left\{ \begin{array}{l} s \text{ odd, } \quad s = \rho + 2k + 1, \quad s > 0 \\ s \text{ even, } \quad s = \rho + 2k, \quad 0 \leq s < \rho \end{array} \right\}.$$

Moreover T is a scalar multiple of $M'S_2$.

n even:

s is in the set

$$\{s \text{ odd}, s = \rho + 2l, s > 0\}$$

Moreover T is a scalar multiple of $M'S_1$.

It will turn out that the s , specified above, give indeed rise to a parametrization of the relative discrete series representations, provided we take s odd. So in the case n odd, the even s , $0 \leq s < \rho$, do not contribute to the discrete spectrum. The proof of this fact is obtained by performing the spectral resolution of the Laplace-Beltrami operator \square . This is the contents of section 5.

5. The Plancherel formula for $SL(n, \mathbb{R})/GL(n-1, \mathbb{R})$

The Plancherel formula is obtained by determining the spectral resolution of the self-adjoint extension $\tilde{\square}$ of the Laplace-Beltrami operator \square on X . Our method is similar to Faraut's [5, Première démonstration de la formule de Plancherel].

We shall only provide the calculations for n even; for n odd they are essentially the same. So from now on n is even, unless otherwise stated.

Let $\lambda = s^2 - \rho^2 (s \in \mathbb{C})$ and define functions $S_{0,s}$ and $S_{1,s}$ on $\mathbb{R} \setminus \{0, 1\}$ as follows:

$$S_{0,s}(t) = \begin{cases} {}_2F_1(\frac{1}{2}(\rho + s), \frac{1}{2}(\rho - s); \rho; 1 - t) & (t > 0) \\ \frac{1}{2} [{}_2F_1(\frac{1}{2}(\rho + s), \frac{1}{2}(\rho - s); \rho; 1 - t - i0) \\ \quad + {}_2F_1(\frac{1}{2}(\rho + s), \frac{1}{2}(\rho - s); \rho; 1 - t + i0)] & (t < 0) \end{cases}$$

$$S_{1,s}(t) = \begin{cases} {}_2F_1(\frac{1}{2}(\rho + s), \frac{1}{2}(\rho - s); 1; t) & (t < 1) \\ \frac{1}{2} [{}_2F_1(\frac{1}{2}(\rho + s), \frac{1}{2}(\rho - s); 1; t + i0) \\ \quad + {}_2F_1(\frac{1}{2}(\rho + s), \frac{1}{2}(\rho - s); 1; t - i0)] & (t > 1). \end{cases}$$

Then $S_{0,s}$ and $S_{1,s}$ correspond to a basis of $D'_{\lambda,H}(X)$, provided $\lambda \neq 4r(r + \rho)$, $r \in \mathbb{N}$.

Observe that $S_{0,s}$ and $S_{1,s}$ are even in s .

The above definition of $S_{1,s}$ differs a factor $(-1)^{\rho-1} \frac{\Gamma(\frac{1}{2}(\rho + s))\Gamma(\frac{1}{2}(\rho - s))}{\Gamma(\rho - 1)}$ from the definition in [7, Section 8], which we used in the previous section.

The new definition is more convenient to work with here.
 If $s_r = \rho + 2r$ ($r \in \mathbb{N}$), then

$$S_{0,s_r} = (-1)^r \frac{\Gamma(r+1)\Gamma(\rho)}{\Gamma(\rho+r)} S_{1,r}.$$

For these s_r , define

$$U_r = \frac{d}{ds} \left[(-1)^{r-1} \frac{\Gamma(\frac{1}{2}(s+\rho))}{\Gamma(\frac{1}{2}(s-\rho+2))\Gamma(\rho)} S_{0,s} + S_{1,s} \right]_{s=s_r}.$$

Then S_{1,s_r} and U_r correspond to a basis of $D'_{\lambda,H}(X)$ with $\lambda = 4r(r+\rho)$. We shall use this basis, which differs by constants from the one in the previous section.

Let $\zeta_{0,s}$ and $\zeta_{1,s}$ be the H -invariant eigendistributions of \square on X defined in [7, section 4].

Let S_s^0 and S_s^1 be the continuous linear forms on $\mathcal{X} = M(D(X))$ defined by $M'S_s^0 = \zeta_{0,s}$ and $M'S_s^1 = \zeta_{1,s}$.

PROPOSITION 5.1 [7, Theorem 8.5]: *If $\text{Im } s \neq 0$ then*

$$S_s^0 = A_{0,0}(s)S_{0,s} + A_{0,1}(s)S_{1,s}$$

$$S_s^1 = A_{1,0}(s)S_{0,s} + A_{1,1}(s)S_{1,s}$$

with

$$A_{0,0}(s) = \frac{2^{-\rho+1}}{\Gamma(\frac{1}{4}(s-\rho+2))^2 \Gamma(\frac{1}{4}(-s-\rho+2))^2 \cos \pi \frac{1}{2}(s)}$$

$$A_{0,1}(s) = A_{0,0}(s) \left[\Gamma(\rho) \Gamma(\frac{1}{2}(s-\rho+2)) \Gamma(\frac{1}{2}(-s-\rho+2)) \right. \\ \left. \times \sin \pi(\frac{1}{2}(s+\rho)) \right] \pi^{-1}$$

$$A_{1,0}(s) = -2^{-1-\rho} \left[\cos \pi \frac{1}{2}s \right] \left[\Gamma(\frac{1}{4}(s-\rho+4))^2 \Gamma(\frac{1}{4}(-s-\rho+4))^2 \right. \\ \left. \times \sin^2 \pi(\frac{1}{4}(s+\rho)) \sin^2 \pi(\frac{1}{4}(s-\rho)) \right]^{-1}$$

$$A_{1,1}(s) = A_{1,0}(s) \left[\Gamma(\rho) \Gamma(\frac{1}{2}(s-\rho+2)) \Gamma(\frac{1}{2}(-s-\rho+2)) \right. \\ \left. \times \sin \pi(\frac{1}{2}(s+\rho)) \right] \pi^{-1}.$$

By analytic continuation the above proposition remains true for all $s \in \mathbb{C}$, $s \neq \pm s_r$ ($s_r = \rho + 2r$, $r \in \mathbb{N}$). Taking the limit $s \rightarrow s_r$ in the above expressions yield:

LEMMA 5.2:

(1) for r even

$$S_{s_r}^0 = 0$$

$$\frac{d}{ds} S_s^0 \Big|_{s=s_r} = \frac{(-1)^{\frac{1}{2}(\rho+1)} 2^{-\rho-1} \Gamma(r+1) \Gamma(\rho) \Gamma(\frac{1}{2}(\rho+r+1))^2}{\pi \Gamma(\rho+r) \Gamma(\frac{1}{2}(r+1))^2} \times S_{1,s_r}$$

$$S_{s_r}^1 = \frac{(-1)^{\frac{1}{2}(\rho+1)} 2^{2-\rho} \Gamma(r+1) \Gamma(\rho)}{\pi \Gamma(\rho+r) \Gamma(\frac{1}{2}(r+2))^2 \Gamma(\frac{1}{2}(-\rho-r+2))^2} U_r$$

(2) for r odd

$$S_{s_r}^0 = - \frac{(-1)^{\frac{1}{2}(\rho+1)} 2^{2-\rho} \Gamma(r+1) \Gamma(\rho)}{\pi \Gamma(\rho+r) \Gamma(\frac{1}{2}(r+1))^2 \Gamma(\frac{1}{2}(-\rho-r+1))^2} U_r$$

$$S_{s_r}^1 = 0$$

$$\frac{d}{ds} S_s^1 \Big|_{s=s_r} = - \frac{(-1)^{\frac{1}{2}(\rho+1)} 2^{-\rho-1} \Gamma(r+1) \Gamma(\rho) \Gamma(\frac{1}{2}(\rho+2))^2}{\pi \Gamma(\rho+r) \Gamma(\frac{1}{2}(r+2))^2} \times S_{1,s_r}$$

Let us denote by $E(d\lambda)$ the spectral measure of $\tilde{\square}$. For every $h \in C_c(\mathbb{R})$ and $\phi, \psi \in D(X)$ one has

$$\int_{-\infty}^{\infty} h(\lambda) \langle E(d\lambda) \phi, \psi \rangle = - \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} h(\lambda) \langle (R_{\lambda+i\epsilon} - R_{\lambda-i\epsilon}) \phi, \psi \rangle d\lambda$$

where R_λ is the resolvent $(\lambda I - \tilde{\square})^{-1}$, $\text{Im } \lambda \neq 0$.

This formula is the starting point towards the Plancherel formula. R_λ has the following properties:

$$(i) \langle R_\lambda \square \phi, \psi \rangle = \langle R_\lambda \phi, \square \psi \rangle = \bar{\lambda} \langle R_\lambda \phi, \psi \rangle - \langle \phi, \psi \rangle$$

for all $\phi, \psi \in D(X)$.

Since R_λ commutes with the G -action on $L^2(X)$ and because of the continuity of the injection $D(X) \rightarrow L^2(X)$, there is an unique H -invariant distribution T_λ on X satisfying

$$\langle R_\lambda \phi, \psi \rangle = \langle T_\lambda, \tilde{\phi}_0 * \psi \rangle \quad \text{for all } \phi_0, \psi_0 \in D(G)$$

T_λ satisfies:

$$(ii) (\lambda - \square)T_\lambda = \delta(x^0).$$

$$(iii) |\langle T_\lambda, \tilde{\phi}_0 * \psi \rangle| \leq \frac{1}{|\text{Im } \lambda|} \|\phi_2\| \|\psi\|_2 \quad (\phi_0, \psi_0 \in D(G)).$$

The following lemma is crucial, and is also valid for n odd.

LEMMA 5.3: *Let T_λ be an H -invariant distribution on X satisfying $(\lambda - \square)T_\lambda = \delta(x^0)$. There is an unique element $K_\lambda \in \mathcal{X}'$ such that $M'K_\lambda = T_\lambda$.*

PROOF: Put $X_1 = \{x \in X: Q(x) < 1\}$, $X_2 = \{x \in X: Q(x) > 0\}$. On X_1 , T_λ satisfies $(\lambda - \square)T_\lambda = 0$. By [7, section 7] there exists an unique continuous linear form \tilde{K}_λ on $M(D(X_1))$ such that $M'\tilde{K}_\lambda = T_\lambda$ on X_1 . The restriction of T_λ to X_2 is H -invariant, hence $T_\lambda = M'K_\lambda$ for an unique continuous linear form on $M(D(X_2))$ (same reference). Clearly $K_\lambda = \tilde{K}_\lambda$ on $(0, 1)$. This provides the extension of K_λ to an element of \mathcal{X}' satisfying $M'K_\lambda = T_\lambda$. The uniqueness of K_λ is clear, since M is surjective.

Choose K_λ as in Lemma 5.3, such that $M'K_\lambda = T_\lambda$ for $\text{Im } \lambda \neq 0$. Then K_λ is a solution of the equation

$$\lambda K_\lambda - LK_\lambda = \frac{1}{c} B_0$$

with $c = (-1)^{\frac{1}{2}(\rho+1)} \pi^{\rho-1} 4 / \Gamma(\rho)$, $B_0 \in \mathcal{X}'$ defined by $B_0(\phi_0 + \phi_1 \log |t| + \phi_2 \eta) = \overline{\phi_2(1)}$. Thus

$$K_\lambda = a(s)S_{0,s} + b(s)S_{1,s} - \frac{\alpha(s)}{c\mu} Y(t-1)S_{0,s} - \frac{1}{c\mu} E$$

with Y the Heaviside function, $\mu = \rho - 1$, $\lambda = s^2 - \rho^2$,

$$\alpha(s) = - \frac{\Gamma(\frac{1}{2}(\rho + s))\Gamma(\frac{1}{2}(\rho - s))}{\Gamma(\frac{1}{2}(s - \rho + 2))\Gamma(\frac{1}{2}(-s - \rho + 2))\Gamma(\rho - 1)\Gamma(\rho)}$$

and $E \in \mathcal{X}'$ as in [7, section 7].

Since $M'K_\lambda$ satisfies the inequality (iii) for $\text{Im } \lambda \neq 0$, the coefficients of $t^{\frac{1}{2}(s-\rho)}$ and $(-t)^{\frac{1}{2}(s-\rho)}$ in the asymptotic expansion of K_λ for $t \rightarrow \infty$ and $t \rightarrow -\infty$ respectively, vanish for $s \notin \mathbb{R}$, $s \notin i\mathbb{R}$ (cf. section 4). This yields:

$$\begin{aligned}
 a(s) &= -\left[\Gamma\left(\frac{1}{2}(\rho + s)\right)\Gamma\left(\frac{1}{2}(\rho - s)\right)\right] \\
 &\quad \times \left[c\Gamma\left(\frac{1}{2}(s - \rho + 2)\right)\Gamma\left(\frac{1}{2}(-s - \rho + 2)\right)\Gamma(\rho)^2\right. \\
 &\quad \left.\times \sin^2\pi\left(\frac{1}{2}(s - \rho)\right)\right]^{-1} \\
 b(s) &= \frac{\Gamma\left(\frac{1}{2}(\rho - s)\right) \cos \pi\left(\frac{1}{2}(s - \rho)\right)}{c\Gamma\left(\frac{1}{2}(-s - \rho + 2)\right)\Gamma(\rho) \sin^2\pi\left(\frac{1}{2}(s - \rho)\right)}
 \end{aligned}$$

($s \notin \mathbb{R}$, $i\mathbb{R}$).

LEMMA 5.4: *Extending $a(s)$, $b(s)$ and $\alpha(s)$ to meromorphic functions on \mathbb{C} we get, for $\mu \in \mathbb{R}$*

- (i) $\text{Im } a(i\mu) = \text{Im } \alpha(i\mu) = 0$
- (ii) $\text{Im } b(i\mu) = \Gamma\left(\frac{1}{2}(\rho - i\mu)\right)\Gamma\left(\frac{1}{2}(\rho + i\mu)\right) \tanh\frac{1}{2}\mu\pi/c\pi\Gamma(\rho)$

We are now prepared to calculate the spectral resolution of $\tilde{\square}$. This implies a special type of resolution of the identity operator on $L^2(X)$. The resolution contains a continuous and discrete part. The continuous part is given by:

$$\begin{aligned}
 \langle \phi, \phi \rangle_{c.p} &= \frac{(-1)^{\frac{1}{2}(\rho-1)}}{4} \pi^{1-\rho} \int_0^\infty \Gamma\left(\frac{1}{2}(\rho - i\mu)\right)\Gamma\left(\frac{1}{2}(\rho + i\mu)\right) \\
 &\quad \times \tanh\frac{1}{2}\mu\pi \langle M'S_{1,i\mu}, \tilde{\phi}_0 * \phi \rangle d\mu \quad (\phi_0 \in D(G)).
 \end{aligned}$$

The discrete part, denoted by $\langle \phi, \phi \rangle_{d.p}$, corresponds to $\lambda \geq -\rho^2$ and consists of point-measures located at $s_r = \rho + 2r \geq 0$ ($r \in \mathbb{Z}$). An explicit calculation of

$$\lim_{\epsilon \rightarrow 0} -\frac{1}{2\pi i} \langle M'(K_{\lambda+i\epsilon} - K_{\lambda-i\epsilon}), \tilde{\phi}_0 * \phi \rangle \quad (\phi_0 \in D(G))$$

shows

$$\begin{aligned}
 \langle \phi, \phi \rangle_{d.p} &= \sum_{\substack{r < 0 \\ s_r \geq 0}} (-1)^{\frac{1}{2}(\rho-1)} \pi^{-1-\rho} s_r \Gamma(-r)\Gamma(\rho+r) \langle M'S_{1,s_r}, \tilde{\phi}_0 * \phi \rangle \\
 &\quad + \sum_{r \geq 0} (-1)^{\frac{1}{2}(\rho+1+2r)} \pi^{-1-\rho} 2s_r \frac{\Gamma(\rho+r)}{\Gamma(1+r)} \langle M'U_r, \tilde{\phi}_0 * \phi \rangle.
 \end{aligned}$$

Combining these formulae, transforming them to $\zeta_{0,s}$ and $\zeta_{1,s}$ with Proposition 5.1 and Lemma 5.2 and simplifying them afterwards, yields

$$\begin{aligned} \langle \phi, \phi \rangle &= \frac{2^{3\rho-3}\pi^{-\rho}}{\Gamma(\rho)} \left\{ \int_0^\infty \left[\Gamma\left(\frac{1}{4}(\rho + i\mu)\right)^2 \Gamma\left(\frac{1}{4}(\rho - i\mu)\right)^2 \Gamma\left(\frac{1}{2}(1 + i\mu)\right) \right. \right. \\ &\quad \times \Gamma\left(\frac{1}{2}(1 - i\mu)\right) \left. \left. \left[\Gamma\left(\frac{1}{2}i\mu\right) \Gamma\left(-\frac{1}{2}i\mu\right) \right]^{-1} \langle \zeta_{0,i\mu}, \tilde{\phi}_0 * \phi \rangle \, d\mu \right. \right. \\ &\quad + \int_0^\infty \left[\Gamma\left(\frac{1}{4}(\rho + i\mu + 2)\right)^2 \Gamma\left(\frac{1}{4}(\rho - i\mu + 2)\right)^2 \Gamma\left(\frac{1}{2}(1 + i\mu)\right) \right. \\ &\quad \times \Gamma\left(\frac{1}{2}(1 - i\mu)\right) \left. \left. \left[\Gamma\left(\frac{1}{2}i\mu\right) \Gamma\left(-\frac{1}{2}i\mu\right) \right]^{-1} \langle \zeta_{1,i\mu}, \tilde{\phi}_0 * \phi \rangle \, d\mu \right. \right. \\ &\quad + \sum_{\substack{s_r \geq 0 \\ r \text{ odd}}} s_r \Gamma\left(\frac{1}{2}(\rho + r)\right)^2 \Gamma\left(-\frac{1}{2}r\right)^2 \langle \zeta_{0,s_r}, \tilde{\phi}_0 * \phi \rangle \\ &\quad \left. \left. + \sum_{\substack{s_r \geq 0 \\ r \text{ even}}} s_r \Gamma\left(\frac{1}{2}(\rho + r + 1)\right)^2 \Gamma\left(\frac{1}{2}(1 - r)\right)^2 \langle \zeta_{1,s_r}, \tilde{\phi}_0 * \phi \rangle \right\} \right. \\ &\quad \left. (\phi_0 \in D(G)). \right. \end{aligned}$$

We now come to the Plancherel formula for $SL(n, \mathbb{R})/GL(n - 1, \mathbb{R})$, $n \geq 3$. First we have to introduce the analogues of the Harish-Chandra c -function, called c_0 and c_1 .

By [7, section 8] we have for $\phi \in D(X)$ with $\text{Supp } \phi \subset \{x \in X: P(x, \xi^0) > 0\}$

$$\lim_{t \rightarrow \infty} 2^{(s-\rho)} e^{(-s+\rho)t} \zeta_{0,s}^t(\phi_{a_t}) = c_{0,+} \Gamma\left(\frac{1}{4}(s - \rho + 2)\right)_0 \hat{\phi}(\xi^0, s)$$

and

$$\lim_{t \rightarrow \infty} 2^{(s-\rho)} e^{(-s+\rho)t} \zeta_{1,s}^t(\phi_{a_t}) = c_{1,+} \Gamma\left(\frac{1}{4}(s - \rho + 4)\right)_1 \hat{\phi}(\xi^0, s).$$

Put

$$c_0(s) = c_{0,+} \cdot \Gamma\left(\frac{1}{4}(s - \rho + 2)\right)^2 \left(\frac{2^\rho \pi}{\Gamma\left(\frac{1}{2}\rho\right) \Gamma\left(\frac{1}{2}n\right)} \right) \frac{\Gamma\left(\frac{1}{2}(-s - \rho + n)\right)}{\Gamma\left(\frac{1}{2}(s - \rho + n)\right)}$$

and

$$c_1(s) = -c_{1,+} \Gamma\left(\frac{1}{4}(s - \rho + 4)\right)^2 \left(\frac{2^\rho \pi}{\Gamma(\frac{1}{2}\rho)\Gamma(\frac{1}{2}n)} \right) \frac{\Gamma(\frac{1}{2}(-s - \rho + n))}{\Gamma(\frac{1}{2}(s - \rho + n))}.$$

So

$$c_0(s) = \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{4}(\rho + s))^2 \Gamma(\frac{1}{2}(1 + s))},$$

$$c_1(s) = \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{4}(\rho + s + 2))^2 \Gamma(\frac{1}{2}(1 + s))}.$$

THEOREM 5.5: Put $s_r = \rho + 2r$ if n is even, $s_r = \rho + 2r + 1$ if n is odd ($r \in \mathbb{Z}$). The Plancherel formula for $\text{SL}(n, \mathbb{R})/\text{GL}(n - 1, \mathbb{R})$ ($n \geq 3$) is given by

$$\begin{aligned} c \cdot \int_X |\phi(x)|^2 dx &= \frac{1}{2\pi} \int_0^\infty \langle \zeta_{0,i\mu}, \tilde{\phi}_0 * \phi \rangle \frac{d\mu}{|\tilde{c}_0(i\mu)|^2} \\ &+ \frac{1}{2\pi} \int_0^\infty \langle \zeta_{1,i\mu}, \tilde{\phi}_0 * \phi \rangle \frac{d\mu}{|c_1(i\mu)|^2} \\ &+ \sum_{\substack{s_r \geq 0 \\ r \text{ odd}}} \langle \zeta_{0,s_r}, \tilde{\phi}_0 * \phi \rangle \text{Res} \left[\frac{1}{c_0(s)c_0(-s)}, s = s_r \right] \\ &+ \sum_{\substack{s_r \geq 0 \\ r \text{ even}}} \langle \zeta_{1,s_r}, \tilde{\phi}_0 * \phi \rangle \text{Res} \left[\frac{1}{c_1(s)c_1(-s)}, s = s_r \right] \end{aligned}$$

for all $\phi_0 \in D(G)$, where $c = \frac{\Gamma(\rho)}{2^{3\rho-4}\pi^{1-\rho}}$.

PROOF: We have to show that the positive-definite spherical distributions $\zeta_{0,i\mu}$, $\zeta_{1,i\mu}$ ($\mu \geq 0$), ζ_{0,s_r} ($s_r \geq 0$, r odd) and ζ_{1,s_r} ($s_r \geq 0$, r even), are extremal. Otherwise stated: they correspond in a natural manner to irreducible unitary representations of G , as explained in Section 1. Let $\phi \rightarrow_0 \hat{\phi}$ and $\phi \rightarrow_1 \hat{\phi}$ be the Fourier transforms defined in [7, p. 22] Now

consider $\zeta_{0,i\mu} (\mu \in \mathbb{R})$. One has $\zeta_{0,i\mu}(\tilde{\phi}_0 * \phi) = \int_B |{}_0\hat{\phi}(b, i\mu)|^2 db$ ($\phi_0 \in D(G)$). A similar formula holds for $\zeta_{1,i\mu}$. By [7, section 6] there is a K -invariant function $\phi \in D(X)$ such that the Fourier transform ${}_0\hat{\phi}(\cdot, i\mu) \neq 0$ for all $\mu \in \mathbb{R}$, but ${}_1\hat{\phi}(\cdot, i\mu) = 0$ for all $\mu \in \mathbb{R}$. A similar fact holds for ${}_1\hat{\phi}$: there is a K -finite function $\phi \in D(X)$ such that ${}_1\hat{\phi}(\cdot, i\mu) \neq 0$ for all $\mu \in \mathbb{R}$, but ${}_0\hat{\phi}(\cdot, i\mu) = 0$ for all $\mu \in \mathbb{R}$.

This implies that $\zeta_{0,i\mu}$ and $\zeta_{1,i\mu}$ form a basis for $D'_{\lambda,H}(X)$ for all $\mu \in \mathbb{R}$; here $\lambda = -\mu^2 - \rho^2$. Moreover, if $\zeta \in D'_{\lambda,H}(X)$ is positive definite (as an H -invariant distribution on G), and $\zeta = c_0\zeta_{0,i\mu} + c_1\zeta_{1,i\mu}$ then clearly $c_0 \geq 0$, $c_1 \geq 0$. This implies that $\zeta_{0,i\mu}$ and $\zeta_{1,i\mu}$ correspond to irreducible unitary representations for all $\mu \in \mathbb{R}$. For $\mu \neq 0$ we may take $\pi_{0,i\mu}$ and $\pi_{1,i\mu}$ for these representations [7, Proposition 3.2]. If $\mu = 0$ we can take the natural representation on the closure of $\{j\hat{\phi}(\cdot, 0) : \phi \in D(X)\}$ in $L^2(B)$ for $j = 0, 1$ respectively. We now come to the ζ_{0,s_r} (r odd) and ζ_{1,s_r} (r even). They are extremal because the $T \in D'_{\lambda,H}(X)$ ($\lambda = s_r^2 - \rho^2$), which correspond to a unitary representation with a realization on $L^2(X)$, span a 1-dimensional subspace in $D'_{\lambda,H}(X)$ by Theorem 4.4 (It is possible to realize the corresponding irreducible unitary representations as a subquotient of π_{0,s_r} and π_{1,s_r} respectively.)

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