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ON THE HOMOLOGY AND COHOMOLOGY OF COMPLETE INTERSECTIONS WITH ISOLATED SINGULARITIES

Alexandru Dimca

Let $V$ be a complex projective complete intersection having only isolated singularities at the points $a_i$ for $i = 1, \ldots, k$.

In this paper we investigate to what extent the integral homology and the cohomology algebra of $V$ can be determined from the local information associated to the singularities $(V, a_i)$. The relevant local properties of the singularities $(V, a_i)$ turn out to be contained in their Milnor lattices $L_i$.

In the first section we recall some facts on the homology of a smooth complete intersection. The only new result here is maybe the observation that the middle homology group of such a complete intersection (regarded as a lattice with the intersection form) contains in a natural way a reduced Milnor lattice $\bar{L} = L / \text{Rad} L$ (see (1.4), (1.5)).

In the next section we prove the main result of the paper (Theorem 2.1) which says that the homology of $V$ is determined by a morphism of lattices

$$\varphi_V : L_1 \oplus \ldots \oplus L_k \rightarrow \bar{L}$$

more precisely, by its kernel and cokernel. (We denote by $\oplus$ orthogonal direct sums.)

In fact, the source and the target lattices depend only on the local information at the singular points, the dimension and the multidegree of $V$. The global unknown information on which the homology of $V$ depends (e.g. the position of the singularities on $V$) is contained in the application $\varphi_V$ itself.

The third section is arithmetic in nature. Assuming the lattices $L_i$ nondegenerate, we show that the torsion part $T(V)$ of the homology of $V$ belongs to a finite set of groups which is computable entirely in terms of the lattices $L_i$, using the formalism of discriminant (bilinear) forms associated to lattices.

In particular, we compute the discriminant forms associated to the hypersurface simple singularities $A_m$, $D_m$ and $E_m$. Using these computations, we are able to prove that $T(V) = 0$ in a lot of concrete situations.

The results in this section were worked out jointly with my student A. Nemethi.
In the forth section we compute the homology of two classes of varieties: the cubic surfaces in \( \mathbb{P}^3 \) and some \( 2m \)-dimensional complete intersections of two quadrics. In these favourable cases the arithmetic of the lattices \( L_i \) and \( \overline{L} \) determines completely the application \( \varphi_V \).

In the next section we consider two complete intersections \( V_0 \) and \( V_1 \) with isolated singularities at the points \( a_i \) such that the singularities \( (V_0, a_i) \) and \( (V_1, a_i) \) are isomorphic for \( i = 1, \ldots, k \). We give in this situation a sufficient condition such that \( V_0 \) and \( V_1 \) have the same homology (Proposition 5.1). Also we show how this result can be used to relate the morphisms \( \varphi_V \) to the adjacency of singularities. This device is basic when the lattices \( L_i \) and \( \overline{L} \) give too little information on \( \varphi_V \), as for instance in the simple case of an odd-dimensional \( A_1 \)-singularity.

In the final section we point out how the morphism \( \varphi_V \) determines the cohomology algebra of \( V \) with coefficients in an unitary ring. This provides us with a finer tool for showing the non-homotopy equivalence of some varieties (see (6.3)). Conversely, in the case of surfaces, this result combined with Whitehead classification of simply connected 4-dimensional CW-complexes, can be used to show that certain surfaces are homotopy equivalent (see (6.4), (6.5)).

1. The homology of smooth complete intersections

In this section we recall some facts concerning the (integral) homology groups \( H_i(V) \) of a smooth complete intersection \( V \subset \mathbb{C}P^{n+p} \) with \( \dim V = n \) and multidegree \( (d_1, \ldots, d_p) \).

It is well known that \( H_i(V) = H_i(\mathbb{C}P^n) \) except for \( H_n(V) \) which is a free group of rank \( b_n(V) \), computable in terms of \( d_1, \ldots, d_p \) [13].

If \( P_1 = \ldots = P_p = 0 \) are the homogeneous equations of \( V \), then there is a natural \( S^1 \)-bundle \( p : K \to V \), where \( K = \{ x \in \mathbb{C}^{n+p+1}; |x| = 1, P_1(x) = \ldots = P_p(x) = 0 \} \).

We call \( p \) the Milnor \( S^1 \)-bundle of the variety \( V \) and note that it gives a Gysin sequence in homology

\[
\ldots \to H_i(K) \xrightarrow{p_*} H_i(V) \xrightarrow{u_i} H_{i-1}(V) \to H_{i-1}(K) \to \ldots
\]  

For \( n \) odd, \( u_{n+1} \) is multiplication by \( d = d_1 \cdots d_p = \deg V \). For \( n \) even, the same is true for \( u_n \circ u_{n+2} \).

Take now a hyperplane \( H \) in \( \mathbb{C}P^{n+p} \) such that \( W = V \cap H \) is smooth and let us denote by \( N \) a tubular neighborhood of \( W \) in \( V \). It is easy to see that the associated \( S^1 \)-bundle \( \partial N \to W \) is equivalent to the Milnor \( S^1 \)-bundle of the variety \( W \) (use the equivalence between \( S^1 \)-bundles and complex line bundles).

Next we prove the following basic result.

**Lemma 1.2:** The affine variety \( U = V \setminus W \) is homeomorphic to the Milnor fiber of the singularity \( (X, 0) = (\text{cone over } W, \text{ vertex}) \).
PROOF: Assume that $x_0 = 0$ is an equation for the hyperplane $H$. Then $U \subset \mathbb{C}^{n+p}$ is given by equations $f_i(x) = 0$ for $i = 1, \ldots, p$, where $x = (x_1, \ldots, x_{n+p})$, $f_i(x) = P_i(1, x) = Q_i(x) + Q'_i(x)$ with $Q_i$ a homogeneous polynomial of degree $d_i$ and $Q'_i$ a polynomial of degree $< d_i$.

Note that $Q_1 = \ldots = Q_p = 0$ are the equations of the singularity $(X, 0)$.

Next take $y = \bar{x} \cdot r^{-1}$ for a real number $r$ and denote

$$U_r = \{ y \in \mathbb{C}^{n+p}; f_i(ry) = 0 \text{ for } i = 1, \ldots, p \}.$$ 

It follows that $U_r \cap B_\epsilon \to U \cap B_\epsilon$, where $B_\epsilon = \{ y \in \mathbb{C}^{n+p}; |y| < \epsilon \}$ and $m$ is multiplication by $r$. We have also $y \in U_r$ if and only if

$$Q'_i(y) = Q_i(y) + Q'_i(ry) \cdot r^{-d_i} = 0 \quad \text{for } i = 1, \ldots, p.$$ 

For $r$ big enough, the equation $Q'_i = 0$ is nothing else but a small deformation of the equation $Q_i = 0$.

It follows that $U_r \cap B_\epsilon$ is the Milnor fiber of the isolated singularity of complete intersection $(X, 0)$ [16].

On the other hand, for any algebraic (possibly singular) set $Z \subset \mathbb{C}^N$, it is known that $Z \cap B_\epsilon$ is homeomorphic to $Z$ for $r$ big enough. □

The Mayer-Vietoris sequence corresponding to the cover $V = U \cup N$ contains the morphism

$$j = (j_1, j_2): H_n(U \cap N) \to H_n(U) + H_n(N)$$

and we want to interpret the components $j_i$ of this morphism after we replace $U \cap N$ by $\partial N = K' = \text{the total space of the Milnor } S^1\text{-bundle of } W$, $U$ by $\bar{X} = \text{the Milnor fiber of } (X, 0)$ and $N$ by the variety $W$.

We denote by $L$ the Milnor lattice $(H_n(\bar{X}),(\ ,\ ))$ where the intersection form $(\ ,\ )$ is symmetric for $n$ even and skew-symmetric for $n$ odd. For any such lattice $L$, we denote by $\text{Rad } L$ the sublattice $\{ x \in L; (x, y) = 0 \text{ for any } y \in L \}$ and call the quotient $\bar{L} = L/\text{Rad } L$ with the induced bilinear form the reduced Milnor lattice of the singularity $(X, 0)$.

With these notations, $j_1$ can be identified with the natural inclusion

$$H_n(K') = \text{Rad } L \to L$$

and $j_2$ can be read off the corresponding Gysin exact sequence (1.1). We get thus the exact sequence

$$0 \to \bar{L} + H_n(W) \xrightarrow{s} H_n(V) \to H_{n-1}(K') \xrightarrow{p_*} H_{n-1}(W)$$  (1.3)
COROLLARY 1.4: For n odd, the lattice $(H_n(V), \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ denotes the intersection form, is isomorphic to the reduced Milnor lattice $L$. In particular, $L$ is unimodular.

PROOF: In this case $H_n(W) = 0 = \ker p_*$, as can be seen from the corresponding exact sequence (1.1). $\square$

Note that by recent work of Chmutov [6], the fact that an odd-dimensional hypersurface singularity has unimodular reduced Milnor lattice has interesting consequences on the monodromy group.

For $n$ even, let $h \in H_n(V)$ be the class of the cycle corresponding to the intersection of $V$ with a generic $(p + n/2)$-plane. Alternatively, $h = u_{n+2}(g)$ where $g$ is the canonical generator of $H_{n+2}(V)$.

Then it is well-known that $\langle h, h \rangle = d$ and let us denote $h^\perp$ the orthogonal complement of $h$ with respect to the intersection form $\langle \cdot, \cdot \rangle$.

COROLLARY 1.5: With the notations above, there is a lattice isomorphism $L \cong h^\perp$.

PROOF: When $n$ is even, in the exact sequence (1.3) one has $H_n(W) = \mathbb{Z}$, $\ker p_* = \mathbb{Z}/d\mathbb{Z}$.

The inclusion $W \subset V$ gives an identification $s(H_n(W)) = \mathbb{Z}.h$. The geometric description of the cycle $h$ shows that $s(L) \subset h^\perp$. A simple computation proves that the lattice $\mathbb{Z}h + h^\perp$ has index $d$ in $H_n(V)$ and hence $s(L) = h^\perp$. $\square$

Note that the obvious consequences of (1.5) that $h^\perp$ is an even lattice and that any cycle $c \in h^\perp$ can be represented by an embedded sphere $S^n \subset V$ have been proved by completely different means in [15] (2.1).

On the other hand, it follows from (1.5) that $\text{sign } L = \text{sign } V - 1$, $\text{rk } L = b_n(V) - 1$ and these relations can be used to get upper bounds for the number of singularities which may occur on a complete intersection of given dimension and multidegree [7].

2. Complete intersections with isolated singularities

We assume from now on that $V$ is a complete intersection in $\mathbb{C}P^{n+p}$ with $\dim V = n$, multidegree $(d_1, \ldots, d_p)$ and having only isolated singularities at the points $a_i$ for $i = 1, \ldots, k$. If $H$ is a hyperplane such that $W = V \cap H$ is smooth, we denote again by $(X, 0)$ the singularity defined by the cone over $W$.

Then, exactly as in the proof of (1.1), one can show that $U = V \setminus W$ is homeomorphic to a singular fiber in a deformation of the singularity $(X, 0)$, arbitrary close to the special fiber $X$. 
By a result in the new book of Looijenga [16] (7.13), we deduce that $U$ has the homotopy type of a bouquet of $n$-spheres and there is a natural exact sequence

$$0 \to L_1 \oplus \ldots \oplus L_k \overset{i_V}{\to} L \to H_n(U) \to 0$$

where $L_i$ (resp. $L$) is the Milnor lattice of the singularity $(V, a_i)$ for $i = 1, \ldots, k$ (resp. $(X, 0)$) and the inclusion $i_V$ respects the intersection forms i.e. it is a morphism of lattices.

Let $N$ be a tubular neighbourhood of $W$ in $V \setminus \{a_1, \ldots, a_k\}$. The Mayer-Vietoris sequence corresponding to the cover $V = U \cup N$ contains the morphism

$$j = (j_1, j_2) : H_n(U \cap N) \to H_n(U) + H_n(N).$$

As in the smooth case treated in section 1, we can identify $U \cap N \approx \partial N$ with the total space $K'$ of the Milnor $S^1$-bundle of $W$ and then the component

$$j_2 = p_* : H_n(K') \to H_n(N) = H_n(W)$$

can be read off from the corresponding sequence (1.1).

The other component $j_1$ corresponds to the composition

$$H_n(K') = \text{Rad } L \to L \to H_n(U) = L/\text{im} i_V.$$

In particular, $j_1$ has the same kernel and cokernel as the morphism $\varphi_V$ defined by the composition of the inclusion $i_V$ with the natural projection $L \to \overline{L} = L/\text{Rad } L$. The advantage of replacing the morphism $j_1$ by $\varphi_V$ is that the latter is a morphism of lattices.

We can state now our main result.

**Theorem 2.1:**

(i) $H_i(V) = H_i(\mathbb{C}P^n)$ for $i \neq n, n + 1$

(ii) $H_n(V) = H_n(\mathbb{C}P^n) + \text{coker } \varphi_V$

$H_{n+1}(V) = H_{n+1}(\mathbb{C}P^n) + \text{ker } \varphi_V$, where "+" denotes direct sum of groups.

**Proof:** The point (i) is clear from the fact that $U$ is a bouquet of $n$-spheres and from the Gysin sequence (1.1) applied to $W$.

The second part (ii) must be proved separately for $n$ odd and for $n$ even. We give the details only for the case $n$ even, which is slightly more delicate than the other case.
Assuming $n$ even, one has $H_{n+1}(W) = 0$ and the Mayer-Vietoris sequence above gives $H_{n+1}(V) = \ker j$ and the exact sequence

$$0 \to \text{coker } j \overset{s}{\to} H_n(V) \to H_{n-1}(K') \overset{p_*}{\to} H_{n-1}(W)$$

(2.2)

Now, $j_2 = 0$ and hence $\ker j = \ker j_1 = \ker \varphi_V$, which proves the second part of (ii).

Moreover, $\text{coker } j = \text{coker } \varphi_V + H_n(W)$ and $\ker p_* = \mathbb{Z}/d\mathbb{Z}$.

There is a natural $S^1$-bundle $q : K \to V$ constructed as in section 1 and by a result of Hamm [11] $K$ is $(n-1)$-connected. If $a = i*b$, where $i : V \subset \mathbb{C}P^{n+p}$ and $b \in H^2(\mathbb{C}P^{n+p})$ is the standard generator, then the morphism $u_m : H_m(V) \to H_{m-2}(V)$ which occurs in the Gysin sequence of $q$ is precisely the cap-product with $a$. In particular, from

$$H_n(V) \overset{u_m = u}{\to} H_{n-2}(V) = \mathbb{Z} \to H_{n-1}(K) = 0$$

it follows that there is an element $h_0 \in H_n(V)$ such that $u(h_0) = 1$. We deduce that $H_n(V) = \ker u + \mathbb{Z}h_0$.

Let $g \in H_n(W)$ be a generator and $h = i_0\ast(g)$ where $i_0 : W \subset V$. Then, by comparing the Gysin sequences for $W$ and $V$, it follows that $u(h) = \pm d$ and hence $h = \pm d h_0 + h'$ for some $h' \in \ker u$.

On the other hand, it is clear that $s(\text{coker } \varphi_V)$ is contained in $\ker u$ and hence

$$\text{im } s \subset \ker u + \mathbb{Z}h$$

But these two inclusions must be equalities since the index of $\ker u + \mathbb{Z}h$ in $H_n(V)$ is equal to $d$, which is the index of $\text{im } s$ in $H_n(V)$ by (2.2). 

As trivial consequences of the Theorem we have the following

**Corollary 2.3:**

(i) $H_{n+1}(V)$ is torsion free

(ii) $\chi(V) = \chi(V_0) + (-1)^{n+1} \sum_{i=1, k} \mu(V, a_i)$

where $\chi$ denotes the Euler-Poincaré characteristic, $V_0$ is a smooth complete intersection with the same dimension and multidegree as $V$ and $\mu(V, a_i) = \text{rk } L_i$ is the Milnor number of the singularity $(V, a_i)$.

The difficulty in applying Theorem (2.1) in concrete cases consists in the fact that one does not known how to identify the embedding $i_V$ starting, let's say, from the equations of $V$. The rest of the paper is devoted to various devices to circumvent this difficulty.
Remark 2.4: The reduced Milnor lattice $\overline{L}$ is the same for all the complete intersections $V$ of given dimension and multidegree (and for all choices of a smooth hyperplane section $W$) by $\mu$-constant deformations arguments. Sometimes, information about $\overline{L}$ can be obtained using (1.4) and (1.5).

3. Singularities with nondegenerate Milnor lattices

First we recall some basic definitions concerning lattices. By a lattice $(M, (\cdot, \cdot))$ we mean a finitely generated free abelian group $M$ together with a bilinear form $(\cdot, \cdot)$ on $M$ which is either

(i) symmetric and even i.e. $(x, x) \equiv 0 \pmod{2}$ for any $x \in M$, or
(ii) skew-symmetric.

We call the lattice $M$ nondegenerate if $\text{Rad} \ M = 0$. In this case the natural homomorphism of groups

$$i_M : M \to M^* = \text{Hom} (M, \mathbb{Z}), \quad i_M (x) = (x, \cdot)$$

is an embedding and the discriminant group $D(M)$ of $M$ is by definition the finite group $\text{coker} \ i_M$.

The natural number $\det M$ equal to the order of the group $D(M)$ is an important numerical invariant of the lattice $M$.

In particular, the lattice $M$ is called unimodular if $\det M = 1$. If $M$ is a sublattice of another lattice $N$ such that $\text{rk} M = \text{rk} N$ (equivalently, the index $[N: M]$ is finite) then one has the equality

$$\det M = [N: M]^2 \det N. \quad (3.1)$$

Now we come back to our main problem (and to the notations from the previous section).

Lemma 3.2: (i) If the Milnor lattices of all the singular points $a_i \in V$ are nondegenerate, then $H_{n+1}(V) = H_{n+1}(\mathbb{C}P^n)$ and the rank of $H_n(V)$ is $b_n(V) = b_n(V_0) - \sum_{i=1}^{k} \mu(V, a_i)$.

(ii) If the Milnor lattices of all the singular points $a_i \in V$ are unimodular, then in addition $H_n(V)$ is a free group.

Proof: This result can be easily derived either from (2.1) or using the simple observation that $V$ is a $\mathbb{Q}$-homology (resp. a $\mathbb{Z}$-homology) manifold in case (i) (resp. in case (ii)). Then the Poincaré duality over $\mathbb{Q}$ (resp. over $\mathbb{Z}$) and the formula for $\chi(V)$ in (2.3) give the result.

Examples 3.3: If $n = \text{dim} \ V$ is even, then the Milnor lattices of the simple hypersurface singularities $A_m$, $D_m$, $E_6$, $E_7$ and $E_8$ are nondegen-
erate. More precisely: \( \det A_m = m + 1 \), \( \det D_m = 4 \), \( \det E_m = 9 - m \) and hence, in particular, \( E_8 \) is unimodular. (Note that we use the same symbol for a singularity and its Milnor lattice!). These and many other examples of even-dimensional hypersurface singularities with nondegenerate Milnor lattices can be found in Ebeling paper [9].

In odd dimensions, we can use the stabilization of singularities (i.e. addition of a square to the defining equation of a hypersurface singularity) and the relation between the corresponding intersection matrices [10] to show that the lattices \( A_{2m}, E_6 \) and \( E_8 \) are unimodular, while the lattices corresponding to the other simple hypersurface singularities are degenerate.

The first examples in this range of dimensions of nondegenerate lattices which are not unimodular are those corresponding to the singularities \( T_{p,q,r}(p, q, r \text{ odd}) \) and \( Q_{k,i}(i \text{ even}) \) which have \( \det T_{p,q,r} = \det Q_{k,i} = 4 \) [6].

Further examples can be found in the papers of Brieskorn [3] and Hamm [12].

We assume from now on in this section that all the singularities \( (V, a_i) \) have nondegenerate Milnor lattices \( L_i \). Then the morphism \( \varphi_V \) is an embedding and the only unknown part of the homology of \( V \) is the finite torsion group

\[ T(V) = \text{Tors } H_n(V). \]

We show now that the lattices \( L_i \) put strong arithmetic restrictions on the group \( T(V) \).

The arithmetic problem is the following: given a nondegenerate lattice \( M \), to describe the set of finite groups

\[ T(M) = \{ \text{Tors } (N/M); \quad N \text{ is a supralattice of } M, \text{i.e. } M \subset N \}. \]

It is a simple observation that in this definition we can take only supralattices \( N \) with \( \text{rk} N = \text{rk} M \). Using this fact and the formula (3.1) we get the following simple, but useful result.

**Corollary 3.4:** If \( F \in T(M) \), then \( |F|^2 \) divides \( \det M \), where \( |F| \) denotes the order of \( F \). Moreover, if we are in the symmetric case and sign \( (M) \neq 0 \pmod{8} \), then \( |F|^2 \neq \det M \).

**Proof:** Since \( |F| = [N : M] \), the first assertion is clear. The second part follows from the fact that an even symmetric lattice \( N \) with \( \text{sign}(N) \neq 0 \pmod{8} \) cannot be unimodular [17]. \( \square \)

To describe more accurately the set \( T(M) \) we can use the bilinear discriminant form of \( M \), which we define now. The bilinear form on \( M \)
can be extended in a natural way to a bilinear form $M^* \times M^* \to \mathbb{Q}$ and this induces a bilinear form $b : D(M) \times D(M) \to \mathbb{Q}/\mathbb{Z}$ which is called the bilinear discriminant form of $M$[8].

If $N$ is a supralattice of $M$ such that $F(N) = N/M$ is a finite group, then there is a natural inclusion of $F(N)$ in $D(M)$ as an isotropic subgroup i.e. $b|F(N) \times F(N) = 0$. Moreover, this correspondence defines a bijection between supralattices $N$ of $M$ with $F(N)$ finite and isotropic subgroups in $D(M)$ [8] (1.4.4). Hence $T(M)$ is the same as the set of isotropic subgroups in $D(M)$ and can be computed if the bilinear discriminant form of $M$ is known. And this discriminant form can be computed in most of the cases using the obvious fact $D(M_1 \oplus M_2) = D(M_1) \oplus D(M_2)$.

In the skew-symmetric case, any nondegenerate lattice $M$ is an orthogonal direct sum of elementary lattices $M_d = \mathbb{Z}\langle e_1, e_2 \rangle$ with $(e_1, e_1) = 0$ and $(e_1, e_2) = d$. Direct computations show that $D(M_d) = (\mathbb{Z}/d\mathbb{Z})^2$ and $b((\hat{a}_1, \hat{b}_1), (\hat{a}_2, \hat{b}_2)) = (a_1b_2 - a_2b_1)d^{-1} \in \mathbb{Q}/\mathbb{Z}$. In particular, it follows that $T(M) = 0$ if and only if $M$ is unimodular.

In the symmetric case the results are much more interesting. The bilinear discriminant form $b$ of an even lattice $M$ is determined by the corresponding quadratic form $q : D(M) \to \mathbb{Q}/2\mathbb{Z}$, $q(x + M) = (x, x) + 2\mathbb{Z}$ and the isotropic subgroups $F \subset D(M)$ are the subgroups $F$ for which $q|F = 0$.

In the next results we determine the quadratic forms corresponding to the singularities $A_m$, $D_m$ and $E_m$ and give some direct consequences.

The proofs consist of simple but tedious computations involving the corresponding intersection forms and are not given here.

**Proposition 3.5:**

(i) $D(A_m) = \mathbb{Z}/(m + 1)\mathbb{Z}$ and $q(\hat{1}) = -m(m + 1)^{-1}$.

(ii) $T(A_m) = \{ \mathbb{Z}/e\mathbb{Z} ; e | m + 1 \text{ and } m(m + 1)e^{-2} \text{ is an even integer} \}$.

As an example, for $1 \leq m \leq 20$ the only nontrivial sets $T(A_m)$ are the following

- $T(A_7) = T(A_{15}) = \{ 0, \mathbb{Z}/2\mathbb{Z} \}$
- $T(A_8) = T(A_{17}) = \{ 0, \mathbb{Z}/3\mathbb{Z} \}$.

**Proposition 3.6:**

(i) For $m$ even, $D(D_m) = (\mathbb{Z}/2\mathbb{Z})^2$ and the generators $u_1, u_2$ of $D(D_m)$ can be chosen such that $q(u_1) = -1$ and $q(u_2) = q(u_1 + u_2) = 0$, $6/4, 1, 1/2$ according to the cases $m \equiv 0, 2, 4, 6 \pmod{8}$.

(ii) For $m$ odd, $D(D_m) = \mathbb{Z}/4\mathbb{Z}$ and $q(\hat{1}) = 7/4, 5/4, 3/4, 1/4$ according to the cases $m \equiv 1, 3, 5, 7 \pmod{8}$.

In particular, $T(D_m) = \{ 0, \mathbb{Z}/2\mathbb{Z} \}$ for $m \equiv 0 \pmod{8}$ and $T(D_m) = \{ 0 \}$ otherwise.
Proposition 3.7:
(i) $D(E_6) = \mathbb{Z}/3\mathbb{Z}$ and $q(\tilde{1}) = 2/3$
(ii) $D(E_7) = \mathbb{Z}/2\mathbb{Z}$ and $q(\tilde{1}) = 1/2$
(iii) $D(E_8) = 0$
In particular, $T(E_m) = \{0\}$.

These results give obviously information on $T(V)$ when the even-di-
dimensional variety $V$ has precisely one singular point of type $A_m$ (resp.
$D_m$ or $E_m$) and some other singularities with unimodular Milnor lattices.
These unimodular lattices and the corresponding singularities will not be
mentioned at all in what follows, since their presence do not affect the
group $T(V)$ as follows from (2.1). But we can also handle with these
techniques the case of several singularities, as is shown in the next two
examples.

Example 3.8: Assume that $V$ has $p$ singularities of type $A_1$, $q$ singulari-
ties of type $D_m$ (for various even integers $m$) and $r$ singularities of type
$E_7$. Then $T(V) = (\mathbb{Z}/2\mathbb{Z})^S$, where $2S = p + 2q + r$. Moreover, if $p + \sum m_i + 7r \neq 0 \pmod{8}$, the inequality above is strict.

Proof: Let $M$ be the lattice $pA_1 \oplus qE_7 \oplus D_{m_1} \oplus \ldots \oplus D_{m_q}$. Then $D(M) = (\mathbb{Z}/2\mathbb{Z})^t$, where $t = p + 2q + r$. Since any subgroup $F \subset D(M)$ is a
$\mathbb{Z}/2\mathbb{Z}$-vector space, it follows that $F = (\mathbb{Z}/2\mathbb{Z})^S$. Then use (3.4).

Example 3.9: If the singularities on $V$ are described by a symbol in the
following list, then $T(V) = 0$:

$2A_1, 3A_1, A_1A_3, A_2A_3, 2A_1A_2, 2A_2A_1$

Proof: Let us treat for example the case $A_1A_3$. First note that we cannot
use only (3.4) to get the result! Using (3.5) we find out that $D(A_1A_3) =
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $q(\tilde{a}, \tilde{b}) = -\frac{1}{3}a^2 - \frac{1}{2}b^2$. A simple verification shows
that $q(\tilde{a}, \tilde{b}) = 0$ in $\mathbb{Q}/2\mathbb{Z}$ if and only if $a = b = 0$. Hence the only
isotropic subgroup in $D(A_1A_3)$ is the trivial group.

4. Two geometric examples

In this section we show that Theorem (2.1) can be used to determine
exactly the (nontrivial) torsion group $T(V)$ for the cubic surfaces in $\mathbb{C}P^3$
and for some types of complete intersections of two quadrics.

The singularities which may occur on a cubic surface $V \subset \mathbb{C}P^3$ (with
isolated singularities) are the following [4]

$A_1, 2A_1, 3A_1, 4A_1; A_2, 2A_2, 3A_2; A_1A_2, A_1A_3, A_1A_4,$
$A_1A_5; 2A_1A_2, 2A_1A_3; 2A_2A_1; A_3, A_4, A_5; D_4, D_5, E_6.$
All the groups of singularities in this list, except the four underlined ones, can give no torsion by the results in the previous section. The next proposition shows that these four cases give indeed nontrivial torsion in homology.

**Proposition 4.1:** The homology of a cubic surface \( V \subset \mathbb{CP}^3 \) with isolated singularities has no torsion, except the following cases.

<table>
<thead>
<tr>
<th>Singularities on ( V )</th>
<th>( T(V) = \text{Tors } H_2(V) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4A_1 )</td>
<td>( \mathbb{Z}/2\mathbb{Z} )</td>
</tr>
<tr>
<td>( 3A_2 )</td>
<td>( \mathbb{Z}/3\mathbb{Z} )</td>
</tr>
<tr>
<td>( A_1A_5 )</td>
<td>( \mathbb{Z}/2\mathbb{Z} )</td>
</tr>
<tr>
<td>( 2A_1A_3 )</td>
<td>( \mathbb{Z}/2\mathbb{Z} )</td>
</tr>
</tbody>
</table>

**Proof:** We give the details only for the case \( 4A_1 \), the other cases being similar.

The reduced Milnor lattice \( \tilde{L} \) is in this situation \( E_6 \). We prove that an embedding \( \varphi : 4A_1 \rightarrow E_6 \) is unique up to an automorphism \( u \) of the lattice \( E_6 \). Assume that \( 4A_1 = \mathbb{Z}\langle f_1, \ldots, f_4 \rangle \) with \( f_i^2 = -2 \), \( (f_i, f_j) = 0 \) for \( i \neq j \) and that \( E_6 = \mathbb{Z}\langle e_1, \ldots, e_6 \rangle \) with the products given by the usual Dynkin diagram in which \( e_1, \ldots, e_5 \) stay in a line and \( e_6 \) is below \( e_3 \) [2]. Since the Weyl group \( \text{Aut}(E_6) \) is transitive on the set of vectors \( v \) with \( v^2 = -2 \), we can take \( \varphi(f_1) = e_6 \). It follows that \( \mathbb{Z}\langle f_2, f_3, f_4 \rangle \) is embedded in \( e_6^\perp \), the orthogonal complement of \( e_6 \).

A basis for \( e_6^\perp \) is given by the vectors \( e_2, e_1, \tilde{e}_3, e_5, e_4 \) where \( \tilde{e}_3 = 2e_3 + e_2 + e_4 + e_6 \) and the corresponding Dynkin diagram is precisely \( A_5 \). Using the explicit description of the lattice (or root system) \( A_5 \) and of its symmetries [2], it follows that the embedding \( \varphi : 3A_1 \rightarrow A_5 \) induced by \( \varphi \) is equivalent up to an automorphism \( \bar{u} \in \text{Aut}(A_5) \) with the obvious embedding \( f_2 \rightarrow e_2, f_3 \rightarrow \tilde{e}_3, f_4 \rightarrow e_4 \).

Moreover, \( \bar{u} \) extends to an \( u \in \text{Aut}(E_6) \) since \( \bar{u} \) is a composition of reflexions and \( u(e_6) = e_6 \).

This gives us \( \text{Tors } (\text{coker } \varphi) = \mathbb{Z}/2\mathbb{Z} \) as required. \( \square \)

**Remark 4.2:** The homology of some singular cubic surfaces have been determined by Barthel and Kaup (see pp. 136, 275-276 in [1]) using completely different methods (e.g. desingularisations and local homology sheaves).

Now we consider two classes of complete intersections \( V \) of two quadrics in \( \mathbb{CP}^{2m+2} \).

**Proposition 4.3:**

(i) If the Segre symbol corresponding to \( V \) is \( [(1,1), \ldots, (1,1), 2(m-k)+3] \) \( k = 0, 1, 2, \ldots, m+1 \), then on the variety \( V \) there are \( 2k \) singular
points of type $A_1$ and a singular point of type $A_{2(m-k)+2}$ for $k \neq m+1$. Moreover, $T(V) = (\mathbb{Z}/2\mathbb{Z})^{k-1}$ for $k \geq 2$ and is trivial for $k = 0, 1$.

(ii) If the Segre symbol corresponding to $V$ is

$$[(1, 1), \ldots, (1, 1), (2(m-k) + 2, 1)] \quad k = 0, \ldots, m$$

then on the variety $V$ there are $2k$ singular points of type $A_1$ and a singular point of type $D_{2(m-k)+3}$ (with the convention $D_3 = A_3$). Moreover $T(V) = (\mathbb{Z}/2\mathbb{Z})^k$.

PROOF: The connection between the Segre symbol of a complete intersection of 2 quadrics $V$ and the type of the singularities on $V$ is explained in [14].

The reduced Milnor lattice $\tilde{L}$ is in this situation $D_{2m+3}$ [14], [15] and the computation of the torsion group $T(V)$ goes essentially as the proof of (4.1), using the following simple observation.

Assume that the lattice $D_{2m+3}$ is the lattice $\mathbb{Z} \langle e_1, \ldots, e_{2m+3} \rangle$ with the corresponding Dynkin diagram having $e_2, e_3, \ldots, e_{2m+3}$ in a line and $e_1$ sitting under $e_3$. Then note that $e_1^\perp$ is the orthogonal direct sum of $\mathbb{Z}e_2$ and $\mathbb{Z} \langle \tilde{e}_3, e_4, \ldots, e_{2m+3} \rangle$ where $\tilde{e}_3 = 2e_3 + e_1 + e_2 + e_4$. Moreover, this last lattice is obviously $D_{2m+1}$ (recall that $D_3 = A_3$). This fact allows one to prove (4.3) by induction on $k$. □

The interested reader can check that similar arguments give the homology of some other types of complete intersections of two quadrics. The simplest case which cannot be decided with these methods is the case of a 4-fold ($m = 2$) with 4 singular points of type $A_1$.

5. Deformations of complete intersections

First we prove a technical result which gives a sufficient condition for the inclusion of Milnor lattices $i_{V'}: L_1 \oplus \ldots \oplus L_k \to L$ to be independent of $V$ in the following sense.

Consider two complete intersections $V_0, V_1 \subset \mathbb{C}P^{n+p}$ of the same dimension $n$ and multidegree $(d_1, \ldots, d_p)$.

Moreover, assume that $V_0$ and $V_1$ have isomorphic isolated singularities at the points $a_1, \ldots, a_m$ and (possibly) some other isolated singularities.

Let us denote by $i_0$ (resp. $i_1$) the restriction of the inclusion $i_{V_0}$ (resp. $i_{V_1}$) defined in section 2 to the direct sum of the Milnor lattices corresponding to the singularities $a_1, \ldots, a_m$.

It is natural to ask when $i_0 = i_1$ (under some identification of the corresponding Milnor lattices).
Consider the vector spaces of polynomials

\[ F = \left\{ f = (f_1, \ldots, f_p); f_i \in \mathbb{C}[x_0, \ldots, x_{n+p}] \text{ homogeneous} \right\} \]

of degree \( d_i \) for \( i = 1, \ldots, p \)

\[ G = \left\{ g = (g_1, \ldots, g_p); g_i \in \mathbb{C}[x_1, \ldots, x_{n+p}], \deg g_i \leq d_i \right\} \]

for \( i = 1, \ldots, p \)

and the natural isomorphism \( h: F \to G \)

\[ h(f) = f(1, x_1, \ldots, x_{n+p}). \]

We can choose the coordinates on \( CP^{n+p} \) such that \( H \cap V_t \) are smooth for \( t = 0, 1 \) where \( H \) is the hyperplane \( x_0 = 0 \). Let \( F_0 \subset F \) be the Zariski open set corresponding to the complete intersections \( W \) such that \( H \cap W \) is smooth. Then a system of equations for \( V_t \) gives a point \( f^t \in F_0 \) \((t = 0, 1)\).

Let \( s_i \) be the order of \( \mathcal{K} \)-determinacy of the singularity \((V_0, a_i)\) \([18]\) for \( i = 1, \ldots, m \) and consider the product of jet spaces

\[ J = J^{s_1}(n + p, p) \mathcal{X} \ldots \mathcal{X} J^{s_m}(n + p, p) \]

and let \( S \subset J \) denote the corresponding product of \( \mathcal{X} \)-orbits.

There is a linear map \( \phi: F \to J \) given by the composition of \( h \) with the map \( G \to J \) which associates to a polynomial map \( g \) its \( s_i \)-jet at the point \( \bar{a}_i \) for \( i = 1, \ldots, m \). Here we assume that \( a_i = (1, \bar{a}_i) \) for \( i = 1, \ldots, m \).

With these notations, we have the following.

**Proposition 5.1:** If the constructible set \( \phi(F) \cap S \) is irreducible, then \( i_0 = i_1 \).

**Proof:** Since \( \phi \) is a linear map, it follows that \( F_t = \phi^{-1}(\phi(F) \cap S) \) is irreducible. Hence the open Zariski set \( F_{0t} = F_0 \cap F_t \) is connected. Take a path \( f^t \in F_{0t} \) for \( t \in [0, 1] \), joining the two points \( f^0 \) and \( f^1 \).

The complete intersection \( V_t \) corresponding to the equation \( f^t = 0 \) has only isolated singularities (since \( V_t \cap H \) is smooth!) and its singularity at the point \( a_t \) is isomorphic to \((V_0, a_i)\) for \( i = 1, \ldots, m \).

To this variety \( V_t \) corresponds an embedding of lattices \( i_t \), defined similarly to \( i_0 \) and \( i_1 \).

In this way we get a continuous family of embeddings and since the set of all homomorphism between two lattices is countable, we must have \( i_t = i_0 \) for any \( t \in [0, 1] \). □
REMARK 5.2: Here are two simple cases when the condition in (5.1) is fulfilled.

(i) If the multidegree \((d_1, \ldots, d_p)\) is big enough compared to the determinacy orders \((s_1, \ldots, s_m)\), then the map \(\phi\) is surjective. Since \(S\) is a smooth connected submanifold in \(J\), it follows that \(\phi(F) \cap S = S\) is irreducible. For example, in the hypersurface case \((p = 1)\) it is enough to have

\[
d_i \geq \sum_{i=1,m} (s_i + 1) - 1
\]

(ii) Assume that we are in the hypersurface case and that the singularities \((V_0, a_i)\) are all of type \(A_1\).

Then \(S\) is a Zariski open subset of a vector space in \(J\), the same is true for \(\phi(F) \cap S\) and hence this last set is irreducible.

Now we present two applications of (5.1). The first one is a direct consequence of Theorem (2.1). With the notation above, assume that all the singularities on \(V_0\) and \(V_1\) different from the points \(a_1, \ldots, a_m\) have unimodular Milnor lattices. If the condition in (5.1) is fulfilled, then it follows easily that \(V_0\) and \(V_1\) have the same Betti number \(b_{n+1}\) and the same torsion part \(T(V_0) = T(V_1)\). Moreover, note that instead of starting with the set of points \(a_1, \ldots, a_m\) we can start with two sets of points \(a_1, \ldots, a_m\) (singular points on \(V_0\)) and \(b_1, \ldots, b_m\) (singular points on \(V_1\)) which are projectively equivalent i.e. there is an automorphism \(u\) of \(\mathbb{CP}^{n+p}\) such that \(u(a_i) = b_i\) for \(i = 1, \ldots, m\).

The second application is more subtle and combines (5.1) and the adjacency of singularities [16] to obtain information on the homology in the presence of singularities with degenerate Milnor lattices. This method will become clear from the following two examples.

EXAMPLE 5.3: Assume that the odd dimensional complete intersection \(V\) has a singular point \(a_1\) of type \(A_1\) and that all the other singular points of \(V\) (if any) have unimodular Milnor lattices.

Then:

(i) \(\ker \varphi_V = 0\), except the case when \(V\) is a hypersurface of degree 2, when \(\ker \varphi_V = \mathbb{Z}\).

(ii) \(\text{coker } \varphi_V\) is torsion free.

PROOF: When \(V\) is a hypersurface of degree 2, one has \(L = 0\) and hence everything is clear.

Assume now that \(V\) is a hypersurface of degree \(d \geq 3\). Then, by (5.1) essentially, \(\ker \varphi_V\) and \(\text{Tors } \text{coker } \varphi_V\) are the same as for the hypersurface \(W\) with affine equation

\[
x_1^d + \ldots + x_{n+1}^d + \alpha_1 x_1^2 + \ldots + \alpha_{n+1} x_{n+1}^2 + \beta x_1^3 = 0
\]
Indeed, for a suitable choice of the constants $\alpha_i, \beta$ the hypersurface $W$ has only one singular point $(1:0:...:0)$ which is of type $A_1$.

On the other hand, if we take $\alpha_1 = 0$ we can arrange that $W$ acquires a single $A_2$ singularity. This shows that the inclusion $i_W: A_1 \rightarrow L$ factorizes as a composition $A_1 \rightarrow A_2 \rightarrow L$. Since $A_2$ is unimodular, it follows that its image in $\overline{L}$ is a direct summand and this ends the proof in this case.

The case of complete intersections can be treated similarly (see [16], (7.18)). □

**EXAMPLE 5.4:** Assume that on the $n$-dimensional ($n \geq 3$ odd) hypersurface $V$ of degree $d \geq d(S)$, the singularities correspond to one of the following symbols

<table>
<thead>
<tr>
<th>$S$</th>
<th>$d(S)$</th>
<th>$2A_1$</th>
<th>$A_1A_3$</th>
<th>$A_3$</th>
<th>$A_5$</th>
<th>$D_4$</th>
<th>$D_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Then $\ker \varphi_V = \text{Tors} \coker \varphi_V = 0$.

**PROOF:** The value of $d(S)$ is computed by the formula in (5.2.i) so that we may apply Proposition (5.1).

Hence it is enough to computer $\ker \varphi_W$ and $\text{Tors} \coker \varphi_W$, where $W$ is any hypersurface of degree $d$ with the singularities prescribed by one of the symbols $S$.

Note that the (affine) equation

$$x_1^d + ... + x_{n+1}^d + C(x_1, x_2, x_3) + \lambda_4 x_4^2 + ... + \lambda_{n+1} x_{n+1}^2 = 0$$

where $C$ is a generic cubic form and the constants $\lambda_i$ are chosen conveniently defines a hypersurface $W_0$ with a single singularity of type $\tilde{E}_6$.

A study of the versal deformation of $\tilde{E}_6$ [5] shows that the singularity $\tilde{E}_6$ deforms to the singularity $E_6$ which in turns deforms to any of the symbols $S$ in the list above (i.e. for any such symbol $S$ there is a fiber in the versal deformation of $E_6$ whose singularities are exactly those prescribed by $S$). Moreover, all these deformations can be performed using only monomials of degree $\leq 3$.

It follows that the corresponding morphisms $\varphi_W$ factorize through the unimodular lattice $E_6$ and this gives the result. □

**REMARK 5.5:** We have said nothing about the case of curves (i.e. when dim $V = 1$). If $V$ is irreducible, its homotopy type can be easily obtained from its normalization $\tilde{V}$ (the singularities of $V$ corresponding essentially to the identification of some points in $\tilde{V}$). We can safely leave the details for the reader.
6. The cohomology algebra and applications

Let $R$ be a commutative unitary ring. Assuming the integral homology $H_*(V)$ of our complete intersection $V$ known, the additive structure of $H^*(V, R)$ follows at once by the Universal Coefficient Theorem.

As to the cup-product, the only difficult point is to identify the pairing induced by it on the middle cohomology group

$$\alpha : H^n(V, R) \times H^n(V, R) \to H^{2n}(V, R) = R.$$ 

Let $U$ be a regular neighborhood of $V$ in $\mathbb{C}P^{n+p}$ (such that $V$ is a deformation retract of $U$). Let $\bar{V}$ denote a smooth complete intersection of the same dimension and multidegree as $V$, which is contained in $U$. The natural morphism

$$j : \bar{V} \subset U \xrightarrow{\psi} V$$

gives a homomorphism at homology level.

Using (1.4), (1.5) and (2.11) we see that via this homomorphism we can identify $H_n(V, R)$ with $H_n(\bar{V}, R)/I$, where

$$I = I(V) = \text{im} \varphi_V \otimes R \subset H_n(\bar{V}) \otimes R = H_n(\bar{V}, R)$$

Note that the Poincaré isomorphism

$$D : H_n(\bar{V}, R) \to \text{Hom}(H_n(\bar{V}, R), R) = H^n(\bar{V}, R)$$

is given by $D(u) = \langle u, \cdot \rangle$, where $\langle \ , \ \rangle$ is the intersection form on $H_n(V, R)$. Moreover, $D$ is compatible with the pairings on the source and the target (intersection form and cup-product).

The above description of $H_n(V, R)$ and the equality $H^n(V, R) = \text{Hom}(H_n(\bar{V}, R), R)$ shows that we can identify $H^n(V, R)$ with

$$\{ v \in H^n(\bar{V}, R) ; \ v|I = 0 \}.$$ 

Thus we get the following.

**Proposition 6.1:** The pairing $\alpha$ is isomorphic with the pairing induced by the intersection form $\langle \ , \ \rangle$ on the orthogonal complement of $I$ in $H_n(\bar{V}, R)$.

In particular, $H^n(V) = H^n(V, \mathbb{Z})$ is torsion free and hence is a lattice as defined in section 3.

Some information on this lattice can be easily derived from (6.1), using some general facts on lattices:
COROLLARY 6.2: (i) If the lattice \((I, \langle , \rangle | I)\) is nondegenerate, then \(H^n(V)\) is also nondegenerate, and

\[
\det H^n(V) = \det I \cdot |T(V)|^{-2}
\]

(ii) If moreover \(n\) is even, then

\[
\text{sign } H^n(V) = \text{sign } \bar{V} - \text{sign } I.
\]

When the Milnor lattices \(L_i\) are nondegenerate, then \(I = L_1 \oplus \ldots \oplus L_k\) and hence \(\det I = \prod \det L_i\) and \(\text{sign } I = \Sigma \text{sign } L_i\).

In some simple cases (e.g. those of section 4, or when \(T(V) = 0\) and one has uniqueness results as in Example (6.4) below) the Proposition 6.1 determines effectively the lattice \(H^n(V)\).

As an application, we consider the problem of deciding if certain varieties are or not homotopy equivalent.

EXAMPLE 6.3: Consider the complete intersections \(V_1, V_2\) and \(V_3\) (of the even dimension \(n\) and same multidegree) having the following type of singularities: \(3A_1, A_1A_2\) and respectively \(A_3\). Then the varieties \(V_i\) have the same homology and even the same real cohomology algebras. But these varieties have distinct integral cohomology algebras, as follows from the table (use (6.2.i))

<table>
<thead>
<tr>
<th>(i)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\det H^n(V_i))</td>
<td>8</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Hence \(V_i\) is not homotopy equivalent to \(V_j\) for \(i \neq j\).

EXAMPLE 6.4: Let \(X_1\) and \(X_2\) be complete intersections of even dimension \(n\) and same multidegree, having both the same type of singularities, namely \(3A_1\) (or \(A_1A_2\), or \(A_3\)). Then \(X_1\) and \(X_2\) have the same integral cohomology algebra. If moreover \(n = 2\), \(X_1\) is homotopy equivalent to \(X_2\).

PROOF: First note that \(T(X_1) = T(X_2) = 0\) by the results in section 4 and hence the inclusions \(I(X_i) \subset H^n(X_i)\) are primitive.

By Proposition 6.1, the discriminant form of the lattice \(H_i = H^n(X_i)\) is equal to the discriminant form \((D, -q)\) of the lattice \(I(X_i)\) with reversed sign ([17], 1.6.2). The lattices \(H_i\) are even iff \(d = \deg X_i\) is even. By Nikulin uniqueness results ([17], 1.13.3 and 1.16.10), the lattices \(H_i\) are isomorphic if:

(a) \(H_i\) are indefinite.

(b) Either \(d\) is even and \(\text{rk } H_i \geq l(D) + 2\), or \(d\) is odd and \(\text{rk } H_i \geq l(D) + 3\), where \(l(D)\) denotes the minimal number of generators of...
D. These conditions are fulfilled in most of the cases, the exceptions being covered in section 4.

When \( n = 2 \) and since there is no torsion, Whitehead's Theorem on the homotopy classification of 4-complexes (to be found for instance in [1], Chap. 2 together with many examples) can be easily applied. 

**Example 6.5:** Let \( V_1 \) be a cubic surface in \( \mathbb{CP}^3 \) with a single \( E_6 \) singularity. Let \( V_2 \) be the projective cone over the twisted cubic curve \( C \) in \( \mathbb{CP}^3 \) (i.e. \( C \) is the image of the Veronese embedding \( v_3 : \mathbb{CP}^1 \rightarrow \mathbb{CP}^3 \)).

Then \( V_1 \) and \( V_2 \) are homotopy equivalent.

**Proof:** By the Whitehead Theorem mentioned above, it is enough to show that \( V_1 \) and \( V_2 \) have the same integral (torsion free) cohomology algebra. The cohomology algebra for \( V_2 \) is computed in [1, p. 72] while the cohomology algebra for \( V_1 \) is easily derived from (6.1).

It is interesting to note that \( V_1 \) and \( V_2 \) are not homeomorphic (compare the local fundamental groups at the singular points!)

Similarly, the projective cone over the image of the Veronese embedding \( v_4 : \mathbb{CP}^1 \rightarrow \mathbb{CP}^4 \) and a complete intersection of two quadrics in \( \mathbb{CP}^4 \) with a single \( D_5 \) singularity are homotopy equivalent.

**References**


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