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ON ARITHMETIC QUOTIENTS OF THE SIEGEL UPPER HALF SPACE OF DEGREE TWO

Joachim Schwermer

§0. Introduction

In the arithmetic theory of automorphic forms with respect to congruence subgroups Γ of the special linear group $SL_2(k)$ over an algebraic number field the study of various cohomology groups attached to Γ is very useful. This approach combines analytic, arithmetic and algebraic-geometric methods and has led, for example, to the results of Langlands [14] or to those of Harder [8], [9], [10] on arithmetic properties of special values of L -functions attached to algebraic Hecke characters.

As one aspect of this study one analyzes the de Rham cohomology groups of the associated arithmetic quotient $\Gamma \backslash X$ (where X denotes the corresponding symmetric space) and their relation to automorphic forms. Here the connection with the theory of Eisenstein series as developed by Selberg and Langlands [15] is of particular interest. First of all, this theory allows us to construct cohomology classes which are represented by values of Eisenstein series and which completely describe the cohomology of $\Gamma \backslash X$ at infinity, i.e. that part of the cohomology of the Borel-Serre compactification $\Gamma \backslash \bar{X}$ of $\Gamma \backslash X$ which restricts non-trivially to the cohomology of its boundary $\partial(\Gamma \backslash \bar{X})$. Moreover, these Eisenstein cohomology classes have interesting algebraic or arithmetic properties which led, for example, to the results mentioned above.

One may ask if and how these ideas can be generalized to other groups. There are some results in particular cases (cf. [7], [10], [21], [22], [23]) but for groups of \mathbb{Q} -rank greater than one there is no complete understanding of the Eisenstein cohomology. In particular, those groups are of interest for which the de Rham cohomology of Γ has an interpretation in étale theory i.e. as cohomology of an underlying arithmetic variety.

The object of this paper is to begin the study of these ideas in the case of an arithmetic torsion free subgroup $\Gamma \subset G(\mathbb{Z})$, $G = Sp_4$ the symplectic group of degree two. The associated symmetric space X is the Siegel upper half space of degree two, and the corresponding arithmetic quotient $\Gamma \backslash X$ is a 6-dimensional non-compact complete Riemannian mani-

fold of finite volume. By the general construction of Borel and Serre [3] it can be viewed as the interior of a compact manifold $\Gamma \backslash \bar{X}$ with corners. Its boundary $\partial(\Gamma \backslash \bar{X})$ is a disjoint union of faces $e'(P)$ corresponding to the Γ -conjugacy classes of proper parabolic \mathbb{Q} -subgroups of G . The $G(\mathbb{Q})$ -conjugacy classes of these parabolics fall into three classes, two conjugacy classes \mathcal{P}_1 and \mathcal{P}_2 of maximal parabolic subgroups and one class \mathcal{P}_0 of minimal ones. Then the first result is that

$$\partial(\Gamma \backslash \bar{X}) = \bar{Y}_1 \bigcup_{Y_0} \bar{Y}_2 \quad (1)$$

where Y_i is the union of the $e'(P)$ with $P \in \mathcal{P}_i$ (modulo Γ) and its compactifications \bar{Y}_1, \bar{Y}_2 are 5-dimensional manifolds with common boundary Y_0 .

By an analysis of the Mayer-Vietoris sequence associated to the decomposition (1) we determine in 2.7 the cohomology of the boundary $\partial(\Gamma \backslash \bar{X})$. More precisely, if $\Gamma = \Gamma(m)$, $m \geq 3$, is a full congruence subgroup of $Sp_4(\mathbb{Z})$ we give a description of $H^*(\partial(\Gamma \backslash \bar{X}), \mathbb{C})$ as a representation space for the finite group ${}_fG = Sp_4(\mathbb{Z})/\Gamma(m)$. It consists of induced representations from subspaces of the cohomology of the individual faces in $\partial(\Gamma \backslash \bar{X})$ (which is described in 2.5.) together with some “non-geometric” pieces. Out of this result we derive a formula (2.9) for the dimension of the image of the restriction

$$r^q: H^q(\Gamma \backslash \bar{X}; \mathbb{C}) \rightarrow H^q(\partial(\Gamma \backslash \bar{X}); \mathbb{C}) \quad (2)$$

of the cohomology of $\Gamma \backslash \bar{X}$ onto the cohomology of its boundary. The formula involves only the dimensions of the spaces of cusp forms of weight $k = 2, 3, 4$ with respect to the congruence subgroup of level m in $SL_2(\mathbb{Z})$ and the number of $\Gamma(m)$ -conjugacy classes of parabolic \mathbb{Q} -subgroups P in \mathcal{P}_i , $i = 0, 1, 2$, all of which can be explicitly computed in terms of m . We are very brief in the proofs of §2 since the investigation just described runs methodologically along similar lines as for SL_3/\mathbb{Q} (cf. [16], [22] §7) though there are some new features.

As indicated, the theory of Eisenstein series can now be used to construct by analytical means a subspace $H_{\text{Eis}}^*(\Gamma \backslash \bar{X}, \mathbb{C})$ in $H^*(\Gamma \backslash \bar{X}, \mathbb{C})$ which is generated by Eisenstein cohomology classes (i.e. classes with a representative given by a regular value of an Eisenstein series attached to classes in $H^*(\partial(\Gamma \backslash \bar{X}), \mathbb{C})$ or a residue of such) and which restricts isomorphically under the restriction r^* : $H^*(\Gamma \backslash \bar{X}, \mathbb{C}) \rightarrow H^*(\partial(\Gamma \backslash \bar{X}), \mathbb{C})$ onto the image of r^* . This result leads to a direct sum decomposition of the cohomology of Γ

$$H^*(\Gamma \backslash \bar{X}, \mathbb{C}) = H_I^*(\Gamma \backslash \bar{X}, \mathbb{C}) \oplus H_{\text{Eis}}^*(\Gamma \backslash \bar{X}, \mathbb{C}) \quad (3)$$

where $H_I^*(\Gamma \backslash \bar{X}, \mathbb{C})$ denotes the image in $H^*(\Gamma \backslash \bar{X}, \mathbb{C})$ of the cohomol-

ogy of $\Gamma \backslash X$ with compact supports under the natural map. The space $H_{\text{Eis}}^*(\Gamma \backslash \bar{X}, \mathbb{C})$ decomposes naturally as a direct sum

$$H_{\text{Eis}}^*(\Gamma \backslash \bar{X}, \mathbb{C}) = H_{\text{max}}^*(\Gamma \backslash \bar{X}, \mathbb{C}) \oplus H_{\text{min}}^*(\Gamma \backslash \bar{X}, \mathbb{C}) \tag{4}$$

where H_{max}^* (resp. H_{min}^*) is built up by Eisenstein cohomology classes attached to cuspidal classes (cf. 3.1.) on the faces $e'(P)$ corresponding to the Γ -conjugacy classes of maximal (resp. minimal) parabolic \mathbb{Q} -subgroups of G .

However, the actual construction of the classes in $H_{\text{Eis}}^*(\Gamma \backslash \bar{X}, \mathbb{C})$ has to be done on different methodological levels depending on the type of the cuspidal class on $e'(P)$ we start with. For building up $H_{\text{max}}^4(\Gamma \backslash \bar{X}, \mathbb{C})$ it is sufficient to consider each face $e'(P)$, $P \in \mathcal{P}_i/\Gamma$, $i = 1, 2$, individually; here I can avail myself of the general results [22], I concerning the construction of Eisenstein classes in this setting and their restrictions to the cohomology of the faces in $\partial(\Gamma \backslash \bar{X})$. But in dealing with the other cases I have to consider the faces $e'(P)$, $P \in \mathcal{P}_i/\Gamma$, $i = 0, 1, 2$, simultaneously, and this can only be done in the adelic language in a satisfactory manner. This is mostly due to the fact that one has to take residues of Eisenstein series into account. Unfortunately, the analogue of [22] in the adelic setting is not yet at hand (but in preparation). Therefore, I shall confine myself in this paper to construct in detail $H_{\text{max}}^4(\Gamma \backslash \bar{X}; \mathbb{C})$ (cf. §3). However, for the convenience of the reader I shall briefly sketch in §4 the complete construction of $H_{\text{Eis}}^*(\Gamma \backslash \bar{X}, \mathbb{C})$, describe its image under the restriction to the cohomology of the boundary and indicate the ideas of proof in an adelic setting. I point out that the methods used in §3 (cf. [22]) are sufficient to construct completely the Eisenstein cohomology $H_{\text{Eis}}^*(\Gamma \backslash \bar{X}, \tilde{E})$ if we are dealing with a twisted coefficient system \tilde{E} given by a finite dimensional representation E of $Sp_4(\mathbb{Q})$ with sufficiently regular highest weight. One has a decomposition as in (3) also for $H^*(\Gamma \backslash X; \tilde{E})$. In both cases the results of §2 about the exact size of the cohomology of Γ at infinity have to be used to obtain (3) (cf. 4.5.).

Conventions. (1) The algebraic groups considered are linear and can be identified with algebraic subgroups of some $GL_n(\mathbb{C})$. We mainly use the notations in [1]. For a (Zariski)-connected \mathbb{Q} -group H we denote by $H = H(\mathbb{R})$ the group of real points of H . An arithmetic subgroup of H is a subgroup of $H(\mathbb{Q})$ which is commensurable with $\psi(H) \cap GL_n(\mathbb{Z})$ for some injective morphism $\psi: H \rightarrow GL_n$ defined over \mathbb{Q} . We put ${}^0H = \cap \ker \chi^2$ for a connected \mathbb{Q} -group H where χ runs through the group $X_{\mathbb{Q}}(H)$ of \mathbb{Q} -morphisms from H to GL_1 ([3], 1.2.). The group ${}^0H(\mathbb{R})$ of real points contains each arithmetic subgroups of H and each compact subgroup of $H(\mathbb{R})$.

(2) Let M be a smooth manifold, and E a finite-dimensional vector space over \mathbb{C} . Then $C^\infty(M, E)$ denotes the space of smooth functions

with values in E , and $\Omega^q(M, E)$ the space of smooth E -valued differential q -forms on M ($q = 0, 1, \dots$). Let $\Omega^*(M, E)$ be the direct sum of the $\Omega^q(M, E)$ endowed with exterior differentiation. If $E = \mathbb{C}$ it will be omitted from the notation; this applies also to the associated de Rham cohomology groups and the singular cohomology as well.

(3) We denote by \mathbf{A} (resp. \mathbf{A}_f) the ring of adèles (resp. finite adèles) of the number field \mathbb{Q} .

§1. Preliminaries on Sp_4 , roots and parabolics

1.1. Let G be the \mathbb{Q} -split algebraic \mathbb{Q} -group Sp_4/\mathbb{Q} , i.e. the symplectic group of degree two. The group $G(\mathbb{R})$ of real points of G will be denoted by G and we have

$$G = Sp_4(\mathbb{R}) = \{ \alpha \in GL_4(\mathbb{R}) \mid \alpha' J \alpha = J \} \quad \text{with } J = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}.$$

We fix as maximal compact subgroup of G the group $K = G \cap O(4)$, i.e. the group of orthogonal symplectic matrices.

Let P be a parabolic subgroup of G defined over \mathbb{Q} , N its unipotent radical and $\kappa: P \rightarrow P/N = M$ the canonical projection. A split component of $P = P(\mathbb{R})$ is by definition a subgroup A of a Levi subgroup of P such that A is mapped isomorphically via κ onto the identity component of the group $S_P(\mathbb{R})$ where S_P is the maximal central \mathbb{Q} -split torus of M . By A_P we denote the unique split component of P which is stable under the Cartan involution Θ associated to K . We then let $M = Z_G(A_P)$ be the unique Θ -stable Levi subgroup of P , and we denote by 0M the inverse image of ${}^0M(\mathbb{R})$ under the isomorphism $M \xrightarrow{\sim} P/N = M(\mathbb{R})$ induced by κ . We then have $P = M \cdot N$ as a semidirect product, $P = A_P \times {}^0P$ and $M = {}^0M \times A_P$. Since M is Θ -stable, one has that $K_P := K \cap P = K \cap {}^0M$ is a maximal compact subgroup of 0M , M and P .

We fix the Cartan subalgebra \mathfrak{h} of $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{R})$ formed by the matrices

$$h = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & -h_1 & \\ & & & -h_2 \end{pmatrix} \in \mathfrak{g}, \quad h_i \in \mathbb{R}, \tag{1}$$

and we let $\Phi = \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the set of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. Its elements will also be viewed as roots of $G_{\mathbb{C}}$ with respect to $H = Z_G(\mathfrak{h})$. We choose an ordering on Φ in such a way that the weights of $\mathfrak{h}_{\mathbb{C}}$ in the orthogonal complement \mathfrak{p} of \mathfrak{k} in \mathfrak{g} with respect to the Killing form are positive, and denote by Δ (resp. Φ^+) the set of simple (resp. positive) roots with respect to the chosen ordering. Then $\Delta = \{ \alpha_1, \alpha_2 \}$, where

$\alpha_1(h) = h_1 - h_2$ and $\alpha_2(h) = 2h_2$ with $h \in \mathfrak{h}$ as in (1). The Weyl group W of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ is then generated by the simple reflections w_i associated to α_i . We recall that we have

$$w_i(\alpha_j) = -\alpha_j \quad \text{resp.} \quad \begin{cases} w_1(\alpha_2) = \alpha_2 + 2\alpha_1 \\ w_2(\alpha_1) = \alpha_1 + \alpha_2 \end{cases} \quad (2)$$

Since G is split over \mathbb{Q} we may (and will) identify Φ and $\Phi_{\mathbb{R}}$.

1.2. The set of parabolic \mathbb{Q} -subgroups of G will be denoted by \mathcal{P} . The conjugacy classes of elements in \mathcal{P} are parametrized by the subsets J of Δ . In particular, if Q is a minimal parabolic \mathbb{Q} -subgroup of G , then it is conjugate to the standard one $P_{\beta} = P_0$

$$P_0 = \left\{ \begin{pmatrix} a_1 & u_1 & v & s \\ 0 & a_2 & s & u_2 \\ & & a_1^{-1} & 0 \\ 0 & & -u_1 & a_2^{-1} \end{pmatrix} \in G \right\}$$

whose decomposition $P_0 = {}^0M_0 \cdot A_0 \cdot N_0$ is given by

$$A_0 = \left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1} \end{pmatrix} \mid a_i \in \mathbb{R}^+ \right\} \quad (2)$$

$${}^0M_0 = \left\{ \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \epsilon_1 & \\ & & & \epsilon_2 \end{pmatrix} \in G \mid \epsilon_i = \pm 1 \right\}$$

$$N_0 = \left\{ \begin{pmatrix} 1 & u_1 & v & s \\ 0 & 1 & s & u_2 \\ & & 1 & 0 \\ & & -u_1 & 1 \end{pmatrix} \in G \mid u_i, v, s \in \mathbb{R} \right\}.$$

If Q is a maximal parabolic \mathbb{Q} -subgroup of G , then it is conjugate to a standard one

$$P_i := P_{\Delta - \{\alpha_i\}} = Z(H_{\Delta - \{\alpha_i\}}) \cdot N_i \supset P_0$$

given as the semidirect product of the unipotent radical

$$N_1 = \left\{ \begin{pmatrix} 1 & u_1 & v & s \\ 0 & 1 & s & 0 \\ & 0 & 1 & 0 \\ & & -u_1 & 1 \end{pmatrix} \in G \right\}$$

$$\text{resp. } N_2 = \left\{ \begin{pmatrix} 1 & 0 & v & s \\ 0 & 1 & s & u_2 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \in G \right\}$$

by the centralizer of $H_{\Delta - \{\alpha_i\}}$, $i = 1, 2$, where we denote $H_J = (\bigcap_{\alpha \in J} \ker \alpha)^0 \subset H$ for a subset J of Δ . Note that the characters of H in N_i , $i = 1, 2$, are exactly the positive roots which contain at least one simple root not in $\Delta - \{\alpha_i\}$. The unique Θ -stable split component is given by

$$A_1 = \left\{ \begin{pmatrix} a & & & \\ & 1 & & \\ & & a^{-1} & \\ & & & 1 \end{pmatrix} \mid a \in \mathbb{R}^+ \right\}$$

$$\text{resp. } A_2 = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^+ \right\}.$$

Since M_1 is the direct product $SL_2(\mathbb{R}) \times GL_1(\mathbb{R})$, resp. $M_2 \cong GL_2(\mathbb{R})$ we have

$${}^0M_1 = \left\{ \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & x_1 & 0 & u_2 \\ 0 & 0 & \epsilon & 0 \\ 0 & x_2 & 0 & x_3 \end{pmatrix} \in G \mid \begin{pmatrix} x_1 & u_2 \\ x_2 & x_3 \end{pmatrix} \in SL_2(\mathbb{R}), \epsilon = \pm 1 \right\},$$

$${}^0M_2 = \left\{ \begin{pmatrix} A & \\ & {}_tA^{-1} \end{pmatrix} \in G \mid A \in SL_2^\pm(\mathbb{R}) \right\}.$$

The set of simple roots of $M_i = Z(A_i) = Z(H_{\Delta - \{\alpha_i\}})$ is $\Delta_{M_i} = \{\alpha_j\}$, $i \neq j$, $i, j = 1, 2$.

Since a maximal parabolic \mathbb{Q} -subgroup P of G is conjugate to its opposite \bar{P} , the class $\mathcal{C}(P)$ of maximal parabolic \mathbb{Q} -subgroups Q associated to P coincides with the conjugacy class of P . (For a definition of the notion ‘‘associate’’ we refer to II, 4 [12].)

§2. The boundary of the Borel-Serre compactification and its cohomology

2.1. Let Γ be a torsion free arithmetic subgroup of $G(\mathbb{Q}) = Sp_4(\mathbb{Q})$. The group Γ operates properly and freely on the associated symmetric space $X = G/K$, and the quotient $\Gamma \backslash X$ is a 6-dimensional non-compact $K(\Gamma, 1)$ -manifold of finite volume. This quotient $\Gamma \backslash X$ may be identified to the interior of a compact manifold $\Gamma \backslash \bar{X}$ with corners [3]; the inclusion $\Gamma \backslash X \rightarrow \Gamma \backslash \bar{X}$ is a homotopy equivalence. The boundary $\partial(\Gamma \backslash \bar{X})$ is a disjoint union of a finite number of faces $e'(Q)$ which correspond bijectively to the Γ -conjugacy classes of proper parabolic \mathbb{Q} -subgroups of G . For a given P in \mathcal{P} with $P = MAN$ as in 1.1. the projection $\kappa: P \rightarrow P/N$ induces an isomorphism $\mu: M \xrightarrow{\sim} P/N$. We put $\Gamma_P = \Gamma \cap P$ resp. $\Gamma_N = \Gamma \cap N$. Then the projection $\Gamma_M = \kappa(\Gamma_P)$ is an arithmetic subgroup of P/N , and $K_M = \kappa(K \cap P)$ is a maximal compact subgroup of P/N which is isomorphic to $K \cap M$ via μ . The quotient $Z_M = ({}^0P/N)/K_M$ is again a symmetric space which will also be viewed as ${}^0M/K \cap M$. Then the face $e'(P)$ corresponding to the Γ -conjugacy class of P is defined as $e'(P) = \Gamma_P \backslash {}^0P/K \cap P$ and it inherits from κ a fiber bundle structure

$$\Gamma_N \backslash N \rightarrow e'(P) = \Gamma_P \backslash {}^0P/K \cap P \rightarrow \Gamma_M \backslash Z_M. \tag{1}$$

The fibers are compact manifolds, and the base space of the fibration may be identified with $\mu^{-1}(\Gamma_M) \backslash {}^0M/K \cap M$.

If P is a maximal parabolic \mathbb{Q} -subgroup of G of type $i = 1, 2$, then $e'(P)$ is a 5-dimensional manifold fibered over a non-compact 2-dimensional manifold $\Gamma_M \backslash Z_M$, (homeomorphic to some arithmetic quotient $\Gamma_i^* \backslash SL_2(\mathbb{R})/SO(2)$) with fiber a 3-dimensional nil-manifold for $i = 1$ (resp. torus for $i = 2$).

If P is a minimal parabolic \mathbb{Q} -subgroup of G of type 0, then $e'(P) = \Gamma_n \backslash N$ is a 4-dimensional compact manifold which can be fibered in two different ways as a fiber bundle over S^1 . For the standard one $P = P_0$ these fibrations $f_i, i = 1, 2$, are induced by the projection maps

$$N_0 \xrightarrow{P_i} U_i, i = 1, 2 \tag{2}$$

$$\begin{pmatrix} 1 & u_1 & v & s \\ 0 & 1 & s & u_2 \\ & 0 & 1 & 0 \\ & & -u_1 & 1 \end{pmatrix} \xrightarrow{p_2} \begin{pmatrix} 1 & u_1 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ 0 & & -u_1 & 1 \end{pmatrix}$$

$$\text{resp. } \xrightarrow{p_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & u_2 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix},$$

of N_0 onto the unipotent radical U_i of the standard Borel subgroup of ${}^0M_i^0$, and one gets

$$\Gamma_{N_i} \backslash N_i \rightarrow e'(P_0) \xrightarrow{f_i} \Gamma_{U_i} \backslash U_i \tag{3}$$

where $\Gamma_{U_i} = \Gamma \cap U_i$.

2.2. We now consider the full congruence subgroup $\Gamma(m) = \{A \in Sp_4(\mathbb{Z}) \mid A \equiv \text{Id} \pmod{m}\}$ of $Sp_4(\mathbb{Z})$ for a given $m \geq 3$. We fix m once and for all and will write $\Gamma = \Gamma(m)$. This also justifies the notation

$${}_fG := \Gamma(m) \backslash Sp_4(\mathbb{Z}) \tag{1}$$

for the finite group, which depends on m . The group $Sp_4(\mathbb{Z})$ operates in a natural way on the Borel-Serre compactification $\Gamma \backslash \bar{X}$ and so also ${}_fG$ acts on $H^*(\Gamma \backslash \bar{X})$ resp. $H^*(\partial(\Gamma \backslash \bar{X}))$. In a similar way the faces $e'(P_i)$, $i = 0, 1, 2$ are acted upon by $P_i \cap Sp_4(\mathbb{Z})$ and we put for $i = 0, 1, 2$

$$\begin{aligned} {}_fP_i &= \Gamma(m)_{P_i} \backslash (Sp_4(\mathbb{Z}) \cap P_i) \quad \text{resp.} \quad {}_fN_i = \Gamma(m)_{N_i} \backslash (Sp_4(\mathbb{Z}) \cap N_i) \\ {}_fM_i &= (\Gamma(m) \cap M_i) \backslash (Sp_4(\mathbb{Z}) \cap M_i). \end{aligned} \tag{2}$$

We have $\Gamma(m)_{P_i} = \Gamma(m)_{M_i} \cdot \Gamma(m)_{N_i}$ in this case and ${}_fP_i$ is a split group extension of ${}_fM_i$ by ${}_fN_i$. Observe that the ${}_fP_i$ are in general *not* parabolic subgroups of ${}_fG$.

According to [3] there is a natural compactification $\overline{e'(P_i)}$ of $e'(P_i)$, $i = 1, 2$, which adds over each cusp of the base $\Gamma_{M_i} \backslash Z_{M_i}$ a 4-dimensional nilmanifold. At the cusp corresponding to the $(\Gamma(m) \cap {}^0M_i)$ -conjugacy class of the standard Borel subgroup of ${}^0M_i^0$ this nilmanifold is exactly the fibered manifold $f_i: e'(P_0) \rightarrow \Gamma_{U_i} \backslash U_i$. In particular, the action of $P_i \cap Sp_4(\mathbb{Z})$ extends to one on $\overline{e'(P_i)}$, and the group ${}_fP_i$ acts transitively on the boundary components of $\overline{e'(P_i)}$. Indeed, one has an ${}_fP_i$ -equivariant diffeomorphism

$$\beta_i: {}_fP_i \times_{{}_fP_0} e'(P_0) \xrightarrow{\sim} \partial(\overline{e'(P_i)}). \tag{3}$$

As here, we now put for $i = 0, 1, 2$ resp. $i = 1, 2$

$$Y_i = {}_fG \times_{{}_fP_i} e'(P_i) \quad \text{resp.} \quad \bar{Y}_i = {}_fG \times_{{}_fP_i} \overline{e'(P_i)} \tag{4}$$

which is a disjoint union of copies of $e'(P_i)$ resp. $\overline{e'(P_i)}$ and has a natural action of ${}_fG$ which extends the one of ${}_fP_i$ on $e'(P_i)$ resp. $\overline{e'(P_i)}$.

The manifolds \bar{Y}_i are compact and one has an ${}_fG$ -equivariant diffeomorphism

$$\alpha_i: Y_0 \xrightarrow{\sim} \partial(\bar{Y}_i) = {}_fG \times_{\substack{P_i \\ e'(P_i)}} \partial(\overline{e'(P_i)}), \quad i = 1, 2, \tag{5}$$

defined by $\text{Id} \times \beta_i$ onto the boundary $\partial(\bar{Y}_i)$. Using the α_i we get a closed 5-dimensional manifold (with an action by ${}_fG$)

$$Y = \bar{Y}_1 \bigcup_{Y_0} \bar{Y}_2 \tag{6}$$

by gluing together \bar{Y}_1 and \bar{Y}_2 along their common boundaries. By the same arguments as in [18], §6 one then obtains

2.3. PROPOSITION: *The boundary $\partial(\Gamma(m) \backslash \bar{X})$ of the Borel-Serre compactification $\Gamma(m) \backslash \bar{X}$ of $\Gamma(m) \backslash X$ is equivariant diffeomorph with respect to the action of ${}_fG$ to the manifold Y given in 2.2.(6).*

2.4. In order to describe $H^*(\partial(\Gamma(m) \backslash \bar{X}))$ and to determine the ${}_fG$ -action on this space we will analyse the Mayer-Vietoris sequence in cohomology attached to the decomposition $\partial(\Gamma(m) \backslash \bar{X}) = \bar{Y}_1 \cup \bar{Y}_2$. Since we have as ${}_fG$ -modules

$$H^*(\bar{Y}_i) = H^*\left({}_fG \times_{\substack{P_i \\ e'(P_i)}} \overline{e'(P_i)}\right) = \text{Ind}_{P_i}^G [H^*(\overline{e'(P_i)})] \tag{1}$$

(where $\text{Ind}_{P_i}^G [\]$ denotes the representation of ${}_fG$ induced from the representation of ${}_fP_i$ on $H^*(\overline{e'(P_i)})$) we will start with a description of the cohomology $H^*(e'(P_i)) = H^*(\overline{e'(P_i)})$ as ${}_fP_i$ -module.

We consider the fibration $\Gamma_{N_i} \backslash N_i \rightarrow e'(P_i) \rightarrow \Gamma_{M_i} \backslash Z_{M_i}$ for $i = 1, 2$ resp. $e'(P_0) = \Gamma_{N_0} \backslash N_0$. The cohomology of $\Gamma_{N_i} \backslash N_i$ can be identified with the cohomology of the Lie algebra \mathfrak{n}_i of N_i . Via this identification $H^*(\mathfrak{n}_i, \mathbb{C}) = H^*(\Gamma_{N_i} \backslash N_i)$ (cf. [22], 2.2.) the natural M_i -module structure on $H^*(\mathfrak{n}_i, \mathbb{C})$ restricts to the action of Γ_{M_i} on $H^*(\Gamma_{M_i} \backslash M_i)$ which inherits therefore by extension a natural M_i -module structure. If we put

$$W^{P_i} = \{ w \in W \mid w^{-1}(\Delta_{M_i}) \subset \Phi^+ \} \tag{2}$$

then there is an isomorphism of M_i -modules ([13], 5.13) due to Kostant

$$H^q(\mathfrak{n}_i, \mathbb{C}) = \bigoplus_{\substack{w \in W^{P_i} \\ l(w) = q}} F_{w(\rho) - \rho}, \quad i = 0, 1, 2, \tag{3}$$

where F_ν denotes an irreducible $M_{i,\mathbb{C}}$ -module with highest weight $\nu \in$

$(\mathfrak{m}_i \cap \mathfrak{h})_{\mathbb{C}}^*$ and $l(w)$ the length of an element in W . One easily checks (using 1.2. and 1.1.(2))

$$\begin{aligned} W^{P_0} &= W \\ W^{P_1} &= \{1, w_1, w_1 \circ w_2, w_1 \circ w_2 \circ w_1\}, \\ W^{P_2} &= \{1, w_2, w_2 \circ w_1, w_2 \circ w_1 \circ w_2\}. \end{aligned} \tag{4}$$

Since $H^*(e'(P_0)) = H^*(\mathfrak{n}_0, \mathbb{C})$ decomposition (3) already determines $H^*(e'(P_0))$ as ${}_fP_0$ -module. The unipotent radical N_0 operates trivially on $H^*(\mathfrak{n}_0, \mathbb{C})$, so the action of ${}_fP_0$ on $F_{w(\rho)-\rho}$ is given by the action of the commutative group ${}_fM_0$ consisting of four elements. Therefore for a given $w \in F_{w(\rho)-\rho}$ contributes to the ${}_fP_0$ -module $H^*(e'(P_0))$ as a one-dimensional representation (η_w, F_{η_w}) of ${}_fP_0$ which is obtained by evaluating $w(\rho) - \rho$ on 0M_0 , hence on ${}_fM_0$ and extending it trivially to ${}_fN_0$, i.e. we have as ${}_fP_0$ -module

$$H^*(e'(P_0)) = \bigoplus_{w \in W} F_{\eta_w}. \tag{5}$$

Associated to the fibration 2.1.(1) of the faces $e'(P_i)$, $i = 1, 2$, there is a spectral sequence which converges to the cohomology of $e'(P_i)$, and whose E_2 -term is given by

$$E_2^{p,q} = H^p(\Gamma_{M_i} \backslash Z_{M_i}, H^q(\mathfrak{n}_i, \mathbb{C})).$$

Since the base space is of type $\Gamma' \backslash H$ ($H =$ upper half plane, $\Gamma' \subset SL_2(\mathbb{Z})$ of finite index) and $H^p(\Gamma' \backslash H, \tilde{E}) = 0$, $p > 1$, for an arbitrary coefficient system \tilde{E} we have $E_2^{p,q} = 0$ for $p > 1$, $q > 3$ and the spectral sequence degenerates at E_2 . We obtain

$$H^r(e'(P_i)) = \bigoplus_{p+q=r} H^p(\Gamma_{M_i} \backslash Z_{M_i}, H^q(\mathfrak{n}_i, \mathbb{C})).$$

The M_i -module structure of $H^q(\mathfrak{n}_i, \mathbb{C})$ is easily determined by means of 2.4.(3). One gets

2.5. PROPOSITION: *The cohomology $H^*(e'(P_i))$ of the face $e'(P_i)$, $i = 1, 2$, is given as ${}_fP_i$ -module by*

$$H^0(e'(P_i)) = \mathbb{C} \quad H^q(e'(P_i)) = 0, \quad q > 4$$

$$H^1(e'(P_i)) = H^1(\Gamma_{M_i} \backslash Z_{M_i}; \mathbb{C})$$

$$H^2(e'(P_i)) = H^1(\Gamma_{M_i} \backslash Z_{M_i}; H^1(\mathfrak{n}_i, \mathbb{C}))$$

$$\begin{aligned}
 H^3(e'(P_i)) &= H^1(\Gamma_{M_i} \backslash Z_{M_i}; H^2(n_i, \mathbb{C})) \oplus \\
 &\quad \oplus H^0(\Gamma_{M_i} \backslash Z_{M_i}, H^3(n_i, \mathbb{C})) \\
 H^4(e'(P_i)) &= H^1(\Gamma_{M_i} \backslash Z_{M_i}; H^3(n_i, \mathbb{C})).
 \end{aligned}$$

If we denote by $E_k(i)$ resp. $E_k^\sigma(i)$ the k -dimensional irreducible representation of 0M_i with ${}^0M_1 \cong SL_2(\mathbb{R}) \times \{\pm 1\}$ and ${}^0M_2 \cong SL_2^\pm(\mathbb{R}) = SL_2(\mathbb{R}) \rtimes \{\pm 1\}$ where (-1) operates trivially resp. non-trivially then we have the following isomorphisms of 0M_i -modules:

$$\begin{aligned}
 i = 1: \quad & H^0(n_1, \mathbb{C}) \cong \mathbb{C} & q = 0 \\
 & H^q(n_1, \mathbb{C}) \cong E_2^\sigma(1) & q = 1, 2 \\
 & H^3(n_1, \mathbb{C}) \cong \mathbb{C} & q = 3 \\
 i = 2: \quad & H^0(n_2, \mathbb{C}) \cong \mathbb{C} & q = 0 \\
 & H^1(n_2, \mathbb{C}) \cong E_3(2) & q = 1 \\
 & H^2(n_2, \mathbb{C}) \cong E_3^\sigma(2) & q = 2 \\
 & H^3(n_2, \mathbb{C}) \cong E_1^\sigma(2) & q = 3.
 \end{aligned}$$

The action of ${}_fP_i$ on $H^p(\Gamma_{M_i} \backslash Z_{M_i}, H^q(n_i, \mathbb{C}))$ is the pullback of the action of ${}_fM_i$ on these cohomology spaces obtained by the natural action of ${}^0M_i \cap Sp_4(\mathbb{Z})$ on $\Gamma_{M_i} \backslash Z_{M_i}$, resp. the action on $H^q(n_i, \mathbb{C})$ just described.

2.6. In order to determine $H^*(\partial(\Gamma(m) \backslash \bar{X}))$ we now study the kernel resp. cokernel of the morphism $\alpha_1^* \oplus \alpha_2^*$ in the Mayer-Vietoris sequence

$$\begin{aligned}
 \rightarrow H^q(\partial(\Gamma \backslash \bar{X})) &\rightarrow H^q(\bar{Y}_1) \oplus H^q(\bar{Y}_2) \xrightarrow{\alpha_1^* \oplus \alpha_2^*} H^q(Y_0) \rightarrow \\
 &\rightarrow H^{q+1}(\partial(\Gamma \backslash \bar{X})) \tag{1}
 \end{aligned}$$

attached to the decomposition $\partial(\Gamma \backslash \bar{X}) = \bar{Y}_1 \cup \bar{Y}_2$ of the boundary.

Since $\partial(\Gamma \backslash \bar{X})$ is connected we get for degree 0 by 2.4.(1) and (5) resp. 2.5. the following short exact sequence of ${}_fG$ -modules

$$0 \rightarrow \mathbb{C} \rightarrow \bigoplus_{i=1}^2 \text{Ind}_{/P_i}^G[\mathbb{C}] \rightarrow \text{Ind}_{/P_0}^G[\mathbb{C}] \rightarrow \text{coker}(\alpha_1^0 \oplus \alpha_2^0) \rightarrow 0. \tag{2}$$

The ${}_fG$ -module $\text{coker}(\alpha_1^0 \oplus \alpha_2^0)$ is uniquely determined by (2) and will be denoted (analogue the Steinberg module) by $St(m)$.

Before we consider the situation in degree 1 we recall that the space $e'(P_0) = \Gamma_{N_0} \backslash N_0$ can be fibered in two different ways (2.1.(3))

$$\Gamma_{N_i} \backslash N_i \rightarrow e'(P_0) = Y_0 \xrightarrow{f_i} \Gamma_{U_i} \backslash U_i, \quad i = 1, 2.$$

The associated spectral sequence written in Lie algebra cohomology terms

$$E_2^{p,q} = H^p(\mathfrak{n}_0/\mathfrak{n}_i, H^q(\mathfrak{n}_i)) \Rightarrow H^{p+q}(\mathfrak{n}_0) \tag{3}$$

degenerates at E_2 . This is a consequence of Kostants result (cf. 2.4.(3)), for example.

We now consider the morphism in degree 1 (using 2.4.(1) and 2.5.)

$$\alpha_1^1 \oplus \alpha_2^1: \bigoplus_{i=1}^2 \text{Ind}_{\mathcal{P}_i}^G [H^1(\overline{e'(P_i)})] \rightarrow \text{Ind}_{\mathcal{P}_0}^G [\text{Ind}_{\mathcal{P}_0}^{\mathcal{P}_i} [H^1(e'(P_0))]] \tag{4}$$

where the right hand side is also a sum of two terms by 2.4.(3)–(5). By an analysis of the \mathcal{P}_i -module structure on both sides given in 2.4.(5) resp. 2.5. and (3) it turns out that $\alpha_1^1 \oplus \alpha_2^1$ is the sum of the two \mathcal{P}_i -morphisms induced from the \mathcal{P}_i -morphisms

$$\begin{aligned} H^1(\overline{e'(P_i)}) &= H^1(\Gamma_{M_i} \backslash \overline{Z}_{M_i}, H^0(\mathfrak{n}_i)) \rightarrow \\ &\rightarrow \text{Ind}_{\mathcal{P}_0}^{\mathcal{P}_i} [H^1(\Gamma_{U_i} \backslash U_i, H^0(\mathfrak{n}_i))] \end{aligned} \tag{5}$$

By construction this is exactly the restriction map of $H^1(\Gamma_{M_i} \backslash \overline{Z}_i)$ to the cohomology of the boundary of the base space $\Gamma_{M_i} \backslash Z_i$. Since as a vectorspace $H^2(\Gamma_{M_i} \backslash \overline{Z}_{M_i}, \partial(\Gamma_{M_i} \backslash \overline{Z}_{M_i})) \cong \mathbb{C}$ the cokernel of this restriction is a onedimensional representation $[\tau_i]$ of \mathcal{P}_i . The kernel is, of course, the image of the cohomology with compact support of $\Gamma_{M_i} \backslash Z_{M_i}$ and can be identified with the cusp cohomology $H_{\text{cusp}}^1(\Gamma_{M_i} \backslash Z_{M_i})$ (cf. 3.1. or [2] 5.5 for definition). Therefore we have a short exact sequence of \mathcal{P}_i -modules

$$\begin{aligned} 0 \rightarrow H_{\text{cusp}}^1(\Gamma_{M_i} \backslash Z_{M_i}) \rightarrow H^1(\Gamma_{M_i} \backslash Z_{M_i}) \rightarrow \text{Ind}_{\mathcal{P}_0}^{\mathcal{P}_i} [H^1(\Gamma_{U_i} \backslash U_i)] \rightarrow \\ \rightarrow [\tau_i] \rightarrow 0 \end{aligned} \tag{6}$$

and we obtain

$$\ker(\alpha_1^1 \oplus \alpha_2^1) \cong \bigoplus_{i=1}^2 \text{Ind}_{\mathcal{P}_i}^G [H_{\text{cusp}}^1(\Gamma_{M_i} \backslash Z_{M_i})] \tag{7}$$

$$\text{coker}(\alpha_1^1 \oplus \alpha_2^1) \cong \bigoplus_{i=1}^2 \text{Ind}_{\mathcal{P}_i}^G [\tau_i] \tag{8}$$

where the action of \mathcal{P}_i on $H_{\text{cusp}}^1(\Gamma_{M_i} \backslash Z_{M_i})$ is the pullback of the natural action of \mathcal{M}_i on the cusp cohomology.

The situation in degree 2 is quite similar to the previous case. By 2.4.(1) and 2.5. we have to consider

$$\alpha_1^2 \oplus \alpha_2^2: \bigoplus_{i=1}^2 \text{Ind}_{fP_i}^G [H^2(\overline{e'(P_i)})] \rightarrow \text{Ind}_{fP_0}^G [\text{Ind}_{fP_0}^{P_i} [H^2(e'(P_0))]].$$

Then again using (3), 2.4.(5) and 2.5. this morphism is the sum of the two fG -homomorphisms induced from the fP_i -morphisms

$$\begin{aligned} H^2(\overline{e'(P_i)}) &= H^1(\Gamma_{M_i} \backslash \overline{Z}_{M_i}, H^1(n_i)) \rightarrow \\ &\rightarrow \text{Ind}_{fP_0}^{P_i} [H^1(\Gamma_{U_i} \backslash U_i, H^1(n_i))]. \end{aligned} \tag{9}$$

The right hand side is the cohomology of the boundary of the compactification $\Gamma_{M_i} \backslash \overline{Z}_{M_i}$ of the base space. Since $H^1(n_i)$ is not the trivial M_i -module \mathbb{C} (resp. \mathbb{C}°) (cf. 2.5.) this map is surjective and determined by the short exact sequence of fP_i -modules

$$\begin{aligned} 0 \rightarrow H_{\text{cusp}}^1(\Gamma_{M_i} \backslash Z_{M_i}, H^1(n_i)) &\rightarrow H^1(\Gamma_{M_i} \backslash Z_{M_i}, H^1(n_i)) \rightarrow \\ &\rightarrow \text{Ind}_{fP_0}^{P_i} [H^1(\Gamma_{U_i} \backslash U_i, H^1(n_i))] \rightarrow 0 \end{aligned} \tag{10}$$

This implies

$$\ker(\alpha_1^2 \oplus \alpha_2^2) \cong \bigoplus_{i=1}^2 \text{Ind}_{fP_i}^G [H_{\text{cusp}}^1(\Gamma_{M_i} \backslash Z_{M_i}, H^1(n_i))] \tag{11}$$

$$\text{coker}(\alpha_1^2 \oplus \alpha_2^2) \cong 0. \tag{12}$$

These investigations in degrees 0, 1, 2 allow us already to determine completely the fG -module $H^*(\partial(\Gamma \backslash \overline{X}))$. For later purposes we give a description of the cohomology which reflects the geometric source of the various summands.

2.7. THEOREM: *For a given congruence subgroup $\Gamma = \Gamma(m)$ of $Sp_4(\mathbb{Z})$, $m \geq 3$, the cohomology of the boundary $\partial(\Gamma(m) \backslash \overline{X})$ of the Borel-Serre compactification of $\Gamma(m) \backslash X$ is described as representation space of $fG = Sp_4(\mathbb{Z}/m\mathbb{Z})$ by*

$$H^q(\partial(\Gamma \backslash \overline{X})) \cong \begin{cases} \mathbb{C} & q = 0, 5 \\ 0 & q > 5 \end{cases} \tag{1}$$

$$H^1(\partial(\Gamma \backslash \overline{X})) \cong \bigoplus_{i=1}^2 \text{Ind}_{fP_i}^G [H_{\text{cusp}}^1(\Gamma_{M_i} \backslash Z_{M_i})] \oplus St(m) \tag{2}$$

$$\begin{aligned}
 H^2(\partial(\Gamma \setminus \bar{X})) \cong \bigoplus_{i=1}^2 \text{Ind}_{fP_i}^G \left[H_{\text{cusp}}^1(\Gamma_{M_i} \setminus Z_{M_i}, H^1(n_i)) \right] \oplus \\
 \bigoplus_{i=1}^2 \text{Ind}_{fP_i}^G [\tau_i] \tag{3}
 \end{aligned}$$

$$\begin{aligned}
 H^3(\partial(\Gamma \setminus \bar{X})) \cong \bigoplus_{i=1}^2 \text{Ind}_{fP_i}^G \left[H_{\text{cusp}}^1(\Gamma_{M_i} \setminus Z_{M_i}, H^2(n_i)) \right] \oplus \\
 \bigoplus_{i=1}^2 \text{Ind}_{fP_i}^G \left[H^0(\Gamma_{M_i} \setminus Z_{M_i}, H^3(n_i)) \right] \tag{4}
 \end{aligned}$$

$$\begin{aligned}
 H^4(\partial(\Gamma \setminus \bar{X})) \cong \bigoplus_{i=1}^2 \text{Ind}_{fP_i}^G \left[H_{\text{cusp}}^1(\Gamma_{M_i} \setminus Z_{M_i}, H^3(n_i)) \right] \oplus St(m). \tag{5}
 \end{aligned}$$

(For notation resp. a description of the M_i -module structure of $H^q(n_i)$ and the action of fP_i on the various terms $H_{\text{cusp}}^p(\Gamma_{M_i} \setminus Z_{M_i}, H^q(n_i))$ we refer to 2.5.) The fG -module $St(m)$ is defined by the short exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow \bigoplus_{i=1}^2 \text{Ind}_{fP_i}^G [\mathbb{C}] \rightarrow \text{Ind}_{fP_0}^G [\mathbb{C}] \rightarrow St(m) \rightarrow 0. \tag{6}$$

The boundary $\partial(\Gamma(m) \setminus \bar{X})$ is a compact connected 5-dimensional manifold. By Poincaré-duality this implies (1). Assertion (2) (resp. (3)) follows from 2.6.(1), (7) (resp. 2.6.(11), (8)). Again, by Poincaré-duality (2) (resp. (3)) and 2.5 then imply also (5) and (4). Indeed, by determining kernel and cokernel of $\alpha_1^* \oplus \alpha_2^*$ also in degree 3 and 4 as above these last two statements can also be seen directly. In particular, one checks by 2.5. that the 1-dimensional representation $H^0(\Gamma_{M_i} \setminus Z_{M_i}, H^3(n_i))$ of fP_i corresponds to $[\tau_i]$.

2.8. In view of this result there are some observations of interest concerning the natural restriction

$$r^*: H^*(\Gamma \setminus \bar{X}) \rightarrow H^*(\partial(\Gamma \setminus \bar{X})). \tag{1}$$

It is clear that r^0 is an isomorphism. The positive solution of the congruence subgroup problem for Sp_4 (cf. [19]) shows that the commutatorfactorgroup $\Gamma/[\Gamma : \Gamma]$ is finite. On one hand, this implies that $H_1(\Gamma \setminus X) = \Gamma/[\Gamma : \Gamma] \otimes \mathbb{C}$ vanishes and via Poincaré-duality one gets that r^4 is surjective. On the other hand, one obtains $H^1(\Gamma \setminus \bar{X}) = 0$, and

therefore r^1 is trivial. Moreover, there is a dual pairing on $H^*(\partial(\Gamma \backslash \bar{X}))$ induced by duality (cf. [5], p. 305) such that the image of $H^*(\Gamma \backslash \bar{X})$ under r^* is its own annihilator; in particular one gets for $i = 0, 1, \dots$

$$\dim \text{Im } r^{5-i} = \dim H^i(\partial(\Gamma / \bar{X})) - \dim \text{Im } r^i \tag{2}$$

which shows that $\text{Im } r^3$ and $\text{Im } r^2$ are related to each other.

We now consider the congruence subgroup $\Gamma(m)$, $m \geq 3$. Then Γ_{M_i} can be viewed as the full congruence subgroup $\Gamma(2, m)$ of level m in $SL_2(\mathbb{Z})$. The Eichler-Shimura isomorphism (cf. [25], 8.2)

$$H_{\text{cusp}}^1(\Gamma(2, m) \backslash H, E_k) \cong S_{k+1}(\Gamma(2, m)) \oplus \overline{S_{k+1}(\Gamma(2, m))} \tag{3}$$

relates the cusp cohomology of $\Gamma(2, m)$ with coefficients in the representation of $SL_2(\mathbb{R})$ of dimension k to the space $S_{k+1}(\Gamma(2, m))$ of holomorphic resp. antiholomorphic cuspidal forms of weight $k + 1$ on the upper half plane H with respect to $\Gamma(2, m)$. The dimension of the spaces on the right hand side are known ([25], §2). Using this and (2) we obtain the following dimension formulas for the image of the restriction map r^* .

2.9. PROPOSITION: *Let $\Gamma = \Gamma(m)$, $m \geq 3$, be the congruence subgroup of level m of $Sp_4(\mathbb{Z})$. The dimensions of the images of the restrictions r^* : $H^*(\Gamma \backslash \bar{X}) \rightarrow H(\partial(\Gamma \backslash \bar{X}))$ are given by*

$$\dim \text{Im } r^q = \begin{cases} 1 & q = 0 \\ 0 & q = 1 \text{ or } q \geq 5 \end{cases} \tag{1}$$

$$\dim \text{Im } r^2 + \dim \text{Im } r^3 = \sum_{i=1}^2 p_i(m)(1 + 2 \dim S_{k_i}(\Gamma(2, m))) \tag{2}$$

$$\text{with } k_i = \begin{cases} 3 & i = 1 \\ 4 & i = 2 \end{cases}$$

$$\begin{aligned} \dim \text{Im } r^4 &= \sum_{i=1}^2 (p_i(m) 2 \cdot \dim S_2(\Gamma(2, m))) \\ &+ (p_0(m) - (p_1(m) + p_2(m)) + 1) \end{aligned} \tag{3}$$

where $p_j(m) = |{}_f P_j \backslash_f G|$ denotes the number of $\Gamma(m)$ -conjugacy classes of parabolic \mathbb{Q} -subgroups of $Sp_4(\mathbb{R})$ of type j ($j = 0, 1, 2$).

§3. Eisenstein cohomology I

We will now use the theory of Eisenstein series ([12], [15]) to construct a subspace $H^4_{\max}(\Gamma \backslash \bar{X})$ in $H^4(\Gamma \backslash \bar{X})$ which is generated by regular Eisenstein cohomology classes and which restricts isomorphically onto the cuspidal cohomology of the faces $e'(Q)$ where Q runs through a set of representatives for the maximal parabolic \mathbb{Q} -subgroups of G modulo conjugation by Γ (3.3., 3.4.). We assume some familiarity with the results in [22], I. The other cases will be dealt with in §4.

3.1. *Eisenstein series.* Given a parabolic \mathbb{Q} -subgroup P of G with $P = {}^0MAN$ the cohomology of the corresponding face $e'(P)$ in the boundary of the Borel-Serre compactification $\Gamma \backslash \bar{X}$ of $\Gamma \backslash X$ is given as (cf. 2.5.)

$$H^*(e'(P)) = H^*(\Gamma_M \backslash Z_M, H^*(\mathfrak{n})).$$

It contains as a natural subspace the cusp cohomology

$$H^*_{\text{cusp}}(e'(P)) := H^*_{\text{cusp}}(\Gamma_M \backslash Z_M, H^*(\mathfrak{n})) \tag{1}$$

of $e'(P)$. By definition ([2], 5.5 or [22], 1.6), this subspace is usually viewed in terms of relative Lie algebra cohomology as the image of the injective homomorphism

$$\begin{aligned} H^*({}^0\mathfrak{m}, K_M; L^2_0(\Gamma_M \backslash {}^0M)^\infty \otimes H^*(\mathfrak{n})) &\rightarrow \\ &\rightarrow H^*({}^0\mathfrak{m}, K_M; C^\infty(\Gamma_M \backslash {}^0M) \otimes H^*(\mathfrak{n})) = \\ &= H^*(\Gamma_M \backslash Z_M, H^*(\mathfrak{n})). \end{aligned} \tag{2}$$

Here $L^2_0(\Gamma_M \backslash {}^0M)^\infty$ denotes the $({}^0\mathfrak{m}, K_M)$ -module of C^∞ -vectors in the representation space $L^2_0(\Gamma_M \backslash {}^0M)$ of cuspidal square-integrable functions on $\Gamma_M \backslash {}^0M$ acted upon by 0M via right translations. For the notion of relative Lie algebra cohomology we refer to [4], I. However, this subspace may also be interpreted in terms of $H^*(\mathfrak{n})$ -valued differential forms whose coefficients are $H^*(\mathfrak{n})$ -valued cuspidal functions. In particular, the cusp cohomology $H^*_{\text{cusp}}(e'(P))$ can be identified with the space of harmonic cuspidal \mathbb{C} -valued differential forms on $e'(P)$, i.e. those whose coefficients are cuspidal (see [2], §5).

Let $0 \neq [\phi] \in H^*_{\text{cusp}}(e'(P))$ be a non-trivial cuspidal cohomology class represented by a harmonic cuspidal form $\phi \in \Omega^*(e'(P))$. Recall that we have topologically $\Gamma_p \backslash X = e'(P) \times A_p$. For a given $\Lambda \in \mathfrak{a}^*_\mathbb{C}$ we can associate to ϕ via the differential form $\phi_\Lambda = \phi a^{\Lambda+\rho}$ in $\Omega^*(\Gamma_p \backslash X)$ the Eisenstein series (cf. [22], §4)

$$E(\phi, \Lambda) = \sum_{\Gamma_p \backslash \Gamma} \gamma \circ \phi_\Lambda. \tag{3}$$

This Eisenstein series is first defined for all Λ in

$$(\mathfrak{a}_{\mathbb{C}}^*)^+ = \{ \Lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid (\operatorname{Re} \Lambda, \alpha) > (\rho_P, \alpha), \alpha \in \Delta(P, A) \} \tag{4}$$

and is holomorphic in that tube where $\Delta(P, A)$ denotes the set of simple roots of P with respect to A and the element $\rho_P \in \mathfrak{a}^*$ is defined by $\rho_P(a) = (\det \operatorname{Ad} a|_{\mathfrak{n}})^{1/2}$, $a \in A$. Via analytic continuation it admits a meromorphic extension to all of $\mathfrak{a}_{\mathbb{C}}^*$. We refer to [12], [15] for the general theory of Eisenstein series. If $\Lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ is fixed and $E(\phi, \Lambda)$ is holomorphic at this point, then evaluating the Eisenstein series in Λ_0 gives a \mathbb{C} -valued, Γ -invariant differential form on X , i.e. we obtain $E(\phi, \Lambda_0) \in \Omega^*(\Gamma \backslash X)$. In fact, by 4.11 [22] there is a special point Λ_ϕ uniquely determined by ϕ such that this construction provides us with a *closed harmonic* form $E(\phi, \Lambda_\phi)$ if $E(\phi, \Lambda)$ is holomorphic at this point Λ_ϕ . In particular, this form represents a non-trivial cohomology class $[E(\phi, \Lambda_\phi)]$ in $H^*(\Gamma \backslash X)$.

3.2. We now assume that $P = {}^0MAN$ is a maximal parabolic \mathbb{Q} -subgroup of G . The space $L^2_0(\Gamma_M \backslash {}^0M)$ of square integrable cuspidal functions on $\Gamma_M \backslash {}^0M$ decomposes into a direct Hilbert sum of closed irreducible 0M -invariant subspaces H_π with finite multiplicities $m(\pi, \Gamma_M)$ ([12], 1. §2). If V_π denotes the isotopic component of $\pi \in {}^0\hat{M}$ we may write

$$L^2_0(\Gamma_M \backslash {}^0M) = \bigoplus_{\pi \in {}^0\hat{M}} \tilde{V}_\pi. \tag{1}$$

By definition of the cusp cohomology, 2.4.(3) and [22] 1.6 we then have a finite sum decomposition

$$H^*_{\text{cusp}}(e'(P)) = \bigoplus_{\pi \in {}^0\hat{M}} \bigoplus_{w \in W^P} H^*({}^0\mathfrak{m}, K_M; V_\pi \otimes F_{w(\rho) - \rho}). \tag{2}$$

Now, the irreducible unitary representations of 0M which may contribute non-trivially for a fixed $w \in W^P$ to the right hand side are well-known. Let $F_k = E_k$ (resp. E_k^σ) as in 2.5. a k -dimensional irreducible representation of 0M_i which is isomorphic to $Sl_2(\mathbb{R}) \times (\pm 1)$ for $i = 1$ resp. to $SL_2^\pm(\mathbb{R})$ for $i = 2$, and we assume $k > 1$ for the moment. If (π, H_π) is then an irreducible unitary representation of 0M_i which is not equivalent to a discrete series representation D_{k+1}^\pm for $i = 1$ (resp. D_{k+1} for $i = 2$) of lowest K_{M_i} -type $k + 1$ (cf. [27], I, §4) we have

$$H^*({}^0\mathfrak{m}_i, K_{M_i}; H_\pi^\infty \otimes F_k) = 0 \tag{3}$$

and for the discrete series representations D_{k+1}^\pm resp. D_{k+1} we have

$$H^*({}^0\mathfrak{m}_i, K_{M_i}; H_{\pi_{k+1}}^\infty \otimes F_k) = \begin{cases} \mathbb{C} & q = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

with $\pi_{k+1} = D_{k+1}^\pm$ for $i = 1$ (resp. $= D_{k+1}$ for $i = 2$). This formula holds also for $k = 1$ (i.e. $F_1 = \mathbb{C}$ or \mathbb{C}^σ). But in this case also the trivial representation \mathbb{C} resp. \mathbb{C}^σ contribute non-trivially in degree 0 (resp. 2). For all other representations non-equivalent to these three ones there is again a vanishing result as in (3). By means of 2.5. the right hand side of (2) then reduces for a fixed $w \in W^P$ of length $l(w)$ with $F_{w(\rho)-\rho} \cong F_k$ to

$$H_{\text{cusp}}^{1+l(w)}(\Gamma_M \backslash Z_M; F_{w(\rho)-\rho}) \cong H^1({}^0\mathfrak{m}, K_M; V_{\pi_{k+1}} \otimes F_k). \tag{5}$$

We observe, that this reflects, more or less, in a representation theoretical version the Eichler-Shimura isomorphism (cf. 2.8.). In particular, for $\Gamma = \Gamma(m)$ the multiplicity $m(D_k^\pm, \Gamma_M)$ is given as the dimension of the space $S_k(\Gamma(2, m))$ of cusp forms with respect to $\Gamma(2, m) \subset SL_2(\mathbb{Z})$ of weight k in the classical sense.

We will now discuss the use of Eisenstein series associated to cusp forms in $H_{\text{cusp}}^4(e'(P))$ to construct cohomology classes in $H^4(\Gamma \backslash X)$. As a final result, we will obtain a complete description of the cohomological contribution to $H^4(\Gamma \backslash X)$ by the cusp cohomology of the faces in $\partial(\Gamma \backslash \bar{X})$ of minimal codimension. To deal with this problem in the degrees 2 and 3 deserves a different approach (cf. §4).

3.3. THEOREM: *Let $\Gamma \subset Sp_4(\mathbb{Z})$ be a torsion free subgroup of finite index; let P be a maximal parabolic \mathbb{Q} -subgroup of $Sp_4(\mathbb{R})$. If $[\phi] \in H_{\text{cusp}}^*(e'(P))$ is a non-trivial cuspidal cohomology class of degree 4 then the Eisenstein series $E(\phi, \Lambda)$, $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$, associated to $[\phi]$ is holomorphic at the point $\Lambda_\phi = -w_P(\rho)|_{\mathfrak{a}} = \rho_P$ (uniquely determined by ϕ). The form $E(\phi, \Lambda_\phi)$ is closed and harmonic and represents a non-trivial class in $H^4(\Gamma \backslash X)$ whose image under the restriction r_Q^* : $H^*(\Gamma \backslash \bar{X}) \rightarrow H^*(e'(Q))$ to a face $e'(Q)$ in $\partial(\Gamma \backslash \bar{X})$ is given by*

$$r_Q^4([E(\phi, \Lambda_\phi)]) = \begin{cases} [\phi] & \text{for } Q \text{ } \Gamma\text{-conjugate to } P \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

PROOF: We can assume that P is standard. For the cusp cohomology of the face $e'(P)$ in degree 4 we have by 2.5. resp. 3.2.

$$H_{\text{cusp}}^4(e'(P)) = H_{\text{cusp}}^1(\Gamma_M \backslash Z_M, H^3(\mathfrak{n}_1)) \tag{2}$$

with $H^3(\mathfrak{n}_1) = F_{w_P(\rho)-\rho}$ where w_P denotes the longest element in W^P , and since $F_{w_P(\rho)-\rho} \cong \mathbb{C}$ resp. \mathbb{C}^σ as 0M -module we get

$$H_{\text{cusp}}^4(e'(P)) \cong H^1({}^0\mathfrak{m}, K_M; V_{\pi_2} \otimes F_{w_P(\rho)-\rho}). \tag{3}$$

i.e. the non-trivial class $[\phi]$ is necessarily of type (π_2, w_P) in the sense of 3.2. [22]. The special point Λ_ϕ uniquely determined by ϕ is then given as

$$\Lambda_\phi = -w_P(\rho) |_{\mathfrak{a}} = \rho |_{\mathfrak{a}} = \rho_P; \tag{4}$$

it lies outside the tube $(\mathfrak{a}_{\mathbb{C}}^*)^+$ of absolute convergence of the associated Eisenstein series $E(\phi, \Lambda)$. However, since π_2 is a tempered representation the result 6.4.(2) [22] applies and $E(\phi, \Lambda)$ is holomorphic at Λ_ϕ . By 4.11. [22], the form $E(\phi, \Lambda_\phi)$ is closed and harmonic and represents a non-trivial class $[E(\phi, \Lambda_\phi)]$ in $H^4(\Gamma \backslash X)$.

We recall that the image of $[E(\phi, \Lambda_\phi)]$ under the restriction r_Q^* is given as $[E(\phi, \Lambda_\phi)_Q] |_{e'(Q)}$ i.e. equal to the restriction to $e'(Q)$ of the class $[E(\phi, \Lambda_\phi)_Q]$ represented by the constant Fourier coefficient $E(\phi, \Lambda_\phi)_Q \in \Omega^*(\Gamma_Q \backslash X)$ of $E(\phi, \Lambda_\phi)$ along Q ([22], 1.10). The theory of the constant term (cf. [12], II, 4., 5., [22], 4.7.) then implies that $E(\phi, \Lambda_\phi)_Q$ vanishes if Q is not associated to P . Since this condition is in this case equivalent to the fact that Q is not $\mathbf{G}(\mathbb{Q})$ -conjugate to P we obtain

$$r_Q^*([E(\phi, \Lambda_\phi)]) = 0 \quad \text{for } Q \text{ not } \mathbf{G}(\mathbb{Q})\text{-conjugate to } P. \tag{5}$$

If Q is associated to P then by definition the finite set $W(A_P, A_Q)$ of isomorphisms of A_P onto A_Q induced by inner automorphisms of G defining a \mathbb{Q} -isomorphism of $\mathbf{M}_P(\mathbb{R})$ onto $\mathbf{M}_Q(\mathbb{R})$ is not empty; we have

$$r_Q^*([E(\phi, \Lambda_\phi)]) = \sum_{s \in W(A_P, A_Q)} [c(s, \Lambda_\phi)_{s\Lambda_\phi}(\phi_{\Lambda_\phi})] |_{e'(Q)} \tag{6}$$

where $c(s, \Lambda_\phi)_{s\Lambda_\phi}: \Omega^*(\Gamma_P \backslash X) \rightarrow \Omega^*(\Gamma_Q \backslash X)$ is an ‘‘intertwining’’ operator defined in 4.10. [22].

As pointed out in 1.2. the maximal parabolic P is conjugate to its opposite \bar{P} . This implies in particular that the set $W(A_P, A_P)$ consists of two elements. If we put

$$s_1 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \quad \text{resp. } s_2 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \tag{7}$$

then we have (in the notation $A_i = A_{P_i}$)

$$W(A_i, A_i) = \{1, s_i\}. \quad i = 1, 2. \tag{8}$$

Since a maximal parabolic \mathbb{Q} -subgroup Q in the associated class $\mathcal{C}(P)$ of P is conjugate to P , there is an element $g \in \mathbf{G}(\mathbb{Q})$ with $P^g = Q$, and $A_Q = A_P^g$ is a split component of Q . We then obtain

$$W(A_i, A_Q) = gW(A_i, A_i) = \{\mathrm{Int} \ g |_{A_i}, \mathrm{Int} \ g \circ s_i |_{A_i}\}. \tag{9}$$

We now deal with the case that Q is Γ -conjugate to P . By 7.7.(1) [3] we then have $e'(P) = e'(Q)$ and we can assume $P = Q$. The summand in the sum (6) corresponding to the element 1 in $W(A_i, A_i)$ then contributes by the class $[\phi]$ we started with (cf. 4.9., 4.12(6) in [22]). The other summand $[\mathbf{c}(s_i, \Lambda_\phi)_{s_i\Lambda_\phi}(\phi_{\Lambda_\phi})] |_{e'(P)}$ in (6) is a cohomology class in $H_{\text{cusp}}^*(e'(P))$ of weight $v_{s_i}(\rho) - \rho |_{\mathfrak{a}_i}$ (in the sense of [22], 3.2.) where v_{s_i} is a uniquely determined element in W^P with $v_{s_i}(\rho) |_{\mathfrak{a}_i} + s_i\Lambda_\phi = 0$ (and a second condition not of interest here, cf. [22], 4.10.(11)); that is, this class is contained in $H_{\text{cusp}}^1(\Gamma_M \backslash Z_M; F_{v_{s_i}(\rho) - \rho})$. Since $s_i\Lambda_\phi = -\Lambda_\phi$ this condition is equivalent to

$$v_{s_i}(\rho) |_{\mathfrak{a}_i} = -w_{P_i}(\rho) |_{\mathfrak{a}_i} \tag{10}$$

and one sees that $v_{s_i} = 1 \in W^P$. Since this occurs only in degree 1 in $H_{\text{cusp}}^*(e'(P_i))$ (cf. 2.5.) this second summand (indeed, the representing differential form) vanishes. So we have

$$r_Q^*([E(\phi, \Lambda_\phi)]) = [\phi] \quad \text{for } Q \text{ } \Gamma\text{-conjugate to } P. \tag{11}$$

We now assume that Q in $\mathcal{C}(P)$ is not Γ -conjugate to P , and we write s for the element $\text{Int } g |_{A_i}$ in (9). By definition (4.8. [22]) the operator $\mathbf{c}(s, \Lambda_\phi)_{s\Lambda_\phi}$ is given as a sum over terms which are parametrized by the set $\Gamma(s) = \Gamma \cap Pg^{-1}Q$. For an element $\gamma = pg^{-1}q$ in $\Gamma(s)$, $p \in P$, $q \in Q$, we have $\gamma^{-1}P\gamma = q^{-1}g^{-1}Pg^{-1}q = Q$; but this contradicts our assumption that Q is not Γ -conjugate to P , and therefore we have that $\Gamma(s)$ is empty. In dealing with the second summand in (6) one observes as above that it has a weight which does not occur in degree 4 of $[E(\phi, \Lambda_\phi)]$. So we finally get

$$r_Q^*([E(\phi, \Lambda_\phi)]) = 0 \quad \text{for } Q \text{ not } \Gamma\text{-conjugate to } P, Q \in \mathcal{C}(P). \tag{12}$$

3.4. COROLLARY: *Let P be a maximal parabolic \mathbb{Q} -subgroup of G , and let $\mathcal{C}(P)$ be its associate class of parabolic \mathbb{Q} -subgroups. Let $H_{\mathcal{C}(P)}^4(\Gamma \backslash X)$ be the subspace in $H^4(\Gamma \backslash X)$ which is generated by the Eisenstein cohomology classes $[E(\phi, \Lambda_\phi)]$ constructed as in 3.3. for all R in a set of representatives of $\Gamma \backslash \mathcal{C}(P)$ and all non-trivial classes $[\phi] \in H_{\text{cusp}}^4(e'(R))$. Then $H_{\mathcal{C}(P)}^4(\Gamma \backslash X)$ is mapped isomorphically under the restriction*

$$r_{\mathcal{C}(P)}^4: H^4(\Gamma \backslash \bar{X}) \rightarrow \bigoplus_{Q \in \Gamma \backslash \mathcal{C}(P)} H^4(e'(Q)) \tag{1}$$

onto $\bigoplus_{Q \in \Gamma \backslash \mathcal{C}(P)} H_{\text{cusp}}^4(e'(Q))$.

§4. Eisenstein cohomology II

We now briefly sketch how the ideas of §3 can be pursued in an adelic setting in order to construct by means of Eisenstein series a section to the restriction $r^*: H^*(\Gamma \backslash \bar{X}) \rightarrow H^*(\partial(\Gamma \backslash \bar{X}))$. The main result is given in 4.3. We have to assume some familiarity with [15], [22]. We denote by $\Gamma = \Gamma(m)$ the congruence subgroup of $Sp_4(\mathbb{Z})$ of level m , $m \geq 3$.

4.1. *The space $H^*_{\min}(\Gamma)$.* Let $G(\mathbf{A})$ be the group of adelic points of $G = Sp_4/\mathbb{Q}$, and let $L \subset G(\mathbf{A}_f)$ be the open compact subgroup of congruence level m . We have $\Gamma \backslash X \cong G(\mathbb{Q}) \backslash G(\mathbf{A})/K \cdot L$. Let $T \subset P_0$ be the maximal split torus, and denote by $\hat{T}_{\mathbf{A}}$ the set of characters

$$\chi: T(\mathbb{Q}) \backslash T(\mathbf{A})/T(\mathbb{R})^0 \cdot (T(\mathbf{A}_f) \cap L) \rightarrow \mathbb{C}^*.$$

Choosing a form $\phi_w \in \Omega^{(w)}(e'(P_0))$ with $\mathbb{C}[\phi_w] \cong F_{w(\rho) - \rho}$ (cf. 2.4.(3)) the cohomology of Y_0 can then be written as

$$H^*(Y_0) = \bigoplus_{w \in W} \mathbb{C} \phi_w \otimes \left(\bigoplus_{\chi \in \hat{T}(w)} \text{Ind}'_{fB}^G [{}_f\chi] \right) \tag{1}$$

where $\hat{T}(w) := \{ \chi \in \hat{T}_{\mathbf{A}} \mid \chi|_{T(\mathbb{R})} = w(\rho) - \rho|_{T(\mathbb{R})} \}$ and ${}_f\chi: {}_fB/{}_fN_0 \rightarrow \mathbb{C}^*$ denotes the character on the standard Borel subgroup ${}_fB$ of ${}_fG$ determined by χ . If $m = \prod p^{v_p}$ then ${}_fG$ is the direct product of the groups $G_p := Sp_4(\mathbb{Z}/p^{v_p}\mathbb{Z})$; for subgroups of ${}_fG$ we will use analogous notation. We will also write ${}_f\chi = \otimes \chi_p$. The torus ${}_fT$ can be naturally written as a product of 1-dimensional tori ${}_fT_i$, $i = 1, 2$, and we have accordingly ${}_f\chi = (\chi_i)_i$ (resp. $\chi_p = (\chi_{p,i})_i$). For a given $\chi \in \hat{T}_{\mathbf{A}}$ we put $\Theta(\chi_p) := \{ j \in \{1, 2\} \mid \chi_{p,j} = 1 \}$. For each p one can construct then (cf. [22], 9.2) a suitable decomposition of $\text{Ind}_{B_p}^{G_p}[\chi_p]$ of (not necessarily irreducible) G_p -modules $V_{\Omega(\chi_p)}^{\Theta(\chi_p)}$ parametrized by the subsets $\Omega(\chi_p) \subseteq \Theta(\chi_p)$. We call a class in (1) of the form

$$\phi_{\mathbf{A}} = \phi_w \otimes \bigoplus_{p|m} v_p, \quad w \in W, \quad \chi \in \hat{T}(w), \quad v_p \in V_{\Omega(\chi_p)}^{\Theta(\chi_p)} \tag{2}$$

a class of type $(w, \chi \parallel \{ \Omega(\chi_p) \}_p)$. As in 3.1. there is then attached to such a $\phi_{\mathbf{A}}$ an Eisenstein series $E(\phi_{\mathbf{A}}, \Lambda)$, $\Lambda \in \mathfrak{a}_{0\mathbb{C}}^*$, first defined for Λ with $(\text{Re } \Lambda, \alpha) > (\rho, \alpha)$, $\alpha \in \Delta(P_0, A_0)$, and holomorphic there, but can be analytically continued to a meromorphic function on $\mathfrak{a}_{0\mathbb{C}}^*$. The singularities lie along hyperplanes of the form $r = \{ \Lambda \in \mathfrak{a}_{0\mathbb{C}}^* \mid (\Lambda, \alpha) = \mu, \alpha \in \Phi(P_0, A_0) \}$ and only finitely many r meet $\{ \Lambda \in \mathfrak{a}_{0\mathbb{C}}^* \mid (\text{Re } \Lambda, \alpha) > 0, \alpha \in \Delta(P_0, A_0) \}$. The residue $\text{Res}_r E(\phi_{\mathbf{A}}, \Lambda)$ of $E(\phi_{\mathbf{A}}, \Lambda)$ along such an r is a meromorphic function on r (cf. [15] p. 171).

Given a non-trivial cohomology class $\phi_{\mathbf{A}} \in H^*(Y_0)$ as in (2) of type $(w, \chi \parallel \{\Omega(\chi_p)\}_p)$ one obtains the following results:

If $w = w_0$ is the longest element in W and $\cap_p \Omega(\chi_p) = \emptyset$, then $E(\phi_{\mathbf{A}}, \Lambda)$ is holomorphic at $\Lambda_0 = \rho$, and $E(\phi_{\mathbf{A}}, \rho)$ is a closed harmonic form on $\Gamma \backslash X$ representing a non-trivial class in $H^*(\Gamma \backslash \bar{X})$ with $r^4([E(\phi_{\mathbf{A}}, \rho)]) = [\phi_{\mathbf{A}}]$. (3)

If $w = w'_p$ is the longest element in W^{P_i} and $\cap \Omega(\chi_p) = \{j\}$, $j \neq i$, then there is a uniquely determined singular hyperplane r_i of $E(\phi_{\mathbf{A}}, \Lambda)$ such that $\text{Res}_{r_i} E(\phi_{\mathbf{A}}, \Lambda)$ is holomorphic in $\Lambda_0 = \rho$. The form $\text{Res}_{r_i} E(\phi_{\mathbf{A}}, \Lambda_0)$ is closed and harmonic and represents a non-trivial class in $H^3(\Gamma \backslash \bar{X})$. (4)

THEOREM: Let $H^q_{\min}(\Gamma \backslash \bar{X})$ be the subspace in $H^q(\Gamma \backslash \bar{X})$ generated by the Eisenstein cohomology classes constructed in (3) for $q = 4$ (resp. (4) for $q = 3$). Under the restriction $r^*: H^*(\Gamma \backslash \bar{X}) \rightarrow H^*(\partial(\Gamma \backslash \bar{X}))$ the space $H^q_{\min}(\Gamma \backslash \bar{X})$ restricts isomorphically onto the subspace $St(m)$ for $q = 4$ (resp. $\oplus \text{Ind}_{P_i}^G[H^0(\Gamma_{M_i} \backslash Z_{M_i}, H^3(n_i))]$ for $q = 3$) in the cohomology of the boundary (cf. 2.7).

Since a class in $H^0(\Gamma \backslash X) \cong \mathbb{C}$ is also obtained by taking a successive residue of an Eisenstein series attached to a class in $H^0(Y_0)$ we may put $H^0_{\min} = H^0$. We recall $H^1(\Gamma \backslash X) = 0$.

The proofs of (3) and (4) rely on an explicit computation of the constant Fourier coefficient $E(\phi_{\mathbf{A}}, \Lambda)_{P_0} = \sum_W \mathbf{c}_{\mathbf{A}}(s, \Lambda)(\phi_{\mathbf{A}})$ along P_0 and the intertwining operators involved; they can be written as a product $\mathbf{c}_{\infty}(s, \Lambda) \otimes \prod \mathbf{c}_q(s, \Lambda)$ of local factors. The infinite part is determined along the same lines as in the case SL_n/\mathbb{Q} dealt with in [25]. In answering the question if $E(\phi_{\mathbf{A}}, \Lambda)$ has a pole at $\Lambda_0 = -w(\rho)$ (cf. [22], 4.11) or not the mutual influence of the poles of the operator $\otimes_{q \mid m} \mathbf{c}_q(s, \Lambda)$ which is for $s \neq 1$ a product of L -factors of the form $L(\chi_i, z)/L(\chi_i, z + 1)$ $z \in \mathbb{C}$, and the “zeros” of the operator $\otimes_{q \mid m} \mathbf{c}_q(s, \Lambda)$ interpreted as a f - G -morphism $\text{Ind}_{fB}^{fG}[f\chi] \rightarrow \text{Ind}_{fB}^{fG}[f\chi]$ is decisive. For example, in the case dealt with in (3) with $\Lambda_0 = \rho$ and $s = s_i$ a simple reflection if $\cap_{p \mid m} \Theta(\chi_p) = \{i\}$ then the global character χ_i is trivial, but the condition $\cap_{p \mid m} \Omega(\chi_p) = \emptyset$ ensures that the finite part $\otimes v_p$ of $\phi_{\mathbf{A}}$ lies in the kernel of $\otimes_{p \mid m} \mathbf{c}_p(s_i, \Lambda_0)$ which implies that the pole of $L(\chi_i, z)$ at $z = 1$ plays globally no role for $\mathbf{c}_{\mathbf{A}}(s_i, \Lambda)$. This study of $E(\phi_{\mathbf{A}}, \Lambda)_{P_0}$ and a suitable characterization of $St(m)$ (resp. $\oplus_i \text{Ind}[H^0(\Gamma_{M_i} \backslash Z_{M_i}, H^3(n_i))]$) as subspace in $H^*(Y_0)$ with respect to the decomposition given above by the $V_{\Omega(\chi_p)}^{\Theta(\chi_p)}$ lead then together with some “weight” arguments as used in the proof of 3.3. to the results. There is some analogy to the case SL_3/\mathbb{Q} described in [22].

4.2. *The space $H_{\max}^*(\Gamma)$.* Let \mathcal{C}_i be a set of representatives for the set of Γ -conjugacy classes $\Gamma \backslash \mathcal{C}(P_i)$, $i = 1, 2$. As in 4.1. there is an adelic interpretation of the space $H_{\text{cusp}}^*(\partial_i) := \oplus H_{\text{cusp}}^*(e'(P))$, $P \in \mathcal{C}_i$, and one attaches to a class in this space an adelic Eisenstein series $E(\psi_{\mathbf{A}}, \Lambda)$ $\Lambda \in \mathfrak{a}_{i,\mathbf{C}}^*$. Recall that by 3.4. there exists in degree 4 in $H^4(\Gamma \backslash X)$ a subspace $H_{\max}^4(\Gamma \backslash \bar{X}) := \oplus_i H_{\mathcal{C}_i}^4(\Gamma \backslash X)$ generated by Eisenstein cohomology classes constructed as in 3.3. which restricts under r^4 isomorphically onto $\oplus_i H_{\text{cusp}}^4(\partial_i)$. In degrees 2 and 3 one has by 2.5., 3.2. for the cusp cohomology of a face $e'(P)$

$$H_{\text{cusp}}^{1+q}(e'(P)) = H^1(0_{\mathfrak{m}}, K_M; V_{\pi_i} \otimes H^q(\mathfrak{n})), P \in \mathcal{C}_i, q = 1, 2, \tag{1}$$

where π_i is an (anti-) holomorphic discrete series representation D_3^{\pm} of ${}^0M \cong SL_2(\mathbb{R}) \times \{\pm 1\}$ with lowest K_M -type 3 for $i = 1$ (resp. a discrete series representation D_4 of ${}^0M \cong SL_2^{\pm}(\mathbb{R})$ with lowest K_M -type 4 for $i = 2$).

PROPOSITION: *Let $\psi_{\mathbf{A}}^i \in H_{\text{cusp}}^3(\partial_i)$, $i = 1, 2$, be a non-trivial cuspidal cohomology class of degree 3; if the attached Eisenstein series $E(\psi_{\mathbf{A}}^i, \Lambda)$, $\Lambda \in \mathfrak{a}_{i,\mathbf{C}}^*$, is holomorphic at $\Lambda_i = (1/i + 1)\rho_{P_i}$, then the form $E(\psi_{\mathbf{A}}^i, \Lambda_i)$ is closed and harmonic and represents a non-trivial class in $H^3(\Gamma \backslash X)$ with $r^3([E(\psi_{\mathbf{A}}^i, \Lambda_i)]) = \psi_{\mathbf{A}}^i$.*

This is proved by the adelic version of the arguments given in the proof of 3.3.

In order to decide if $E(\psi_{\mathbf{A}}^i, \Lambda)$ is holomorphic at Λ_i or not one analyzes the constant Fourier coefficient along P_i and the intertwining operator involved. The answer depends on the (non-) vanishing of certain Euler products attached to $\psi_{\mathbf{A}}^i$ in the sense of Langlands at special points. For details we refer to [24] where also precise criteria in terms of the type (e.g. CM) of the cuspidal automorphic form $\psi_{\mathbf{A}}^i$ are given under which $E(\psi_{\mathbf{A}}^i, \Lambda)$ has a pole at Λ_i . Now it turns out that there is a close relation between the Eisenstein cohomology classes which can be constructed in degree 3 as above and the ones in degree 2. This is already suggested by the dimension formula $\dim \text{Im } r^2 + \dim \text{Im } r^3 = \dim H^q(\partial(\Gamma \backslash \bar{X}))$, $q = 2, 3$, (cf. 2.7., 2.8.(2)). One obtains that for a class $\psi_{\mathbf{A}}$ in $\oplus H_{\text{cusp}}^3(\partial_i)$ which cannot be lifted to $H^3(\partial(\Gamma \backslash \bar{X}))$ via Eisenstein series there exists a class $\psi_{\mathbf{A}}'$ which provides via a residue of the attached Eisenstein series $E(\psi_{\mathbf{A}}', \Lambda)$ a non-trivial cohomology class $[\text{Res}_{\Lambda_i} E(\psi_{\mathbf{A}}', \Lambda)]$ of degree 2 in $H^2(\Gamma \backslash X)$. As a consequence of the scalar product formula ([12], IV, §8) this class is square-integrable; it restricts up to a scalar to $\psi_{\mathbf{A}}'$.

We denote by $H_{\max}^q(\Gamma \backslash \bar{X})$, $q = 2, 3, 4$, the subspace of $H^q(\Gamma \backslash \bar{X})$ generated by the Eisenstein cohomology classes constructed above for $q = 2, 3$ resp. by 3.3. for $q = 4$.

4.3. THEOREM: Let $\Gamma = \Gamma(m)$, $m \geq 3$, be the congruence subgroup of level m of $Sp_4(\mathbb{Z})$; denote by $H_{\text{Eis}}^*(\Gamma \backslash \bar{X}) := H_{\max}^*(\Gamma \backslash \bar{X}) \oplus H_{\min}^*(\Gamma \backslash \bar{X})$ the subspace in $H^*(\Gamma \backslash \bar{X})$ generated by the Eisenstein cohomology classes constructed in 3.3., 4.1., 4.2. Then one has a direct sum decomposition

$$H^*(\Gamma \backslash \bar{X}) = H_{\dagger}^*(\Gamma \backslash \bar{X}) \oplus H_{\text{Eis}}^*(\Gamma \backslash \bar{X}) \tag{1}$$

where $H_{\dagger}^*(\Gamma \backslash \bar{X})$ is the image in $H^*(\Gamma \backslash X)$ of the cohomology with compact supports. The Eisenstein cohomology $H_{\text{Eis}}^*(\Gamma \backslash \bar{X})$ maps under the restriction $r^*: H^*(\Gamma \backslash \bar{X}) \rightarrow H^*(\partial(\Gamma \backslash \bar{X}))$ isomorphically onto the image of r^* . Its dimension is given in 2.9. For the interior cohomology $H_{\dagger}^*(\Gamma \backslash \bar{X})$ one has

$$\begin{aligned} H_{\dagger}^4(\Gamma \backslash \bar{X}) &= H_{(2)}^4(\Gamma \backslash \bar{X}) \quad \text{resp.} \\ H_{\text{cusp}}^3(\Gamma \backslash \bar{X}) &= H_{\dagger}^3(\Gamma \backslash \bar{X}) = H_{(2)}^3(\Gamma \backslash \bar{X}) \end{aligned} \tag{2}$$

where $H_{(2)}^*(\Gamma \backslash \bar{X})$ is the subspace of $H^*(\Gamma \backslash X)$ given by classes represented by closed square integrable forms, and $H_{\text{cusp}}^*(\Gamma \backslash \bar{X})$ denotes the cusp cohomology of Γ (cf. 3.1.). Each class in $H^*(\Gamma \backslash X)$ has a harmonic automorphic form as a representative.

Using the long exact cohomology sequence of the pair $(\Gamma \backslash \bar{X}, \partial(\Gamma \backslash \bar{X}))$ this follows by the construction of $H_{\text{Eis}}^*(\Gamma \backslash \bar{X})$ because we obtained exactly as many classes with non-trivial restriction as the size of the cohomology of Γ “at infinity” (determined in 2.7., 2.9.) allows us to get. Observe that $H_{\text{Eis}}^q(\Gamma \backslash \bar{X})$, $q > 2$, consists by construction of non-square integrable classes. This implies (2); here we use also (cf. [16], 1.6.) that only a discrete series representation of G can contribute non-trivially to $H_{(2)}^3(\Gamma \backslash \bar{X})$ (cf. 4.4.).

4.4. Since used before we include the finite list (up to equivalence) of irreducible unitary representations of G with non-trivial (\mathfrak{g}, K) -cohomology; it can be derived from [28]. The representations (2) correspond e.g. to the square integrable Eisenstein cohomology classes of degree 2 (cf. 4.2.). First of all, there is the trivial representation \mathbb{C} which has nontrivial cohomology $H^q(\mathfrak{g}, K; \mathbb{C})$ in degrees 0, 2, 4, 6. By ([11], §41, Thm. 16) there are four discrete series representations ω_{dis}^j , $j = 1, \dots, 4$, whose infinitesimal character coincides with the one of \mathbb{C} (cf. [6], [20]). One has ([4], II, 5.3., 5.4.)

$$H^q(\mathfrak{g}, K; \omega_{dis}^{\infty}) = \mathbb{C} \quad \text{for } q = 3 = (1/2) \dim X, \tag{1}$$

it vanishes otherwise. Moreover, there are the three Langlands quotients

$$J(P_i, \sigma_i, \Lambda_i), \quad \Lambda_i = 1/(i+1)\rho_{P_i} \quad (i = 1, 2), \quad \sigma_1 = D_3^\pm, \quad \sigma_2 = D_4 \quad (2)$$

having non-trivial cohomology only in degree 2 and 4; they occur as subquotients of the reducible induced representations $\mathrm{Ind}_{P_i, \sigma_i, \Lambda_i}$. This completes the list.

4.5. *Twisted coefficients.* Considering the cohomology of $\Gamma = \Gamma(m)$ with coefficients in the local system given by a finite dimensional representation (τ, E) one obtains an analogue of the decomposition 4.3.(1). In particular, if the highest weight of τ is sufficiently regular, $H_{\mathrm{Eis}}^q(\Gamma \backslash X, E)$ restricts via r^q isomorphically onto $\oplus_i H_{\mathrm{cusp}}^3(\partial_i, E)$ for $q = 3$ resp. $H^4(Y_0, E) \oplus \oplus_i H_{\mathrm{cusp}}^4(\partial_i, E)$ for $q = 4$ and vanishes otherwise. This can be shown by means of the methods used in §3 since the question if $E(\phi, \Lambda)$ is holomorphic at the special point Λ_ϕ (cf. 3.1.) is easily answered because the sufficiently twisted coefficients force Λ_ϕ to lie inside the region of absolute convergence of the defining series.

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