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## CONTINUOUS FIBRATIONS AND INVERSE LIMITS OF TOPOSES

Ieke Moerdijk

### Introduction

We will discuss some preservation properties of limits of filtered inverse systems of (Grothendieck) toposes. If  $(\mathcal{E}_i)_i$  is such a system, with geometric morphisms  $f_{ij}: \mathcal{E}_i \rightarrow \mathcal{E}_j$  ( $i \geq j$ ), and  $\mathcal{E}^\infty$  is the inverse limit with projection morphisms  $\pi_i: \mathcal{E}^\infty \rightarrow \mathcal{E}_i$ , we say that a property of geometric morphisms is preserved if whenever each  $f_{ij}$  has the property then so does each  $\pi_i$ . It will be proved that some of the important properties of geometric morphisms are preserved by filtered inverse limits, notably surjections, open surjections, hyperconnected geometric morphisms, connected locally connected geometric morphisms, and connected atomic morphisms (definitions and references will be given below).

Filtered inverse limits of toposes have been considered by Grothendieck and Verdier (SGA 4(2), Exposé IV, §8). Here we will take a slightly different – more ‘logical’ – approach, by exploiting more explicitly the possibility of regarding a topos as a set theoretic universe (for a *constructive* set theory, without excluded middle and without choice), and a geometric morphism  $\mathcal{F} \rightarrow \mathcal{E}$  of toposes as a topos  $\mathcal{F}$  constructed in this universe  $\mathcal{E}$ . This will also bring some parallels with iterated forcing (with finite supports) in set theory to the surface.

Already in the first section this parallel becomes apparent when we show that any geometric morphism of toposes can be represented by a morphism of the underlying sites which possesses some special properties that will be very useful for studying inverse limits. Such a morphism will be called a *continuous fibration*. In the second section, we will characterize some properties of geometric morphisms in terms of continuous fibrations. This enables us to prove the preservation properties for inverse sequences of toposes (sections 3 and 4). In the final section we will show how the results can be generalized to arbitrary (small) filtered systems.

Throughout this paper,  $\mathcal{S}$  denotes a fixed base topos, all toposes are assumed to be Grothendieck toposes (over  $\mathcal{S}$ ), and all geometric morphisms are taken to be bounded (over  $\mathcal{S}$ ). Moreover, 2-categorical

details will be suppressed for the sake of clarity of exposition. Such details are not really relevant for the preservation properties under consideration, but the meticulous reader should add a prefix ‘pseudo-’ or a suffix ‘up to canonical isomorphism’ at the obvious places.

### Acknowledgements

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### 1. Iteration and continuous fibrations

If  $\mathcal{E}$  is a topos and  $\mathbb{C}$  is a site in  $\mathcal{E}$ , we write  $\mathcal{E}[\mathbb{C}]$  for the category  $\text{Sh}_{\mathcal{E}}(\mathbb{C})$  of sheaves on  $\mathbb{C}$ , made in  $\mathcal{E}$ . So  $\mathcal{E}[\mathbb{C}]$  is a (Grothendieck) topos over  $\mathcal{E}$ , and every topos over  $\mathcal{E}$  is of this form. Now suppose we have geometric morphisms

$$\mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{S},$$

where  $\mathcal{E} = \mathcal{S}[\mathbb{C}]$  and  $\mathcal{F} = \mathcal{E}[\mathbb{D}]$ , for sites  $\mathbb{C}$  in  $\mathcal{S}$  and  $\mathbb{D}$  in  $\mathcal{E}$ . We will construct a site  $\mathbb{C} \times \mathbb{D}$  in  $\mathcal{S}$  such that  $\mathcal{F} \rightarrow \mathcal{S}$  is equivalent to  $\mathcal{S}[\mathbb{C} \times \mathbb{D}] \rightarrow \mathcal{S}$ , and  $\mathcal{F} \rightarrow \mathcal{E}$  corresponds to a flat continuous functor  $T: \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{D}$ . The *objects* of  $\mathbb{C} \times \mathbb{D}$  are pairs  $(C, D)$ , with  $C$  an object of  $\mathbb{C}$  and  $D$  an object of  $\mathbb{D}$  over  $C$  ( $D \in \mathbb{D}_0(C)$ , where  $\mathbb{D}_0: \mathbb{C}^{op} \rightarrow \mathcal{S}$  is the sheaf on  $\mathbb{C}$  of objects of  $\mathbb{D}$ ). *Morphisms*  $(C, D) \rightarrow (C', D')$  of  $\mathbb{C} \times \mathbb{D}$  are pairs  $(f, g)$ ,  $f: C \rightarrow C'$  in  $\mathbb{C}$ , and  $g: D \rightarrow D' | f$  over  $C$  (i.e.  $g \in \mathbb{D}_1(C)$ , where  $\mathbb{D}_1$  is the sheaf on  $\mathbb{C}$  of morphisms of  $\mathbb{D}$ , and  $D' | f = \mathbb{D}_0(f)(D')$ , the restriction of  $D'$  along  $f$ ). Composition is defined in the obvious way: if  $(f, g): (C, D) \rightarrow (C', D')$  and  $(f', g'): (C', D') \rightarrow (C'', D'')$ , then  $(f', g') \circ (f, g)$  is the pair  $(f' \circ f, (g' | f) \circ g)$ . The *Grothendieck topology* on  $\mathbb{C} \times \mathbb{D}$  is defined by:  $\{(C_i, D_i \xrightarrow{(f_i, g_i)} (C, D))\}_i$  covers iff the subsheaf  $S$  of  $\mathbb{D}_1$  at  $C$  generated by the conditions  $C_i \Vdash g_i \in S$  satisfies  $C \Vdash S$  covers  $D$ . (It looks as if we only use the topology of  $\mathbb{D}$ , but the topology of  $\mathbb{C}$  comes in with the definition of  $S$  as the subsheaf  $S \in \mathcal{P}(\mathbb{D}_1)(C)$  “generated by” these conditions.) One easily checks that this indeed defines a Grothendieck topology on  $\mathbb{C} \times \mathbb{D}$ .

If  $X$  is an object of  $\mathcal{S}[\mathbf{C}]$ , with a sheaf structure  $(E, |)$  for  $\mathbb{D}$  ( $E: X \rightarrow \mathbb{D}_0$  and  $|: X \times_{\mathbb{D}_0} \mathbb{D}_1 \rightarrow X$  the maps of extent and restriction), we can construct a sheaf  $\tilde{X}$  on  $\mathbf{C} \times \mathbb{D}$  as follows:  $\tilde{X}(C, D) = \{x \in X(C) \mid E_C(x) = D\}$ , and for  $(f, g): (C', D') \rightarrow (C, D)$  and  $x \in \tilde{X}(C, D)$ ,  $\tilde{X}(f, g)(x) = X(f)(x) \mid g$ .

Conversely, if  $Y$  is a sheaf on  $\mathbf{C} \times \mathbb{D}$ , we first construct a sheaf  $\bar{Y}$  on  $\mathbf{C}$ , by

$$\bar{Y}(C) = \perp \{Y(C, D) \mid D \in \mathbb{D}_0(C)\}$$

and restrictions along  $C' \xrightarrow{f} C$  given by

$$\bar{Y}(f)(y) = Y(f, 1)(y),$$

where  $y \in Y(C, D)$ ,  $1$  the identity on  $D \mid f$  (over  $C'$ ). The sheaf  $\bar{Y}$  carries a canonical sheaf structure for  $\mathbb{D}$  in  $\mathcal{S}[\mathbf{C}]$ : extent is given by the components

$$E_C: \bar{Y}(C) \rightarrow \mathbb{D}_0(C), \quad y \mapsto D \quad \text{if } y \in Y(C, D),$$

while restrictions are given by the components

$$|_C: \bar{Y}(C) \times_{\mathbb{D}_0(C)} \mathbb{D}_1(C) \rightarrow \bar{Y}(C):$$

if  $y \in \bar{Y}(C)$  with  $E_C(y) = D$ , and  $C \Vdash f: D' \rightarrow D$ , then

$$y \mid_C f = Y\left(\left((C, D') \xrightarrow{(1, f)} (C, D)\right)\right)(y) \in Y(C, D').$$

It is clear that  $X \mapsto \tilde{X}$  and  $Y \mapsto \bar{Y}$  define functors which are inverse to each other (up to natural isomorphism). Moreover, we have a canonical projection functor  $P: \mathbf{C} \times \mathbb{D} \rightarrow \mathbf{C}$ , and this yields

1.1. PROPOSITION:  $\mathcal{F}$  is equivalent to  $\mathcal{S}[\mathbf{C} \times \mathbb{D}]$ , in other words

$$\mathcal{S}[\mathbf{C}][\mathbb{D}] \simeq \mathcal{S}[\mathbf{C} \times \mathbb{D}]$$

and the geometric morphism  $\mathcal{F} \rightarrow \mathcal{E}$  is induced by the functor  $P: \mathbf{C} \times \mathbb{D} \rightarrow \mathbf{C}$ .

□

If we assume that the category (underlying)  $\mathbb{D} \in \mathcal{E} = \mathcal{S}[\mathbf{C}]$  has a terminal object  $1 \in \mathbb{D}_0$ , with components  $1_C \in \mathbb{D}_0(C)$ , then  $P: \mathbf{C} \times \mathbb{D} \rightarrow \mathbf{C}$  has a right adjoint

$$T: \mathbf{C} \rightarrow \mathbf{C} \times \mathbb{D}, \quad T(C) = (C, 1_C).$$

$T$  is a flat and continuous functor of sites, so it induces a geometric morphism  $\mathcal{F} = \mathcal{E}[\mathbb{D}] \rightarrow \mathcal{E}$  by Diaconescu's theorem, and this is again the given geometric morphism that we started with.

Even without assuming that  $\mathbb{D}$  has a terminal object (in  $\mathcal{E}$ ),  $P$  locally has a right adjoint. For each object  $(C, D) \in \mathbb{C} \times \mathbb{D}$ , the functor

$$P/(C, D): \mathbb{C} \times \mathbb{D}/(C, D) \rightarrow \mathbb{C}/C$$

has a right adjoint

$$T_{(C,D)}: \mathbb{C}/C \rightarrow \mathbb{C} \times \mathbb{D}/(C, D), \quad T_{(C,D)}(C' \xrightarrow{f} C) = (C', D|f).$$

So  $P/(C, D) \circ T_{(C,D)}$  is the identity on  $\mathbb{C}/C$ . If  $\mathbb{C}$  and  $\mathbb{D}$  have terminal objects 1, then  $T = T_{(1,1)}$ .

Thus we have obtained a fibration  $\mathbb{C} \times \mathbb{D} \rightarrow \mathbb{C}$ . In general, if we are given a geometric morphism  $\mathcal{F} \rightarrow \mathcal{E}$  and a site  $\mathbb{C}$  in  $\mathcal{S}$  for  $\mathcal{E}$ , we can express  $\mathcal{F}$  as sheaves on a site (with a terminal object) in  $\mathcal{E}$ , and then "take this site out" to obtain a site in the base topos  $\mathcal{S}$ , as described in Proposition 1.1. In this way, we get the following theorem (by taking for  $\mathbb{D}$  in 1.2 the site  $\mathbb{C} \times \mathbb{D}$  of 1.1).

**1.2. THEOREM:** *Let  $\mathcal{F} \rightarrow \mathcal{E}$  be a geometric morphism over  $\mathcal{S}$ , and let  $\mathbb{C}$  be a site for  $\mathcal{E}$ . Then there exists a site  $\mathbb{D}$  in  $\mathcal{S}$  for  $\mathcal{F}$ ,  $\mathcal{F} = \mathcal{S}[\mathbb{D}]$ , such that the geometric morphism  $\mathcal{F} \rightarrow \mathcal{E}$  is induced by a pair of functors*

$$\mathbb{D} \begin{array}{c} \xrightarrow{P} \\ \rightleftarrows \\ \xleftarrow{T} \end{array} \mathbb{C}$$

*with the following properties:  $T$  is right adjoint right inverse to  $P$ ,  $T$  is flat and continuous, and for each  $D \in \mathbb{D}$ ,  $P/D: \mathbb{D}/D \rightarrow \mathbb{C}/PD$  has a flat and continuous right adjoint right inverse  $T_D$ .  $\square$*

We will refer to a pair  $\mathbb{D} \rightleftarrows \mathbb{C}$  with these properties as a *continuous fibration*. The geometric morphism

$$p: \mathcal{S}[\mathbb{D}] \rightarrow \mathcal{S}[\mathbb{C}]$$

that such a pair induces is described in terms of presheaves by

$$p_* = \text{compose with } T$$

$$p^* = \text{compose with } P, \text{ then sheafify.}$$

If  $\mathbb{C}$  and  $\mathbb{D}$  have finite inverse limits, then so does  $\mathbb{C} \times \mathbb{D}$ , and  $P$  and  $T$  both preserve them.  $P$  cannot be assumed to preserve covers, however,

since in that case it would give rise to a pair of geometric morphisms  $\mathcal{F} \rightleftarrows \mathcal{E}$ ; see Moerdijk & Reyes (1984), Theorem 2.2.

1.3. EXAMPLES. (a) (*Iterated forcing in set theory*). If  $\mathbb{P}$  is a poset in  $\mathcal{S}$  (with  $p \leq p'$  iff  $p \rightarrow p'$ , iff “ $p$  extends  $p'$ ”), and  $\mathbb{Q}$  is a poset in  $\mathcal{S}^{\mathbb{P}^{op}}$ , then  $\mathbb{P} \times \mathbb{Q}$  is the poset in  $\mathcal{S}$  of pairs  $(p, q)$  with  $p \in \mathbb{P}$ ,  $p \Vdash q \in \mathbb{Q}$ , and  $(p, q) \leq (p', q')$  iff  $p \leq p'$  and  $p \Vdash q \leq q'$ . If  $\mathcal{E} = \text{Sh}_{\mathcal{S}}(\mathbb{P}, \neg, \neg)$ , and  $\mathbb{Q}$  is a poset in  $\mathcal{E}$ ,  $\mathcal{F} = \text{Sh}_{\mathcal{E}}(\mathbb{Q}, \neg, \neg)$ , then  $\mathcal{F} \simeq \text{Sh}_{\mathcal{S}}(\mathbb{P} \times \mathbb{Q}, \neg, \neg)$ . In other words,  $(\mathbb{P}, \neg, \neg) \times (\mathbb{Q}, \neg, \neg) \cong (\mathbb{P} \times \mathbb{Q}, \neg, \neg)$ .

(b) Let  $G$  be a group in  $\mathcal{S}$  (thought of as a category with one object), and  $\mathcal{E} = \mathcal{S}^G$  be the topos of left  $G$ -sets. If  $H$  is a group in  $\mathcal{E}$ ,  $H$  can be identified with a group in  $\mathcal{S}$  on which  $G$  acts on the left (preserving unit and multiplication of  $H$ ). According to proposition 1.1 above,  $(\mathcal{S}^G)^H \simeq \mathcal{S}^{G \times H}$ , where  $G \times H$  is the product of  $G$  and  $H$  with group action given by  $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 g_1 h_2)$  (since  $G$  acts on the left,  $h_1 g_1 h_2$  can only be read as  $h_1(g_1 h_2)$ ). This is just the *semidirect product* of  $G$  and  $H$ , i.e.

$$(\mathcal{S}^G)^H \simeq \mathcal{S}^{G \times H}.$$

(c) Let  $\mathbb{C}$  be a site in  $\mathcal{S}$ , and  $A$  a locale in  $\mathcal{S}[\mathbb{C}]$ .  $A$  gives rise to the following data (see e.g. Joyal & Tierney (1984)): for each  $C \in \mathbb{C}$  a frame  $A(C)$  in  $\mathcal{S}$ , and for each morphism  $f: C \rightarrow D$  of  $\mathbb{C}$  a frame map  $(\wedge\text{-map}) A(D) \xrightarrow{A(f)} A(C)$  with a left adjoint  $\Sigma_f$ . The formula for internal sups of  $A$  (or rather, in the frame of opens of  $A$ ) is as follows: If  $S \in \Omega^A(C)$  is a subsheaf of  $A$  over  $C$ , then  $V_A(S) \in A(C)$  is the element  $V_{A(C)} \bigcup_{f: D \rightarrow C} \{ \Sigma_f(x) \mid x \in S(D \xrightarrow{f} C) \subseteq A(D) \}$ . So a site  $\mathbb{C} \times A$  for  $\mathcal{S}[\mathbb{C}][A]$  has as objects the pairs  $(a, C)$ ,  $a \in A(C)$ ; as morphisms  $(a, C) \xrightarrow{f} (b, D)$ , where  $C \xrightarrow{f} D$  in  $\mathbb{C}$  and  $a \leq A(f)(b)$  in  $A(C)$ ; and  $(a_i, C_i) \xrightarrow{f_i} (b, D)$  covers iff  $V_{A(D)} \{ \Sigma_{f_i}(a_i) \mid i \} = b$ .

If  $G$  is a topological group and  $A$  is a locale in the topos  $\mathcal{C}(G)$  of continuous  $G$ -sets, one may apply this procedure to a site for  $\mathcal{C}(G)$ , so as to obtain an explicit site for  $\mathcal{C}(G)[A]$  as used in Freyd (1979). (A site for  $\mathcal{C}(G)$  is the atomic site with the following underlying category: objects are quotients  $G/U$ ,  $U$  an open subgroup of  $G$ , and morphisms  $\varphi: G/U \rightarrow G/V$  are maps of left  $G$ -sets, or equivalently, cosets  $gV$  with  $U \subseteq gVg^{-1}$ .)

## 2. Some types of geometric morphisms

We will express some properties of geometric morphisms in terms of their corresponding continuous fibrations of sites in  $\mathcal{S}$ .

We start with the case of surjections. Recall that a geometric morphism  $\mathcal{F} \xrightarrow{\varphi} \mathcal{E}$  is a *surjection* iff  $\varphi^*$  is faithful, iff the localic part of  $\varphi$ ,  $\text{Sh}_{\mathcal{E}}(\varphi_*(\Omega_{\mathcal{F}})) \rightarrow \mathcal{E}$  is a surjection, iff the unique  $\wedge$ -map  $\Omega_{\mathcal{E}} \rightarrow \varphi_*(\Omega_{\mathcal{F}})$  in  $\mathcal{E}$  is monic.

**2.1. LEMMA:** *Let  $\mathcal{F} \xrightarrow{\varphi} \mathcal{E}$  be a geometric morphism, and  $\mathbb{C}$  a site for  $\mathcal{E}$ , i.e.  $\mathcal{E} = \mathcal{S}[\mathbb{C}]$ . Then  $\varphi$  is a surjection iff  $\varphi$  is induced by a continuous fibration  $\mathbb{D} \xrightleftharpoons[T]{P} \mathbb{C}$  with the property that  $P$  preserves covers of objects in the image of  $T$  (i.e., if  $\{D_i \xrightarrow{f_i} TC\}_i$  is a cover in  $\mathbb{D}$ , then  $\{PD_i \xrightarrow{\hat{f}_i} C\}_i$  covers in  $\mathbb{C}$ , where  $Pf_i = \hat{f}_i =$  the transpose of  $f_i$ .)*

**PROOF:** ( $\Rightarrow$ ) let  $\mathbf{A}$  be a site for  $\mathcal{F}$  in  $\mathcal{E}$ , with a terminal object  $1_{\mathbf{A}}$ . As is well-known,  $\mathcal{F} = \mathcal{E}[\mathbf{A}] \rightarrow \mathcal{E}$  is a surjection iff  $\mathcal{E} \models$  “every cover of  $1_{\mathbf{A}}$  is inhabited”. Under the construction  $\mathbb{D} = \mathbb{C} \times \mathbf{A}$  of section 1, this translates precisely into the righthand side of the equivalence stated in the lemma.

( $\Leftarrow$ ) There are several ways of proving this direction. We choose to compute the canonical map of frames  $\Omega_{\mathcal{E}} \xrightarrow{f} \varphi_*(\Omega_{\mathcal{F}})$  in  $\mathcal{E}$  (since we will need to consider this map anyway), and show that it is monic.

Since  $\varphi_*$  comes from composing with  $T$ ,

$$\varphi_*(\Omega_{\mathcal{F}})(C) = \text{closed cribles on } TC \text{ in } \mathbb{D}.$$

The components  $\lambda_C: \Omega_{\mathcal{E}}(C) \rightarrow \varphi_*(\Omega_{\mathcal{F}})(C)$  of the map  $\lambda$  are given by

$$\lambda_C(K) = \text{the closed crible generated by } TK, \text{ or equivalently,}$$

$$\text{by } \{D \xrightarrow{f} TC \mid Pf: PD \rightarrow C \in K\}.$$

We will denote the closure of a crible for a given Grothendieck topology by brackets  $[\cdot]$ , so  $\lambda_C(K) = [TK] = [\{f \mid Pf \in K\}]$ . The (internal) right adjoint  $\rho$  of  $\lambda$  has components

$$\rho_C: \varphi_*(\Omega_{\mathcal{F}})(C) \rightarrow \Omega_{\mathcal{E}}(C)$$

$$\rho_C(S) = \{C' \xrightarrow{f} C \mid Tf \in S\}$$

(this is a closed crible already, since  $T$  preserves covers).

So if  $C' \xrightarrow{f} C \in \rho_C \lambda_C(K)$ , that is  $TC' \xrightarrow{Tf} TC \in \lambda_C(K)$ , then there is a cover  $D_i \xrightarrow{g_i} TC'$  in  $\mathbb{D}$  such that  $P(Tf \circ g_i) \in K$  for each  $i$ . But  $P(Tf \circ g_i) = f \circ Pg_i$ , and  $\{Pg_i: D_i \rightarrow C'\}$  covers by assumption, so  $f \in K$ .  $\square$

Next, let us consider open surjections.  $\mathcal{F} \xrightarrow{\varphi} \mathcal{E}$  is *open* iff  $\varphi^*$  preserves first order logic. As with surjections,  $\varphi$  is open iff its localic part is, iff the unique  $\wedge \vee$ -map  $\Omega_{\mathcal{E}} \rightarrow \varphi_*(\Omega_{\mathcal{F}})$  in  $\mathcal{E}$  has an internal left adjoint. See Johnstone (1980), Joyal & Tierney (1984) for details, and various other equivalents.

2.2. LEMMA: Let  $\mathcal{F} \xrightarrow{\varphi} \mathcal{E}$  and  $\mathcal{E} = \mathcal{S}[\mathbb{C}]$  as in 2.1. Then  $\varphi$  is an open surjection iff  $\varphi$  is induced by a continuous fibration  $\mathbb{D} \rightleftarrows \mathbb{C}$  such that  $P$  preserves covers.

PROOF: ( $\Rightarrow$ ) If  $\varphi$  is open, there is a site  $\mathbf{A}$  for  $\mathcal{F}$  in  $\mathcal{E}$  such that  $\mathcal{E} \models \text{“}\forall A \in \mathbf{A}: \text{all covers of } A \text{ are inhabited”}$  (see Joyal & Tierney (1984)). Moreover, if  $\varphi$  is a surjective, we may assume that  $\mathbf{A}$  has a terminal object. So by constructing  $\mathbb{D} = \mathbb{C} \times \mathbf{A}$ , the implication from left to right is clear.

( $\Leftarrow$ ) We will show that the unique internal frame map in  $\mathcal{E}$ ,  $\lambda: \Omega_{\mathcal{E}} \rightarrow \varphi_*(\Omega_{\mathcal{F}})$ , has an internal left adjoint  $\mu$ . First note that since  $P$  preserves covers, the components of  $\lambda$  (cf. the proof of 2.1) can now be described by

$$\lambda_C(K) = \{ D \xrightarrow{f} TC \mid Pf: PD \rightarrow C \in K \}$$

for  $C \in \mathbb{C}$ ,  $K$  a closed crible on  $C$  in  $\mathbb{C}$ . Now define  $\mu: \varphi_*(\Omega_{\mathcal{F}}) \rightarrow \Omega_{\mathcal{E}}$  by setting for  $C \in \mathbb{C}$  and  $S$  a closed crible on  $TC$ ,

$$\begin{aligned} \mu_C(S) &= [PS], \quad \text{the closed crible generated by} \\ &\{ Pf: PD \rightarrow C \mid f: D \rightarrow TC \in S \}, \end{aligned}$$

$\mu$  is indeed a natural transformation, for suppose we are given  $\alpha: C' \rightarrow C$  in  $\mathbb{C}$ ,  $S \in \Omega_{\mathcal{F}}(TC)$ . We have to check that

$$\begin{array}{ccc} \Omega_{\mathcal{F}}(TC) & \xrightarrow{\mu_C} & \Omega_{\mathcal{E}}(C) \\ \downarrow (T\alpha)^{-1} & & \downarrow \alpha^{-1} \\ \Omega_{\mathcal{F}}(TC') & \xrightarrow{\mu_{C'}} & \Omega_{\mathcal{E}}(C') \end{array}$$

$$\alpha^{-1}(\mu_C(S)) = \mu_{C'}((T\alpha)^{-1}(S)), \text{ i.e.}$$

$$\begin{aligned} &\{ C_0 \xrightarrow{f} C \mid \alpha f \in [PS] \} \\ &= [ \{ Pg: PD \rightarrow C' \mid g: D \rightarrow TC', T\alpha \circ g \in S \} ]. \end{aligned}$$

$\supseteq$  is clear, since  $P(T\alpha \circ g) = \alpha \circ Pg: PD \rightarrow C$ . Conversely, suppose  $C_0 \xrightarrow{f} C' \xrightarrow{\alpha} C \in [PS]$ . Then there is a cover  $C_i \xrightarrow{h_i} C_0$  such that  $\alpha fh_i = P(k_i)$  for some  $k_i: D_i \rightarrow TC \in S$ . But  $k_i$  can be factored as

$$\begin{array}{ccc} D_i & \xrightarrow{k_i} & TC \\ & \searrow u_i & \nearrow T\alpha \\ & & TC' \end{array}$$

such that  $Pu_i = f \circ h_i$ , by adjointness  $P \dashv T$ , from which the inclusion  $\subseteq$  follows immediately.

Finally,  $\mu$  and  $\lambda$  are indeed adjoint functors, since as one easily checks,

$$\mu_C \lambda_C(K) \subset K, \quad \lambda_C \mu_C(S) \supset S,$$

for each  $C \in \mathbb{C}$  and closed cibles  $S$  on  $TC$ ,  $K$  on  $C$ .  $\square$

Recall that  $\mathcal{F} \xrightarrow{\varphi} \mathcal{E}$  is *hyperconnected* iff its localic part is trivial, i.e.  $\varphi_*(\Omega_{\mathcal{F}}) \cong \Omega_{\mathcal{E}}$  in  $\mathcal{E}$  (see Johnstone (1981), Joyal & Tierney (1984)).

2.3. LEMMA: Let  $\mathcal{F} \xrightarrow{\varphi} \mathcal{E}$ ,  $\mathcal{E} = \mathcal{S}[\mathbb{C}]$  be as before. Then  $\varphi$  is hyperconnected iff  $\varphi$  is induced by a continuous fibration  $\mathbb{D} \xrightleftharpoons[T]{P} \mathbb{C}$  such that  $P$  preserves covers, and moreover every unit morphism  $D \xrightarrow{\eta_D} TPD$  is a singleton-cover in  $\mathbb{D}$ .

PROOF: ( $\Rightarrow$ ) Write  $\varphi$  as  $\mathcal{E}[\mathbf{A}] \rightarrow \mathcal{E}$ ,  $\mathbf{A}$  a site for  $\mathcal{F}$  in  $\mathcal{E}$ .  $\varphi$  is an open surjection, so we may assume that it holds in  $\mathcal{E}$  that if  $\{A_i \rightarrow A\}_{i \in I}$  is a cover then  $I$  must be inhabited, and moreover that  $\mathbf{A}$  contains a terminal object.  $\varphi$  is hyperconnected iff the canonical frame map  $\mathcal{P}(1) \rightarrow \varphi_*(\Omega_{\mathcal{F}})$  (= the poset of closed cibles on  $\mathbf{A}$ ) is an isomorphism in  $\mathcal{E}$ , iff its left adjoint

$$\mu(K) = \llbracket K \text{ is inhabited} \rrbracket$$

is an isomorphism. So  $K$  is inhabited iff  $1 \in K$ , and hence every map  $A \rightarrow 1$  is a cover in  $\mathbf{A}$ .  $\Rightarrow$  now follows by taking  $\mathbb{D} = \mathbb{C} \times \mathbf{A}$ , as before.

( $\Leftarrow$ ) Recall  $\lambda: \Omega_{\mathcal{E}} \rightarrow \varphi_*(\Omega_{\mathcal{F}})$  and its right adjoint  $\rho$  from the proofs of 2.1 and 2.2. By 2.1, we have  $\rho\lambda = id$ . Conversely, if  $C \in \mathbb{C}$  and  $S \in \Omega_{\mathcal{F}}(TC)$  is a closed cible on  $TC$  in  $\mathbb{D}$ , then

$$\begin{aligned} \lambda_C \rho_C(S) &= \{D \xrightarrow{f} TC \mid Pf: PD \rightarrow C \in \rho_C(S)\} \\ &= \{D \xrightarrow{f} TC \mid TPD \xrightarrow{TPf} TC \in S\}. \end{aligned}$$

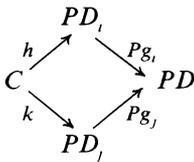
But if  $D \xrightarrow{\eta} \text{TPD}$  covers, then  $TPf: \text{TPD} \rightarrow \text{TC} \in S$  iff  $TPf \circ \eta = D \xrightarrow{f} \text{TC} \in S$ , i.e.  $\lambda\rho = id$ . So  $\varphi_*(\Omega_{\mathcal{F}}) \cong \Omega_{\mathcal{E}}$ .  $\square$

A geometric morphism  $\mathcal{F} \xrightarrow{\varphi} \mathcal{E}$  is *connected* if  $\varphi^*$  is full and faithful.  $\varphi$  is called *atomic* if  $\varphi^*$  is logical, or equivalently, if  $\mathcal{F} = \mathcal{E}[\mathbb{D}]$  for some atomic site  $\mathbb{D}$  in  $\mathcal{E}$  (see Barr & Diaconescu (1980), Joyal & Tierney (1984)). Such an atomic morphism  $\varphi$  is connected iff it is of the form  $\mathcal{E}[\mathbb{D}]$  for some atomic site  $\mathbb{D}$  having a terminal object (1 is an atom in  $\mathcal{F}$ ).

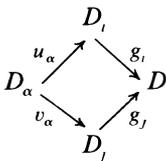
2.4. LEMMA: *Again, let  $\mathcal{F} \xrightarrow{\varphi} \mathcal{E}$  be a geometric morphism,  $\mathbb{C}$  a site for  $\mathcal{E}$  in  $\mathcal{S}$ . Then  $\varphi$  is atomic connected iff  $\varphi$  is induced by a continuous fibration  $\mathbb{D} \begin{smallmatrix} \xrightarrow{P} \\ \xrightarrow{T} \end{smallmatrix} \mathbb{C}$  where  $P$  preserves and reflects covers.*

PROOF: ( $\Rightarrow$ ) Let  $\mathbb{A}$  be an atomic site for  $\mathcal{F}$  in  $\mathcal{E}$ , with a terminal object. It suffices to take  $\mathbb{D} = \mathbb{C} \times \mathbb{A}$ .

( $\Leftarrow$ ) Let us first note that if  $X$  is a sheaf on  $\mathbb{C}$ ,  $X \circ P$  is a sheaf on  $\mathbb{D}$ . In that case  $\varphi^* = \text{compose with } P$ , and  $\varphi_*\varphi^* \cong id$ , so  $\varphi$  is certainly a connected surjection. To see this, suppose  $\{D_i \xrightarrow{g_i} D\}_i$  is a cover in  $\mathbb{D}$  and  $x_i \in X(PD_i)$  is a compatible family of elements – compatible over  $\mathbb{D}$ , that is.  $\{PD_i \xrightarrow{Pg_i} PD\}_i$  is a cover in  $\mathbb{C}$ , so it suffices to show that the family  $\{x_i\}$  is also compatible over  $\mathbb{C}$ . That is, for each  $i, j$  and each commutative square



in  $\mathbb{C}$ ,  $x_i|_h = x_j|_k$ . Since  $\{x_i\}$  is  $\mathbb{D}$ -compatible and  $X$  is a sheaf on  $\mathbb{C}$ , it is sufficient to find a cover  $PD_\alpha \xrightarrow{f_\alpha} C$  and for each  $\alpha$  a commutative square



with  $P(u_\alpha) = hf_\alpha$ ,  $P(v_\alpha) = kg$ . Write  $e: C \rightarrow PD$  for  $Pg_i \circ h = Pg_j \circ k$ , and let  $\tilde{D} \xrightarrow{\tilde{e}} D = T_D(e)$ . Applying  $T_{PD}$ , and  $T_{PD}$ , we find a diagram

$$\begin{array}{ccc}
 \tilde{D}_i & \xrightarrow{\quad} & D_i \\
 \tilde{D}_i & \xrightarrow{\tilde{g}_i} & \tilde{D} \\
 \tilde{D}_j & \xrightarrow{\tilde{g}_j} & \tilde{D} \\
 \tilde{D}_j & \xrightarrow{\quad} & D_j \\
 & & \tilde{D} \xrightarrow{g_i} D
 \end{array}$$

with  $P\tilde{g}_i = P\tilde{g}_j = id_C$ .  $P$  reflects covers, so  $\tilde{D}_i \xrightarrow{\tilde{g}_i} D$  is a cover in  $\mathbb{D}$  (consisting of one element). Pulling back this cover along  $\tilde{g}_j$  yields a family of commutative squares

$$\begin{array}{ccc}
 & \tilde{D}_i & \\
 \tilde{D}_\alpha & \nearrow \tilde{u}_\alpha & \tilde{D} \\
 & \tilde{D}_j & \\
 \tilde{D}_\alpha & \searrow \tilde{v}_\alpha & \tilde{D}_j \\
 & \tilde{D}_j & \\
 & \tilde{D}_j & \nwarrow \tilde{g}_j \\
 & \tilde{D}_j & \nearrow \tilde{g}_i \\
 & \tilde{D}_j &
 \end{array}$$

such that the  $\tilde{v}_\alpha$  cover  $\tilde{D}_j$ . We need only set  $f_\alpha = P(\tilde{g}_i \circ \tilde{u}_\alpha) = P(\tilde{g}_j \circ \tilde{v}_\alpha)$  to obtain the desired cover of  $C$ .

Having the information that  $\varphi^*(X) = X \circ P$  for every  $X \in \mathcal{S}[\mathbb{C}]$ , we easily prove that  $\varphi^*$  is logical.

First,  $\varphi^*$  preserves the subobject classifier: we always have a canonical map  $\Omega_{\mathcal{F}} \xrightarrow{\sigma} \varphi^*(\Omega_{\mathcal{E}})$  in  $\mathcal{F}$  with an adjoint  $\tau$ . In this case the components are described by

$$\sigma_D: \Omega_{\mathcal{F}}(D) \rightarrow \Omega_{\mathcal{E}}(PD)$$

$$\sigma_D(K) = \{C \xrightarrow{f} PD \mid T_D(f) \in K\} = PK$$

$$\tau_D: \Omega_{\mathcal{E}}(PD) \rightarrow \Omega_{\mathcal{F}}(D)$$

$$\tau_D(S) = \{D' \xrightarrow{g} D \mid Pg \in S\}.$$

Note that  $\sigma_D(K)$  as described is indeed a closed crible, and that  $\sigma$  and  $\tau$  are natural. We claim that  $\sigma$  and  $\tau$  are inverse to each other. If  $S$  is a closed crible on  $PD$ , then  $\sigma_D\tau_D(S) = P(\{D' \xrightarrow{g} D \mid Pg \in S\})$ , so clearly  $\sigma_D\tau_D(S) \subset S$ . Conversely if  $C \xrightarrow{f} PD \in S$ , then  $T_D(f) \in \tau_D(S)$  and  $PT_D(f) = f$ , so  $S \subset \sigma_D\tau_D(S)$ . And if  $K$  is a closed crible on  $D$ , we have

$\tau_D \sigma_D(K) = \{ D' \xrightarrow{g} D \mid T_D(Pg) \in K \}$ , so  $K \subset \tau_D \sigma_D(K)$  since there is a factorization

$$\begin{array}{ccc} D' & \xrightarrow{g} & D \\ \downarrow v & & \uparrow T_D(Pg) \\ & T_D(D') & \end{array}$$

But  $P$  maps this morphism  $v: D' \rightarrow T_D(D') := \text{domain } T_D(Pg)$  to the identity on  $PD'$ , so  $v$  covers since  $P$  reflects covers. Thus  $K \supset \tau_D \sigma_D(K)$ .

Finally, we show that  $\varphi^*$  preserves exponentials, i.e. for  $X, Y \in \mathcal{E}$ ,  $\varphi^*(Y^X) \cong \varphi^*(Y)^{\varphi^*(X)}$ . Let us describe the isomorphism  $\alpha: \varphi^*(Y^X) \rightarrow \varphi^*(Y)^{\varphi^*(X)}$  with inverse  $\beta$  explicitly:  $Y^X(C) =$  the set of natural transformations  $X \rightarrow Y$  over  $C$ , so  $\varphi^*(Y^X)(D) = Y^X(PD)$ , where  $\varphi^*(Y)^{\varphi^*(X)}(D) =$  the set of natural transformations  $X \circ P \rightarrow Y \circ P$  over  $D$ . The canonical map  $\alpha: \varphi^*(Y^X) \rightarrow \varphi^*(Y)^{\varphi^*(X)}$  has components  $\alpha_D$  described as follows. Given  $\tau: X \rightarrow Y$  over  $PD$ ,  $\alpha_D(\tau): X \circ P \rightarrow Y \circ P$  over  $D$  is defined by

$$\alpha_D(\tau)_f(x) = \tau_{Pf}(x)$$

where  $D' \xrightarrow{f} D$  and  $x \in X(PD')$ . The components  $\beta_D$  of  $\beta: \varphi^*(Y)^{\varphi^*(X)} \rightarrow \varphi^*(Y^X)$  are described as follows. Given  $\sigma: X \circ P \rightarrow Y \circ P$  over  $D$ ,  $\beta_D(\sigma): X \rightarrow Y$  over  $PD$  is defined by

$$\beta_D(\sigma)_g(x) = \sigma_{T_D(g)}(x),$$

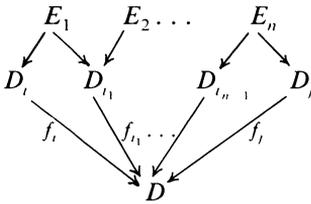
for  $C \xrightarrow{g} PD$ , (so  $T_D(g): D' \rightarrow D$  with  $PD' = C$ ), and  $x \in X(C)$ . It is clear that  $\beta \circ \alpha$  is the identity. The other way round,  $\alpha \circ \beta = id$  follows from the fact that if  $D' \xrightarrow{f} D$ , then the unit  $u: D' \rightarrow \text{domain } T_D(Pf)$ , with  $T_D(Pf) \circ u = f$ , is mapped by  $P$  to the identity on  $PD'$ . For if  $x \in X(PD')$  then  $\alpha_D \circ \beta_D(\sigma)_f(x) = \beta_D(\sigma)_{Pf}(x) \sigma_{T_D(Pf)}(x) = \sigma_f(x)$ , the last equality by naturality of  $\sigma$ , since  $Pu = id$ .  $\square$

Finally, we consider the case of connected locally connected geometric morphisms. This class has been studied in Barr & Paré (1980). For some equivalent descriptions see the Appendix (This Appendix has been added since we will need the main characterization of locally connected morphisms of Barr & Paré in a slightly different form later on, and moreover for the bounded case there is a rather short and self-contained proof of this characterization, which we give in this Appendix.)

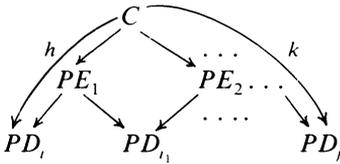
Let  $\mathbb{D} \begin{smallmatrix} P \\ \rightrightarrows \\ T \end{smallmatrix} \mathbb{C}$  be a continuous fibration, and let  $\{D_i \xrightarrow{f_i} D\}_{i \in I}$  be a cover in  $\mathbb{D}$ . Given a commutative square

$$(*) \quad \begin{array}{ccc} & C & \\ h \swarrow & & \searrow k \\ PD_i & & PD_i \\ Pf_i \searrow & & \swarrow Pf_i \\ & PD & \end{array}$$

we call  $D_i$  and  $D_j$  *C-connected* (for the cover  $\{D_i \xrightarrow{f_i} D\}$ ) if there are  $i = i_0, i_1, \dots, i_n = j$  in  $I$  such that there is a commuting zig-zag



the *P*-image of which commutes under *C*:



2.5. LEMMA: Let  $\mathcal{F} \xrightarrow{\varphi} \mathcal{E}$ ,  $\mathcal{E} = \mathcal{S}[\mathbb{C}]$  be as before. Then  $\varphi$  is connected locally connected iff it is induced by a continuous fibration  $\mathbb{D} \begin{smallmatrix} P \\ \rightrightarrows \\ T \end{smallmatrix} \mathbb{C}$  such that *P* preserves covers, and for every cover  $\{D_i \xrightarrow{f_i} D\}$  in  $\mathbb{D}$  and every commutative square of the form  $(*)$ , the family of maps  $C' \rightarrow C$  such that  $D_i, D_j$  are  $C'$ -connected is a cover of *C*.

PROOF: ( $\Rightarrow$ ) Let  $\mathbb{A}$  be a molecular site (see Appendix) in  $\mathcal{E}$ , with terminal object  $1_{\mathbb{A}}$ . A straightforward argument by forcing over  $\mathbb{C}$  shows that  $\mathbb{D} = \mathbb{C} \times \mathbb{A}$  satisfies the required conditions.

( $\Leftarrow$ ) This is really completely similar to the proof of  $\Leftarrow$  of lemma 2.4. The condition on  $\mathbb{D} \begin{smallmatrix} P \\ \rightrightarrows \\ T \end{smallmatrix} \mathbb{C}$  is exactly what we need to show that if *X* is a sheaf on  $\mathbb{C}$ ,  $X \circ P$  is a sheaf on  $\mathbb{D}$ . So again  $\varphi^* = \text{compose with } P$  and  $\varphi_* \varphi^* \cong id$ . So  $\varphi$  is connected. Moreover, it is now straightforward to

check that  $\varphi^*$  commutes with  $\Pi$ -functors (or, since the conditions on  $\mathbb{D} \rightleftarrows \mathbb{C}$  are stable under localization, just check that  $\varphi^*$  preserves exponentials), as in the proof of 2.4.  $\square$

2.6. **REMARK:** Equivalences 2.1–2.5 can be stated in a stronger way, by analyzing which properties of continuous fibrations are actually used in the proofs of the  $\Leftarrow$ -parts. We do not need these stronger versions, the formulation of which we leave to the reader.

### 3. A description of inverse limits

In the special case of inverse sequences of geometric morphisms we can easily express the inverse limit explicitly as sheaves over a site using continuous fibrations. Suppose we are given a sequence

$$\dots \mathcal{E}_{n+1} \xrightarrow{f_n} \mathcal{E}_n \rightarrow \dots \rightarrow \mathcal{E}_1 \xrightarrow{f_0} \mathcal{E}_0$$

of toposes and geometric morphisms over  $\mathcal{S}$ . By the results of sections 1, we can find sites  $\mathbb{C}_n$  for  $\mathcal{E}_n$  in  $\mathcal{S}$ , that is  $\mathcal{E}_n = \mathcal{S}[\mathbb{C}_n]$ , such that  $f_n$  is represented by a continuous fibration

$$\mathbb{C}_{n+1} \begin{array}{c} P_n \\ \rightleftarrows \\ T_n \end{array} \mathbb{C}_n.$$

We will write  $P_{nm}: \mathbb{C}_n \rightarrow \mathbb{C}_m$  for  $P_m \circ \dots \circ P_{n-1}$ ,  $T_{nm}: \mathbb{C}_m \rightarrow \mathbb{C}_n$  for  $T_{n-1} \circ \dots \circ T_m$ . ( $P_{nn}, T_{nn}$  are identity functors.)

We construct a site  $\mathbb{C}^\infty = \text{proj lim } \mathbb{C}_n$  as follows. Objects of  $\mathbb{C}^\infty$  are sequences  $(C_n)_n$ ,  $C_n$  an object of  $\mathbb{C}_n$ , such that  $P_n(C_{n+1}) = C_n$  and  $\exists m \forall n \geq m \ C_{n+1} = T_n(C_n)$ . Morphisms  $(C_n)_n \rightarrow (C'_n)_n$  of  $\mathbb{C}^\infty$  are just sequences  $f = (f_n)_n$ ,  $f_n: C_n \rightarrow C'_n$  a morphism of  $\mathbb{C}_n$ , such that  $P_n(f_{n+1}) = f_n$ .

There is a canonical fibration

$$\mathbb{C}^\infty \begin{array}{c} P_m^\infty \\ \rightleftarrows \\ T_m^\infty \end{array} \mathbb{C}_m$$

for each  $m$ :  $P_m^\infty((C_n)_n) = C_m$ , and for  $C \in \mathbb{C}_m$ ,

$$T_m^\infty(C)_n = \begin{cases} P_{mn}(C) & \text{if } m \geq n \\ T_{nm}(C) & \text{if } m \leq n. \end{cases}$$

$\mathbb{C}^\infty$  is made into a site by equipping it with the coarsest Grothendieck topology making all the functors  $T_m^\infty$  continuous. That is, the topology on  $\mathbb{C}^\infty$  is generated by covers of the form

$$\{T_m^\infty(C_i \xrightarrow{f_i} C)\}_i, \text{ for } \{C_i \xrightarrow{f_i} C\}_i \text{ a cover in } \mathbb{C}_m.$$

Note that this is a *stable* generating system for the topology. This topology makes each pair  $P_m^\infty, T_m^\infty: \mathbb{C}^\infty \rightleftarrows \mathbb{C}_m$  into a continuous fibration. Moreover, the  $P_m^\infty, T_m^\infty$  are coherent in the sense that  $T_{m+1}^\infty \circ T_m = T_m^\infty, P_m \circ P_{m+1}^\infty = P_m^\infty$ .

Let  $\mathcal{E}^\infty = \mathcal{S}[\mathbb{C}^\infty]$ . The functors  $T_m^\infty$  are flat and continuous, so they induce geometric morphisms

$$\pi_m: \mathcal{E}^\infty \rightarrow \mathcal{E}_m$$

over  $\mathcal{S}$ . We claim that  $\mathcal{E}^\infty = \text{proj lim } \mathcal{E}_m$ . Indeed, suppose we are given geometric morphisms  $g_n: \mathcal{F} \rightarrow \mathcal{E}_n$  with  $f_n \circ g_{n+1} = g_n$ , each  $n$  (or really only up to canonical isomorphism, since  $\mathcal{E}^\infty$  is a pseudolimit; but cf. the introduction).  $g_n$  comes from a flat and continuous functor  $\mathbb{C}_n \xrightarrow{G_n} \mathcal{F}$ , and  $G_{n+1} \circ T_n = G_n$ . Define  $G^\infty: \mathbb{C}^\infty \rightarrow \mathcal{F}$  by  $G^\infty((C_n)_n) = G_m(C_m)$ , for  $m$  so large that  $C_{n+1} = T_n(C_n), \forall n \geq m$ . (This does not depend on the choice of  $m$ , since  $G_{n+1} \circ T_n = G_n$ . Moreover, the canonical maps  $G_{m+1}(C_{m+1}) \rightarrow G_m(C_m)$  become isomorphisms eventually, so we could equivalently have set  $G^\infty((C_n)_n) = \text{proj lim } G_m(C_m)$ . This also looks more coherently functorial.) Observe that  $G^\infty$  is a flat and continuous functor  $\mathbb{C}^\infty \rightarrow \mathcal{F}$ , and that  $G^\infty \circ T_m^\infty = G_m$ . So we obtain a geometric morphism

$$g^\infty: \mathcal{F} \rightarrow \mathcal{E}^\infty, \quad \text{with } \pi_m \circ g^\infty = g_m.$$

$g^\infty$  is obviously unique up to natural isomorphism, since  $(C_n)_n = \text{proj lim}_m T_m^\infty(C_m)$  in  $\mathbb{C}^\infty$ , and this inverse limit is eventually constant, so really a finite inverse limit. Therefore, any flat and continuous functor  $H: \mathbb{C}^\infty \rightarrow \mathcal{F}$  with  $H \circ T_m^\infty = G_m$  for all  $m$  must satisfy  $H((C_n)_n) = \text{proj lim}(G_m(C_m))$ . For the record,

3.1. THEOREM: If  $\dots \mathcal{E}_{n+1} \xrightarrow{f_n} \mathcal{E}_n \rightarrow \dots \rightarrow \mathcal{E}_1 \xrightarrow{f_0} \mathcal{E}_0$  is a sequence of geometric morphisms, with  $f_n$  induced by a continuous fibration  $\mathbb{C}_{n+1} \xrightleftharpoons[T_n]{P_n} \mathbb{C}_n$ , then  $\mathbb{C}^\infty$  as constructed above is a site for  $\mathcal{E}^\infty = \text{proj lim } \mathcal{E}_n$ , and the canonical projections  $\mathcal{E}^\infty \xrightarrow{\pi_m} \mathcal{E}_m$  correspond to the continuous fibrations  $\mathbb{C}^\infty \xrightleftharpoons[T_m]{P_m^\infty} \mathbb{C}_m$ .  $\square$

Suppose we are given an inverse sequence of continuous fibrations

$$\dots \mathbb{C}_{n+1} \xrightleftharpoons[T_n]{P_n} \mathbb{C}_n \dots \rightleftarrows \mathbb{C}_2 \xrightleftharpoons[T_1]{P_1} \mathbb{C}_1 \xrightleftharpoons[T_0]{P_0} \mathbb{C}_0$$

as above. Let  $A_n$  be the preordered reflection of  $\mathbb{C}_n$ . That is, the objects of  $A_n$  are the same as those of  $\mathbb{C}_n$ , and we put  $C \leq C'$  in  $A_n$  iff there is a

morphism  $C \rightarrow C'$  in  $\mathbf{C}_n$ . A family  $\{C_i \leq C\}_i$  covers in  $A$  iff there is a covering family  $\{C_i \rightarrow C\}_i$  in  $\mathbf{C}_n$ . So  $\mathcal{S}[\mathbf{C}_n] \rightarrow \mathcal{S}[A_n] \rightarrow \mathcal{S}$  is the hyperconnected – localic factorization of  $\mathcal{S}[\mathbf{C}_n] = \mathcal{E}_n \rightarrow \mathcal{S}$ . There are canonical projection functors  $\mathbf{C}_n \xrightarrow{\tau_n} A_n$  and the continuous fibration  $\mathbf{C}_{n+1} \xrightleftharpoons[T_n]{P_n} \mathbf{C}_n$  is mapped down by  $\tau_n$  to a continuous fibration  $A_{n+1} \rightleftharpoons A_n$ . So we obtain a diagram

$$\begin{array}{ccccccc} \mathbf{C}^\infty & \dots & \rightleftharpoons \mathbf{C}_2 & \rightleftharpoons \mathbf{C}_1 & \rightleftharpoons \mathbf{C}_0 & & \\ \downarrow \tau^\infty & & \downarrow \tau_2 & \downarrow \tau_1 & \downarrow \tau_0 & & \\ A^\infty & \dots & \rightleftharpoons A_2 & \rightleftharpoons A_1 & \rightleftharpoons A_0 & & \end{array}$$

and as is immediate from the construction,  $A^\infty$ , the limit of the lower sequence as described before theorem 3.1, is the preordered reflection of the site  $\mathbf{C}^\infty$ . So  $\mathcal{S}[\mathbf{C}^\infty] \rightarrow \mathcal{S}[A^\infty] \rightarrow \mathcal{S}$  is the hyperconnected – localic factorization of  $\mathcal{E}^\infty \rightarrow \mathcal{S}$ .

Thus we have the following corollary.

**3.2. COROLLARY:** *The localic reflection preserves limits of inverse sequences.*  $\square$

This will be generalized in Section 5.

#### 4. Preservation properties of inverse sequences

We are now ready to prove preservation under limits of inverse sequences of toposes for the classes of geometric morphisms that were analyzed in Section 2. In the next section, this will be generalized to arbitrary filtered systems.

**4.1. THEOREM:** *Let  $\dots \mathcal{E}_{n+1} \xrightarrow{f_n} \mathcal{E}_n \rightarrow \dots \rightarrow \mathcal{E}_1 \xrightarrow{f_0} \mathcal{E}_0$  be a sequence of Grothendieck toposes and geometric morphisms over  $\mathcal{S}$ , and let  $\mathcal{E}^\infty = \text{proj lim } \mathcal{E}_n$  be the inverse limit of this sequence, with canonical projections  $\mathcal{E}^\infty \xrightarrow{\pi_n} \mathcal{E}_n$ . Then*

- (i) *If each  $f_n$  is a surjection, then so is each  $\pi_n$ ;*
- (ii) *If each  $f_n$  is an open surjection, then so is each  $\pi_n$ ;*
- (iii) *If each  $f_n$  is hyperconnected, then so is each  $\pi_n$ ;*
- (iv) *If each  $f_n$  is connected atomic, then so is each  $\pi_n$ ;*
- (v) *If each  $f_n$  is connected locally connected, then so is each  $\pi_n$ .*

**PROOF:** I postpone the proof of (i). For (ii), choose by repeated application of Lemma 2.2 a sequence of sites  $\mathbf{C}_n$  for  $\mathcal{E}_n$ , i.e.  $\mathcal{E}_n = \mathcal{S}[\mathbf{C}_n]$ , such that  $f_n$  is induced by a continuous fibration

$$\mathbf{C}_{n+1} \xrightleftharpoons[T_n]{P_n} \mathbf{C}_n$$

where  $P_n$  preserves covers. By Lemma 2.2 again, we need to show that each projection  $P_n^\infty: \mathbb{C}^\infty \rightarrow \mathbb{C}_n$  as described in Section 3 preserves covers. Since the family of covers generating the topology of  $\mathbb{C}^\infty$  is stable, it is sufficient to show that  $P_n^\infty$  preserves these generating covers. But if  $\{T_m^\infty(C_i) \xrightarrow{T_m^\infty(g_i)} T_m^\infty(C)\}_i$  is a basic cover of  $\mathbb{C}^\infty$  coming from a cover  $\{C_i \xrightarrow{g_i} C\}_i$  in  $\mathbb{C}_m$ , then  $P_n^\infty(T_m^\infty(g_i)) = P_{mn}(g_i)$  if  $n \leq m$ , and  $P_n^\infty(T_m^\infty(g_i)) = T_{nm}(g_i)$  if  $n \geq m$ , so  $P_n^\infty$  preserves this cover since both  $T_{nm}$  and  $P_{nm}$  preserve covers. This proves (ii).

The proofs of (iv) and (v) are entirely similar. For example to show for (iv) that  $P_m^\infty$  reflects covers if all the  $P_n$  do, we reason as follows: suppose we have a family of morphisms  $g^i = (g_n^i)_n$  in  $\mathbb{C}^\infty$  with common codomain,

$$(C_n^i)_n \xrightarrow{g^i} (C_n)_n,$$

and suppose  $\{P_m^\infty(g^i)\}_i$ , that is  $\{C_m^i \xrightarrow{g_m^i} C_m\}_i$ , is a cover in  $\mathbb{C}_m$ . Choose an  $n_0 \geq m$  such that  $C_{n+1} = T_n(C_n)$  for  $n \geq n_0$ .  $P_{n_0 m}$  reflects covers since all the  $P_m$  do, so  $\{C_{n_0}^i \xrightarrow{g_{n_0}^i} C_{n_0}\}_i$  is a cover in  $\mathbb{C}_n$ . Thus  $\{T_{n_0}^\infty(C_{n_0}^i) \xrightarrow{h^i} T_{n_0}^\infty(C_{n_0})\}_i$  is a cover in  $\mathbb{C}^\infty$ , where  $h^i = T_{n_0}^\infty(g_{n_0}^i)$ . But  $T_{n_0}^\infty(C_{n_0}) = (C_n)_n$  since  $m \geq n_0$ , so we need only show that for a fixed  $i$ , the canonical map

$$k^i: (C_n^i)_n \rightarrow T_{n_0}^\infty(C_{n_0}^i), \quad (k_n^i = g_n^i \text{ for } n \leq n_0)$$

is a cover. Choose  $n_i \geq n_0$  so large that  $(C_n^i)_n = T_{n_i}(C_{n_i}^i)$ . Then  $k^i$  is the map

$$k^i = T_{n_i}^\infty(k_{n_i}^i): T_{n_i}^\infty(C_{n_i}^i) \rightarrow T_{n_i}^\infty(T_{n_i, n_0}(C_{n_0}^i)),$$

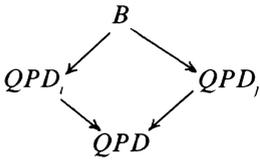
which covers because the map  $C_{n_i}^i \rightarrow T_{n_i, n_0}(C_{n_0}^i)$  covers in  $\mathbb{C}_{n_i}$ , as follows from the fact that  $P_{n_i, n_0}$  reflects covers.

For (v), suppose each continuous fibration  $\mathbb{C}_{n+1} \xrightleftharpoons[T_n]{P_n} \mathbb{C}_n$  is of the form as described in Lemma 2.5. We wish to show for each  $m$  that the continuous fibration  $\mathbb{C} \xrightleftharpoons[T_m^\infty]{P_m^\infty} \mathbb{C}_m$  is again of this form. It suffices to consider the basic covers in  $\mathbb{C}^\infty$  which generate the topology when verifying the conditions of Lemma 2.5. So let  $\{T_n^\infty(C_i \rightarrow C)\}_i$  be such a basic cover, coming from a cover  $\{C_i \rightarrow C\}_i$  in  $\mathbb{C}_n$ . Since  $T_n^\infty = T_{n'}^\infty \circ T_{n'n}$  (any  $n' \geq n$ ) and  $T_{n'n}$  preserves covers, we may without loss assume that  $n \geq m$ . It is now clear that we only need to show that the composite continuous fibration  $\mathbb{C}_n \xrightleftharpoons[T_{nm}]{P_{nm}} \mathbb{C}_m$  again satisfies the conditions of Lemma

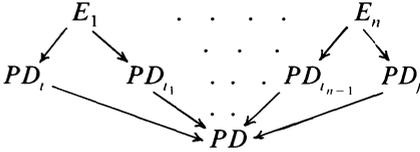
2.5. Thus, arguing by induction, the following lemma completes the proof of case (v) of the theorem.

LEMMA: Let  $\mathbb{D} \begin{smallmatrix} P \\ \rightrightarrows \\ T \end{smallmatrix} \mathbb{C} \begin{smallmatrix} Q \\ \rightrightarrows \\ S \end{smallmatrix} \mathbb{B}$  be two continuous fibrations, each as described in Lemma 2.5. Then the composite continuous fibration  $\mathbb{D} \begin{smallmatrix} QP \\ \rightrightarrows \\ TS \end{smallmatrix} \mathbb{B}$  also satisfies the conditions of Lemma 2.5.

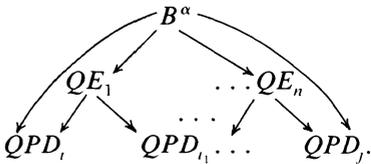
PROOF OF LEMMA: Clearly if  $P$  and  $Q$  preserve covers then so does  $QP$ . Suppose  $\{D_i \rightarrow D\}$  is a cover in  $\mathbb{D}$ , and for fixed indices  $i, j$  we are given  $B \in \mathbb{B}$  and a commutative square



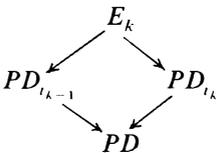
Applying the conditions of Lemma 2.5 to  $\mathbb{D} \rightarrow \mathbb{C}$ , we find a cover  $\{B^\alpha \rightarrow B\}_\alpha$  of  $B$ , and for each  $\alpha$  connecting zig-zags ( $n = n_\alpha$  depending on  $\alpha$ )



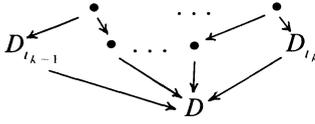
such that their  $Q$ -images commute under  $B^\alpha$ :



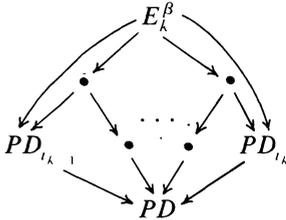
Now fix  $\alpha$ , and apply the conditions of 2.5 to the continuous fibration  $\mathbb{D} \rightrightarrows \mathbb{C}$  for each square ( $0 \leq k \leq n_\alpha$ )



separately. This gives for each  $k \leq n_\alpha$  a cover  $\{E_k^\beta \rightarrow E_k\}$  and connecting chains



the  $P$ -images of which commute under  $E_k^\beta$ :



For each  $k < n_\alpha$ ,  $\{QE_k^\beta \rightarrow QE_k\}$  covers in  $\mathbb{B}$ , and pulling back this cover along  $B^\alpha \rightarrow QE_k$  gives a cover of  $B^\alpha$ . It now suffices to take a common refinement of these  $n_\alpha$  – many covers of  $B^\alpha$ , say  $\{B^{\alpha\gamma} \rightarrow B^\alpha\}_\gamma$ ; then the family of composites  $\{B^{\alpha\gamma} \rightarrow C\}_{\alpha,\gamma}$  is a cover of  $C$  such that  $D_i$  and  $D_j$  are  $B^{\alpha\gamma}$ -connected (each  $\alpha, \gamma$ ), thus proving the lemma.

We now complete the proof of the theorem. (iii) can be proved exactly as (ii), (iv) and (v), but for (i) I do not see an argument of this type. But in any case, (i) and (iii) follow immediately from Corollary 3.2: Let  $A_n$  be the localic reflection of  $\mathcal{E}^n \rightarrow \mathcal{S}$ , so  $A^\infty = \text{proj lim } A_n$  is the localic reflection of  $\mathcal{E}^\infty$ .

$$\begin{array}{ccccc}
 \mathcal{E}^\infty & \overset{f_1}{\dashrightarrow} & \mathcal{E}_1 & \xrightarrow{f_0} & \mathcal{E}_0 \\
 \downarrow & \dots & \downarrow & & \downarrow \\
 \text{Sh}(A^\infty) & \overset{g_1}{\dashrightarrow} & \text{Sh}(A_1) & \xrightarrow{g_0} & \text{Sh}(A_0).
 \end{array}$$

For (iii), suppose each  $f_n$  is hyperconnected, and choose  $m$ . We want to show that  $\pi_m$  is hyperconnected. By working in  $\mathcal{E}_m$ , we may assume that  $m = 0$  and that  $\mathcal{E}_0 = \mathcal{S}$ . Since each  $f_n$  is hyperconnected, each  $A_n$  is the one-point locale 1 (i.e.  $\mathcal{O}(A_n) = \mathcal{P}(1)$ ), hence so is  $A^\infty$ , hence  $\mathcal{E}^\infty \rightarrow \mathcal{S}$  is hyperconnected.

For (i), suppose each  $f_n$  is surjective. To prove that each  $\pi_m$  is surjective, it is again sufficient to assume  $m = 0$  and  $\mathcal{S} = \mathcal{E}_0$ . But then by Proposition IV.4.2 of Joyal & Tierney (1984),  $A^\infty \rightarrow A_0 = 1$  is surjective, hence so is  $\mathcal{E}^\infty \rightarrow \mathcal{S}$ .

This completes the proof of the theorem.  $\square$

## 5. Preservation properties of filtered inverse systems

In this section we will generalize Theorem 4.1 to inverse limits of arbitrary (small) filtered systems of toposes, and prove the main theorem. (Recall that a poset  $I$  is filtered (or directed) if for all  $i, j \in I$  there is a  $k \in I$  with  $k \geq i, k \geq j$ .)

5.1. THEOREM: Let  $(\mathcal{E}_i)_{i \in I}$  be an inverse system of toposes over  $\mathcal{S}$ , indexed by a filtered poset  $I$  in  $\mathcal{S}$ , with transition mappings  $f_{ij}: \mathcal{E}_i \rightarrow \mathcal{E}_j$  for  $i \geq j$ . Let  $\mathcal{E}^\infty = \text{proj lim } \mathcal{E}_i$  be the inverse limit of this system, with projection mappings  $\mathcal{E}^\infty \xrightarrow{\pi_i} \mathcal{E}_i$ . Then

- (i) if each  $f_{ij}$  is a surjection, then so is each  $\pi_i$ ;
- (ii) if each  $f_{ij}$  is an open surjection, then so is each  $\pi_i$ ;
- (iii) if each  $f_{ij}$  is hyperconnected, then so is each  $\pi_i$ ;
- (iv) if each  $f_{ij}$  is connected atomic, then so is each  $\pi_i$ ;
- (v) if each  $f_{ij}$  is connected locally connected, then so is each  $\pi_i$ .

Corollary 3.2 will also be generalized to arbitrary filtered systems. A main ingredient involved in the proofs is the following theorem, which is of independent interest.

5.2. THEOREM: All the types of geometric morphisms involved in Theorem 5.1 are reflected down open surjections. More explicitly, if

$$\begin{array}{ccc}
 \mathcal{E}' & \xrightarrow{q} & \mathcal{E} \\
 (*) \quad \downarrow f' & & \downarrow f \\
 \mathcal{S}' & \xrightarrow{p} & \mathcal{S}
 \end{array}$$

is a pullback of toposes, where  $p$  is an open surjection, then

- (i) if  $f'$  is a surjection, then so is  $f$ ;
- (ii) if  $f'$  is an open surjection, then so is  $f$ ;
- (iii) if  $f'$  is hyperconnected, then so is  $f$ ;
- (iv) if  $f'$  is (connected) atomic, then so is  $f$ ;
- (v) if  $f'$  is (connected) locally connected, then so is  $f$ .

PROOF OF 5.2: (i) is trivial. Note that the types of geometric morphisms in (ii)–(v) are stable under pullback, so by using a localic cover  $\mathcal{S}[A] \rightarrow \mathcal{S}'$  ( $A$  a locale in  $\mathcal{S}$ ,  $\mathcal{S}[A] \rightarrow \mathcal{S}'$  an open surjection; see Diaconescu (1976), Johnstone (1980), Joyal & Tierney (1984)), we may assume that  $\mathcal{S}' \rightarrow \mathcal{S}$  is localic whenever this is convenient.

Case (ii) follows from proposition VII.1.2 of Joyal & Tierney (1984). For case (iii), let  $\mathcal{S}' = \mathcal{S}[A]$ ,  $A$  an open surjective locale in  $\mathcal{S}$ , and

write  $B = f_*(\Omega_{\mathcal{E}})$ ,  $B' = f'_*(\Omega_{\mathcal{E}'})$  for the localic reflections of  $\mathcal{E}$  in  $\mathcal{S}$  and of  $\mathcal{E}'$  in  $\mathcal{S}'$ . Then we have a pullback of open surjective locales

$$\begin{array}{ccc} B' & \rightarrow & B \\ \downarrow & & \downarrow \\ A & \rightarrow & 1 \end{array}$$

in  $\mathcal{S}$ , such that the map  $B' \rightarrow A$  is an isomorphism. So  $\mathcal{O}(B') = \mathcal{O}(A) \otimes \mathcal{O}(B)$ , and in terms of frames we get

$$\begin{array}{ccccc} \mathcal{O}(A) \otimes \mathcal{O}(B) & \xrightarrow{\pi_2^*} & \mathcal{O}(B) & & \\ \uparrow \pi_1^* & & \exists_{\pi_2} & & \uparrow B^* \\ \mathcal{O}(A) & \xrightarrow[A^*]{} & \mathcal{P}(1) & & \downarrow \\ & & \exists_1 & & \end{array}$$

where the maps  $\exists_{(\cdot)}$  are left adjoint to the frame maps  $(\cdot)^*$ , and a Beck condition holds (Joyal & Tierney (1984), §V.4):

$$A^* \exists_B = \exists_{\pi_1} \pi_2^*, \quad B^* \exists_A = \exists_{\pi_2} \pi_1^*.$$

By assumption  $\pi_1^*$  is an isomorphism, so  $\exists_{\pi_1}$  is its inverse. We claim that  $B^*$  is an isomorphism, with inverse  $\exists_B$ .  $\exists_B \circ B^* = id$  since  $B \rightarrow 1$  is an open surjection. The other way round, we have  $\pi_2^* B^* \exists_B = \pi_1^* A^* \exists_B = \pi_1^* \exists_{\pi_1} \pi_2^* = \pi_2^*$ , and  $\pi_2$  is an open surjection, so  $\pi_2^*$  is  $1 - 1$ , so  $B^* \exists_B = id$ . This proves case (iii).

For case (iv), let us first prove that atomicity by itself is reflected. This actually follows trivially from the characterization of atomic maps by Joyal & Tierney (1984), Ch.VIII: a map  $\mathcal{E} \xrightarrow{f} \mathcal{S}$  is atomic iff both  $f$  and the diagonal  $\mathcal{E} \xrightarrow{\Delta} \mathcal{E} \times_{\mathcal{S}} \mathcal{E}$  are open. For consider the following diagram:

$$\begin{array}{ccccc} \mathcal{E}' & \xrightarrow{\Delta'} & \mathcal{E}' \times_{\mathcal{S}'} \mathcal{E}' & \rightarrow & \mathcal{S}' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow{\Delta} & \mathcal{E} \times_{\mathcal{S}} \mathcal{E} & \rightarrow & \mathcal{S}. \end{array}$$

Since  $(*)$  in the statement of the theorem is a pullback and the right-hand square above is a pullback, so is the left-hand square. Therefore by case (ii) of the theorem, if  $\Delta'$  is open so is  $\Delta$ . Hence if  $\mathcal{E}' \xrightarrow{f'} \mathcal{S}'$  is atomic, so is  $\mathcal{E} \xrightarrow{f} \mathcal{S}$ .

Next, we show that if  $f'$  in the diagram  $(*)$  is *stably* connected (i.e. the pullback of  $f'$  along a geometric morphism is still connected, as is the

case for connectedness in combination with local connectedness, and hence in combination with atomicity, as in (iv) and (v) of the theorem), then  $f$  must be connected. To do this, we use descent theory (Ch.VIII of Joyal & Tierney (1984): As  $f'$  is stably connected, so is the map  $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \xrightarrow{f' \times f'} \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}'$ , since we have a pullback

$$\begin{array}{ccc} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' & \rightarrow & \mathcal{E} \\ \downarrow & & \downarrow f \\ \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}' & \rightarrow & \mathcal{S}, \end{array}$$

and similarly  $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \rightarrow \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}'$  is connected. Now clearly, if  $f'^*$  is faithful so is  $f^*$  (this is case (i) above). To show that  $f^*$  is full, let  $\beta: f^*(X) \rightarrow f^*(Y)$  be a map in  $\mathcal{E}$ .  $\mathcal{E}' \xrightarrow{q} \mathcal{E}$  is an open surjection, hence an effective descent morphism, so  $\beta$  is equivalent to a map  $\gamma = q^*(\beta): q^*f^*(X) \rightarrow q^*f^*(Y)$  compatible with descent data in  $\mathcal{E}'$ , i.e.  $\gamma: f'^*p^*(X) \rightarrow f'^*p^*(Y)$ . Descent data lives in  $\mathcal{E}'$ ,  $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}'$ , and  $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}'$ , so since  $f'$ ,  $f' \times f'$ ,  $f' \times f' \times f'$  are all connected, as just pointed out,  $\gamma$  must come from a map  $\delta: p^*(X) \rightarrow p^*(Y)$  compatible with descent data in  $\mathcal{S}'$ ,  $\gamma = f'^*(\delta)$ .  $\mathcal{S}' \xrightarrow{p} \mathcal{S}$  is an effective descent morphism, so  $\gamma = p^*(\alpha)$  for some  $\alpha: X \rightarrow Y$  in  $\mathcal{S}$ . Then also  $\beta = f^*(\alpha)$ . This proves that  $f^*$  is full.

It remains to show that if  $f'$  is locally connected, so is  $f$ . Since  $f$  and  $f'$  are open surjections, it suffices to prove (cf. the Appendix) that for a generating collection of objects  $X \in \mathcal{E}$ , the locale  $B = f_*(X_{\text{dis}})$  (corresponding to the frame  $f_*(\mathcal{P}(X))$  of subobjects of  $X$ ) is locally connected. Thus, assuming  $\mathcal{S}' = \mathcal{S}[A] \xrightarrow{p} \mathcal{S}$  is localic over  $\mathcal{S}$ , it suffices to show for open surjective locales  $A$  and  $B$  in  $\mathcal{S}$  that if  $A \times B \xrightarrow{\pi_1} A$  is locally connected, i.e.  $Sh(A) \models "p^\#(B) \text{ is a locally connected locale}"$ , then  $B$  is locally connected in  $\mathcal{S}$ . We use proposition 3 of the Appendix (or rather, the first lines of its proof). Let  $S, T \in \mathcal{S}$ , and let  $\alpha: f^*(T) \rightarrow f^*(S)$  be a map in  $Sh(B)$ . Consider  $\beta = \pi_2^*(\alpha): \pi_2^*f^*(T) \rightarrow \pi_2^*f^*(S)$ , i.e.  $\beta: \pi_1^*p^*(T) \rightarrow \pi_1^*p^*(S)$  in  $Sh(A \times B) = Sh_{Sh(A)}(p^\#(B))$ . Since  $\pi_1$  is locally connected, it holds in  $Sh(A)$  that  $\beta$  locally (in the sense of  $p^\#(B)$ ) comes from a map  $\gamma: p^*(T) \rightarrow p^*(S)$ . As seen from  $\mathcal{S}$ , this means that there is a cover  $\{U_i \times V_i\}_i$  of  $A \times B$ ,  $U_i \in \mathcal{O}(A)$  and  $V_i \in \mathcal{O}(B)$ , and maps  $\gamma_i: p^*_i(T) \rightarrow p^*_i(S)$  over  $U_i$  such that  $\pi_1^*(\gamma_i) = \beta = \pi_2^*(\alpha)$  in  $Sh(U_i \times V_i)$ :

$$\begin{array}{ccc} Sh(U_i \times V_i) & \xrightarrow{\pi_2} & Sh(V_i) \\ \downarrow \pi_1 & & \downarrow f_i \\ Sh(U_i) & \xrightarrow{p_i} & \mathcal{S}. \end{array}$$

Translating this back into  $\mathcal{S}$ , we have continuous maps (of locales)

$\alpha: V_i \rightarrow S^T$  and  $\gamma_i: U_i \rightarrow S^T$  (where  $S^T$  denotes the product of  $T$  copies of the discrete locale  $S$ , cf. the Appendix), such that  $U_i \times V_i \xrightarrow{\pi_2} V_i \xrightarrow{\alpha} S^T = U_i \times V_i \xrightarrow{\pi_1} U_i \xrightarrow{\gamma_i} S^T$ . The following lemma then shows that  $\alpha$  must locally (namely, for the cover  $\{V_i\}$  of  $B$ ) come from a map  $S^T$  in  $\mathcal{S}$ .

LEMMA: *Let*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_1} & X \\ \downarrow \pi_2 & & \downarrow p \\ Y & \xrightarrow{q} & Z \end{array}$$

*be a pullback of locales in  $S$ , with  $p$  and  $q$  open surjections. Then this square is also a pushout of locales.*

PROOF OF LEMMA: This follows once more from the Beck condition mentioned above: we have a pushout of frame maps  $(.)^*$

$$\begin{array}{ccc} \mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y) & \xrightarrow{\pi_1^*} & \mathcal{O}(X) \\ \pi_2^* \updownarrow \exists_{\pi_2} & & \exists_{\pi_1} \updownarrow p^* \\ \mathcal{O}(Y) & \xrightarrow{q^*} & \mathcal{O}(Z) \end{array}$$

with left adjoints  $\exists_{(-)}$ , and  $q^* \exists_p = \exists_{\pi_2} \pi_1^*$ ,  $p^* \exists_q = \exists_{\pi_1} \pi_2^*$ . Now suppose we have continuous maps  $h: X \rightarrow P$  and  $k: Y \rightarrow P$  into some locale  $P$  such that  $h\pi_1 = k\pi_2$ . It suffices to show that  $h^* = p^* \exists_p h^*$  and  $k^* = q^* \exists_q h^*$ . For the first equality, we have  $\pi_1^* p^* \exists_p h^* = \pi_2^* q^* \exists_q h^* = \pi_2^* \exists_{\pi_2} \pi_1^* h^* = \pi_2^* \exists_{\pi_2} \pi_2^* k^* = \pi_2^* k^* = \pi_1^* h^*$ .  $\pi_1^*$  is  $1 - 1$ , so  $h^* = p^* \exists_p h^*$ . The other equation is verified similarly.

This completes the proof of the lemma.

Applying this argument to each open of  $B$ , we find that  $B$  is a locally connected locale. Thus we have proved Theorem 5.2.  $\square$

5.3. LEMMA: *Let  $I$  be a directed poset in  $\mathcal{S}$ . Then there exists a localic open surjection  $\mathcal{S}[A] \xrightarrow{p} \mathcal{S}$  and a  $\varphi: \mathbb{N} \rightarrow p^*(I)$  in  $\mathcal{S}[A]$  such that  $\varphi$  is cofinal in  $I$ , i.e.  $\mathcal{S}[A] \models \forall i \in p^*(I) \exists n \in \mathbb{N} \varphi(n) \geq i$ .*

PROOF: We adjoin a generic cofinal sequence to the universe in the standard way: Let  $\mathbb{P}$  be the poset of finite sequences  $s = (i_1, \dots, i_n)$  from  $I$  with  $i_1 \leq \dots \leq i_n$ , partially ordered by  $s \leq t$  iff  $s$  extends  $t$ , and let the locale  $A$  be defined by the following covering system on  $\mathbb{P}$ : for each  $(i_1, \dots, i_n) \in \mathbb{P}$  and each  $j \in I$ , the family

$$\{(i_1, \dots, i_n, \dots, i_m) \mid j \leq i_m\}$$

of extensions of  $(i_1, \dots, i_n)$  covers  $(i_1, \dots, i_n)$ . Since  $I$  is directed, this is a stable family of covers (i.e. if  $S$  covers  $s$  and  $t \leq s$  then  $\{r \in S \mid r \leq t\}$  covers  $t$ ), and moreover each cover is inhabited. Thus if  $A$  is the locale defined by this poset with covering system on  $\mathbb{P}$ ,  $\mathcal{S}[A] \rightarrow \mathcal{S}$  is an open surjection.  $\square$

PROOF OF THEOREM 5.1: Let  $(\mathcal{E}_i)_i$  be the given system in  $\mathcal{S}$ , and let  $(\mathcal{F}_i)_i$  denote the corresponding system obtained by change of base along the map  $\mathcal{S}[A] \rightarrow \mathcal{S}$  of 5.3. Let  $\mathcal{F}^\infty$  and  $\mathcal{E}^\infty$  denote the inverse limits. So we have a pullback

$$\begin{array}{ccc} \mathcal{F}^\infty & \rightarrow & \mathcal{E}^\infty \\ \downarrow & & \downarrow \\ \mathcal{S}[A] & \longrightarrow & \mathcal{S} \end{array}$$

$\mathcal{F}^\infty = \text{proj lim}_i \mathcal{F}_i = \text{proj lim}_{\varphi(n)} \mathcal{F}_{\varphi(n)}$ , where  $\varphi$  is the cofinal sequence of lemma 5.3. Since the geometric morphisms of types (ii)–(v) are all preserved by pullback, the result now follows immediately from theorems 5.2 and 4.1 in these cases. The case (i) of surjections follows from Corollary 5.4 below and Proposition IV.4.2 of Joyal & Tierney (1984), just as for inverse sequences (see section 4).  $\square$

5.4. COROLLARY: *The localic reflection preserves limits of filtered inverse systems of toposes.*

PROOF: The hyperconnected–localic factorization is preserved by pullback, so this follows from Corollary 3.2, using a change of base by Lemma 5.3 as in the proof of 5.1.  $\square$

5.5. REMARK: In Joyal & Tierney (1984) it was noted that the localic reflection preserves binary products. By Corollary 5.4 this can be extended to arbitrary (small) products. It is not true, however, that the localic reflection preserves all inverse limits. Here is a simple example of a case where it does not preserve pullbacks: Let  $\mathbb{C}$  be the category with two objects  $C, D$  and only two non-identity arrows  $f$  and  $g: C \rightarrow D$

$$C \cdot \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} \cdot D$$

Let  $\mathbb{D}_1$  be the category (poset) in  $\mathcal{S}^{\mathbb{C}^{op}}$  with  $\mathbb{D}_1(D) = \{0 \leq 1\}$ ,  $\mathbb{D}_1(C) = \{0 \leq 1\}$ ,  $\mathbb{D}_1(f)(1) = \mathbb{D}_1(f)(0) = 1$ ,  $\mathbb{D}_1(g) = id$ . Let  $\mathbb{D}_2$  be exactly the same, but with  $f$  and  $g$  interchanged. Construct the product-poset  $\mathbb{D}_1 \times \mathbb{D}_2$  in  $\mathcal{S}^{\mathbb{C}^{op}}$ . So we have a pullback

$$\begin{array}{ccc} \mathcal{S}^{\mathbb{C} \times (\mathbb{D}_1 \times \mathbb{D}_2)^{op}} & \rightarrow & \mathcal{S}^{\mathbb{C} \times \mathbb{D}_1^{op}} \\ \downarrow & & \downarrow \\ \mathcal{S}^{\mathbb{C} \times \mathbb{D}_2^{op}} & \rightarrow & \mathcal{S}^{\mathbb{C}^{op}} \end{array}$$

The poset-reflection of the diagram

$$\begin{array}{ccc} \mathbb{C} \times (\mathbb{D}_1 \times \mathbb{D}_2) & \rightarrow & \mathbb{C} \times \mathbb{D}_1 \\ \downarrow & & \downarrow \\ \mathbb{C} \times \mathbb{D}_1 & \longrightarrow & \mathbb{C} \end{array}$$

however, does not give a pullback of posets, as is easily verified, and therefore it does not give a pullback of the corresponding locales of downwards closed sets.  $\square$

### Appendix: Locally connected geometric morphisms

Let  $\mathcal{S}$  be an *arbitrary* topos, which we fix as our base topos from now on. In M. Barr & R. Paré (1980) several characterizations are given of toposes which are *locally connected* (they say: *molecular*) over  $\mathcal{S}$ . The purpose of this Appendix is to give an alternative proof of these characterizations for the special case of Grothendieck toposes over  $\mathcal{S}$ . So in the sequel, geometric morphism means *bounded* geometric morphism. Our proof uses locale theory, and familiarity with the paper of Barr and Paré is not presupposed. Also, this Appendix can be read independently from the rest of this paper.

For the case of Grothendieck toposes over  $\mathcal{S}$ , the result of Barr and Paré can be stated as follows.

**THEOREM:** *The following conditions on a geometric morphism  $\gamma: \mathcal{E} \rightarrow \mathcal{S}$  are equivalent:*

(1) *There exists a locally connected site  $\mathbb{C} \in \mathcal{S}$  such that  $\mathcal{E} = \mathcal{S}[\mathbb{C}]$ , the topos of sheaves on  $\mathbb{C}$  (locally connected site is defined below).*

(2) *There exists a site  $\mathbb{C} \in \mathcal{S}$  such that  $\mathcal{E} = \mathcal{S}[\mathbb{C}]$ , with the property that all constant presheaves on  $\mathbb{C}$  are sheaves.*

(3) *The functor  $\gamma^*: \mathcal{S} \rightarrow \mathcal{E}$  (left adjoint to the global sections functor  $\gamma_*: \mathcal{E} \rightarrow \mathcal{S}$ ) has an  $\mathcal{S}$ -indexed left adjoint.*

(4)  *$\gamma^*$  commutes with  $\Pi$ -functors, i.e. for each  $S \xrightarrow{\alpha} T$  in  $\mathcal{S}$  we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{S}/S & \xrightarrow{\gamma^*/S} & \mathcal{E}/\gamma^*(S) \\ \downarrow \Pi_\alpha & & \downarrow \Pi_{\gamma^*(\alpha)} \\ \mathcal{S}/T & \xrightarrow{\gamma^*/T} & \mathcal{E}/\gamma^*(T) \end{array}$$

A geometric morphism satisfying these equivalent conditions is called *locally connected*. Recall that a *site*  $\mathbb{C}$  is called *locally connected* (or

molecular) if every covering sieve of an object  $C \in \mathbb{C}$  is connected (as a full subcategory of  $\mathbb{C}/C$ ) and inhabited.

The implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) are easy. The difficult part is (4)  $\Rightarrow$  (1), and it is here that our approach differs from the one taken by Barr and Paré. To prove (4)  $\Rightarrow$  (1), choose first a site  $\mathbb{C}$  for  $\mathcal{E}$  in  $\mathcal{S}$ , closed under finite limits and subobjects, say. It suffices to show that for each  $C \in \mathbb{C}$ , the *locale* (in  $\mathcal{S}$ )  $\text{Sub}_{\mathcal{E}}(C)$  of subobjects of  $C$  is locally connected (see definition 1 below). If  $S \in \mathcal{S}$ , a section of  $\gamma^*(S)$  over  $C$  is nothing but a continuous map from the locale  $\text{Sub}_{\mathcal{E}}(C)$  into the discrete locale  $S$ , i.e. a global section of the constant object  $S$  in the localic topos  $\text{Sh}(\text{Sub}_{\mathcal{S}}(C))$ , so it follows that for each  $C \in \mathbb{C}$ , the inverse image of the canonical geometric morphism  $\text{Sh}(\text{Sub}_{\mathcal{S}}(C)) \rightarrow \mathcal{S}$  preserves  $\Pi$ -functors. So we only need to show (4)  $\Rightarrow$  (1) of the theorem for locales, i.e. to show proposition 3 below. But let us recall some definitions first.

**DEFINITION 1:** Let  $A$  be a locale in  $\mathcal{S}$ .

- (i) an element  $a \in A$  is called *positive* (written  $\text{pos}(a)$ ) if every cover of  $a$  is inhabited.  $A$  is positive if  $\text{pos}(1_A)$ .
- (ii)  $a \in A$  is called *connected* if  $a$  is positive, and every cover of  $a$  by positive elements is positively connected. ( $A$  cover  $\mathcal{U} = \{b_i\}_i$  of  $a$  is positively connected if for every  $b_i, b_j \in \mathcal{U}$  there is a chain  $b_i = b_{i_0}, \dots, b_{i_n} = b_j$  in  $\mathcal{U}$  such that  $\forall k < n \text{ pos}(b_{i_k} \wedge b_{i_{k+1}})$ .)  $A$  is connected if  $1_A$  is connected.
- (iii)  $A$  is *open* if for every  $a \in A$ ,  $a = \bigvee \{b \leq a \mid \text{pos}(b)\}$ .
- (iv)  $A$  is *locally connected* if for every  $a \in A$ ,  $a = \bigvee \{b \leq a \mid b \text{ is connected}\}$

**LEMMA 2:** Let  $A$  be a positive open locale in  $\mathcal{S}$ . The following conditions are equivalent:

- (1)  $A$  is connected.
- (2) For every  $S \in \mathcal{S}$ , every continuous map  $A \xrightarrow{f} S$  into the discrete locale  $S$  is constant, i.e.  $\exists s \in S f^{-1}(s) = 1_A$ .
- (3) Every continuous map  $A \rightarrow \Omega$  into the discrete locale  $\Omega = \mathcal{P}(1)$  is constant.

**PROOF:** (1)  $\Rightarrow$  (2): Let  $A \xrightarrow{f} S$ , and put  $a_s = f^{-1}(\{s\})$ . Then  $\bigvee_{s \in S} a_s = 1$ , so also  $\bigvee \{a_s \mid \text{pos}(a_s)\} = 1$  since  $A$  is open. This latter cover must be positively connected. But if  $\text{pos}(a_s \wedge a_{s'})$ , then since  $a_s \wedge a_{s'}$  is covered by  $f^{-1}(s) \wedge f^{-1}(s') = f^{-1}(\{s\} \cap \{s'\}) = \{f^{-1}(s) \mid s = s'\}$ , it follows that  $s = s'$ . So  $\exists s \in S a_s = 1$ .

(2)  $\Rightarrow$  (3) is clear.

(3)  $\Rightarrow$  (1): Suppose  $\mathcal{U} = \{b_i\}_{i \in I}$  is a cover of  $1_A$ , with  $\text{pos}(b_i)$  for all  $i$ . Fix  $i_0 \in I$  and define  $p_j \in \Omega$  for each  $j \in I$  to be the value of the

sentence

$$\exists \text{ chain } b_{i_0}, b_{i_1}, \dots, b_{i_n} = b_j \quad \text{in } \mathcal{U} \text{ such that } \forall k < n \\ \text{pos}(b_{i_k} \wedge b_{i_{k+1}}).$$

Now let  $f^{-1}: \mathcal{P}(\Omega) \rightarrow \mathcal{O}(A)$  be defined by

$$f^{-1}(V) = \vee \{ b_j \in \mathcal{U} \mid p_j \in V \}.$$

We claim that  $f^{-1}$  is an  $\wedge \vee$ -map, i.e. defines a continuous map  $A \rightarrow \Omega$ . Indeed,  $f^{-1}$  preserves  $\vee$  by definition, and  $f^{-1}(\Omega) = \vee \mathcal{U} = 1_A$ . To show that  $f^{-1}$  preserves  $\wedge$ , it suffices by openness of  $A$  that  $\text{pos}(b_j \wedge b_{j'})$  implies  $p_j = p_{j'}$ , which is obvious from the definition of  $p_j$ . By (3),  $f$  is constant. But  $b_{i_0} \leq f^{-1}(\{T\})$ , so  $1_A \leq f^{-1}(\{T\})$ , i.e.  $\mathcal{U}$  connected.

Note that from this proof we can extract that for a positive open locale  $A$ , a continuous function  $A \xrightarrow{f} S$  ( $S$  discrete) corresponds to giving a sequence  $\{a_s \mid s \in S\}$  of elements of  $A$  such that  $\vee_{s \in S} a_s = 1$  and  $\text{pos}(a_s \wedge a_{s'}) \Rightarrow s = s'$ . We will call such a cover  $\{a_s \mid s \in S\}$  of  $A$  *discrete*.

**PROPOSITION 3:** *Let  $A$  be a locale in  $\mathcal{S}$ , and suppose that the inverse image functor  $\gamma^*$  of the geometric morphism  $\text{Sh}(A) \rightarrow \mathcal{S}$  preserves  $\Pi$ -functors (hence exponentials). Then  $A$  is locally connected.*

**PROOF:** As is well-known and easy to prove, the assumption implies that  $A$  is open (e.g.  $\gamma^*$  preserves universal quantification, and use Joyal & Tierney (1984), §VII.1.2). We will show that if  $A$  is open and  $\gamma^*$  preserves exponentials, then  $A$  is locally connected.

Preservation of exponentials means that for any  $S, T \in \mathcal{S}$  and any  $a \in A$  (viewed as a sublocale of  $A$ ), an  $f: a \rightarrow S^T$  is continuous for the product locale  $S^T$  iff it is continuous for the discrete local  $S^T$ .

Let for  $a \in A$ ,  $F_a$  be the set of continuous maps  $a \rightarrow \Omega$ . Then we have a canonical map

$$\varphi_a: a \rightarrow \Omega^{F_a}$$

which by assumption is continuous as a map into the discrete locale  $\Omega^{F_a}$ . Thus, writing  $a_\alpha = \varphi_a^{-1}(\alpha)$ ,  $C(a) = \{a_\alpha \mid \alpha \in \Omega^{F_a}\}$  is a discrete cover of  $a$ , and since  $A$  is open, so is  $C^+(a) = \{c \in C(a) \mid \text{pos}(c)\}$ . We claim that the elements of  $C^+(a)$  are the ‘‘connected components’’ of  $a$ .

First, suppose we are given a continuous  $a \xrightarrow{g} S$  into a discrete locale  $S$ . Then for each  $a_\alpha \in C^+(a)$  we have

$$(i) \quad \exists ! s \in S \ a_\alpha \leq g^{-1}(s).$$

Indeed, the set  $U = \{s \in S \mid \text{pos}(a_\alpha \wedge g^{-1}(s))\}$  is inhabited since  $a_\alpha$  is positive and covered by  $\{g^{-1}(s) \mid s \in U\}$ . Suppose  $s \in U$ , and let  $a \rightarrow \Omega$  be the composite

$$a \xrightarrow{g} S \xrightarrow{[\cdot = s]} \Omega.$$

Then  $g^{-1}(s) = f^{-1}(T) = \bigvee \{a_\beta \in C^+(a) \mid \beta_f = T\}$ . Since  $\text{pos}(a_\alpha \wedge g^{-1}(s))$ , there is a  $\beta \in \Omega^{F_a}$  with  $\beta_f = T$  such that  $\text{pos}(a_\alpha \wedge a_\beta)$ , so  $\alpha = \beta$ , so  $\beta_f = T$ , i.e.  $a_\alpha \leq g^{-1}(s)$ . Since  $s \in U$  was arbitrary, this shows that the inhabited set  $U$  can have at most one element, proving (i).

In other words, there is a natural 1 – 1 correspondence

$$(ii) \quad \frac{a \rightarrow S \text{ (of locales in } \mathcal{S}\text{)}}{C^+(a) \rightarrow S \text{ (of "sets" in } \mathcal{S}\text{)}}$$

between continuous maps into discrete locales  $S$  and maps  $C^+(a) \rightarrow S$  in  $\mathcal{S}$ : given  $a \xrightarrow{g} s$ , define  $\varphi: C^+(a) \rightarrow S$  by  $\varphi(c) =$  the unique  $s$  with  $c \leq g^{-1}(s)$  (see (i)); and given  $\varphi: C^+(a) \rightarrow S$  define  $f$  by  $f^{-1}(s) = \bigvee \varphi^{-1}(s)$ .

Now we show that each  $b \in C^+(a)$  is connected. By lemma 2, it suffices to show that each continuous map  $b \rightarrow \Omega$  is constant. This would follow from (i) if we show that a continuous  $b \rightarrow \Omega$  can always be extended to a continuous  $a \rightarrow \Omega$ . So take  $b_0 \in C^+(a)$  and  $b_0 \xrightarrow{g} \Omega$ , corresponding to  $\varphi: C^+(b) \rightarrow \Omega$  by (ii). Write  $a = \bigvee C^+(a) = \bigvee \{c \mid \exists b \in C^+(a): c \in C^+(b)\}$ . The set  $\mathcal{U} = \{c \mid \exists b \in C^+(a) c \in C^+(b)\}$  is a discrete cover (i.e.  $\forall c, c' \in \mathcal{U} \text{ pos}(c \wedge c') \rightarrow c = c'$ ) of  $a$  by positive elements, and  $C^+(b_0) \subset \mathcal{U}$ .  $\Omega$  is injective, so  $\varphi$  can be extended to a function  $\psi: \mathcal{U} \rightarrow \Omega$ , which by discreteness of  $\mathcal{U}$  gives a continuous  $f: a \rightarrow \Omega$ , defined by  $f^{-1}(p) = \bigvee \psi^{-1}(p)$ .  $f$  is the required extension of  $g$ .

This completes the proof of proposition 3, and hence of the implication (4)  $\Rightarrow$  (1) of the theorem.

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