

COMPOSITIO MATHEMATICA

B. DWORK

A. OGUS

Canonical liftings of jacobians

Compositio Mathematica, tome 58, n° 1 (1986), p. 111-131

<http://www.numdam.org/item?id=CM_1986__58_1_111_0>

© Foundation Compositio Mathematica, 1986, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

CANONICAL LIFTINGS OF JACOBIANS

B. Dwork and A. Ogus

Introduction

Perhaps the chief reason why the study of Fermat curves is so rewarding is that their Jacobians are of CM type. Is there a systematic way to construct other curves with this property? In this paper, we examine the limitations of one approach based on a p -adic method – the “canonical lifting” of Serre and Tate [15]. They showed that an ordinary abelian variety A over a perfect field k of characteristic $p > 0$ has a distinguished lifting A_{can} to the Witt ring W of k ; if k is finite, A_{can} is of CM type. Thus it is natural to try to start with an arbitrary ordinary curve X/k and ask if the canonical lifting of its Jacobian is again the Jacobian of some curve. In this paper we show that, when p is odd and the genus is greater than three, the answer is “no” for most curves, even if one works mod p^2 .

Let us briefly sketch our method. We work over an algebraically closed field k of odd characteristic p ; this allows us to use crystalline deformation theory to give a completely linear description of the canonical lifting of an ordinary abelian variety. Let $W_2 = W/p^2W$ and let X/k be a curve of genus $g \geq 2$ with Jacobian J/k . The set of liftings of X/k to W_2 is a torsor under $H^1(X, \Theta_{X,k}) \cong \Gamma(X, \Omega_{X/k}^{\otimes 2})^{\vee}$, and the set of liftings of J/k together with its polarization to W_2 , is a torsor under $\text{Sym}^2\Gamma(X, \Omega_{X/k})^{\vee}$. The cokernel $Q_{X/k}$ of the natural map

$$\mu_x^{\vee}: \Gamma(X, \Omega_X^{\otimes 2})^{\vee} \rightarrow \text{Sym}^2\Gamma(X, \Omega_X)^{\vee}$$

can be thought of as the value at X of the normal bundle to the moduli space \mathcal{M}_g of curves in the moduli space \mathcal{Q}_g of principally polarized abelian varieties. In (2.4) we construct an element β_X of $Q_{X/k}$ which is the obstruction we need: $\beta_X = 0$ iff J_{can}/W_2 is the Jacobian of a curve over W_2 .

Of course, the key point is to show that β_X is generically not zero. Rather than try to construct examples for every g and p , we proceed by “pure thought”. First we show that the Frobenius pullback $F_k^*(\beta_X)$ of β_X can be interpolated in any family of curves Y/T : there is a section $\tilde{\beta}_{Y/T}$

of $F_T^*(Q_{Y/T})$ whose value at each closed pointed t of T is $F_k^*(\beta_{Y(t)})$. Since $Q_{Y/T}$ is coherent, this shows at least that the set Σ of all closed points in T corresponding to curves X with a canonical lifting mod p^2 is *constructible*. When $g = 2$ or when the fibers of Y/T are nonhyperelliptic (as we may safely assume when $g > 2$), $Q_{Y/T}$ is locally free of rank $((g-2)(g-3)/2)$, and then Σ is in fact *closed*.

To show that $\beta_{Y/T}$ is not zero when $g \geq 4$ and Y/T is a versal family of curves, we prove that in fact $\tilde{\beta}_{Y/T}$ cannot be descended to $\Gamma(T, Q_{Y/T}) \rightarrow \Gamma(T, F_T^*Q_{Y/T})$. This is the same as showing that $\nabla\tilde{\beta}_{Y/T} \neq 0$, where ∇ is the canonical connection which exists on the Frobenius pullback of any coherent sheaf [10, (5.1)]. To accomplish this, we first give a very simple formula (3.2) for $\nabla\tilde{\beta}_{Y/T}$ involving the Cartier operator and the multiplication map: $\mu_{Y/T}: \text{Sym}^2\Gamma(Y, \Omega_{Y/T}) \rightarrow \Gamma(Y, \Omega_{Y/T}^{\otimes 2})$.

The rest of our argument is clearest if we make a finite étale covering of T and choose a level p -structure for the Jacobian of Y/T . Using the well-known correspondence between divisors of order p and holomorphic differentials fixed by the Cartier operator [14], we find that $\Gamma(Y, \Omega_{Y/T})^{\mathcal{G}}$ has an \mathcal{O}_T -basis $\{\omega_1, \dots, \omega_g\}$, and the canonical embedding can be viewed as a T -map $Y \rightarrow T \times \mathbb{P}^{g-1}$. Then we see immediately from our formula (3.2) that if $\nabla\tilde{\beta}_{Y/T}$ vanished, $K_{Y/T} =: \ker(\mu_{Y/T})$ would also be invariant under the Cartier operator. In other words, the intersection $Z_{Y/T}$ of all the quadrics in $T \times \mathbb{P}^{g-1}$ containing Y would descend to a “constant” subscheme of T . When $g \geq 5$, Petri’s theorem [16] tells us that a general curve of genus g is equal to the intersection of the quadrics containing it, leading to the absurd conclusion that the generic curve is definable over \mathbb{F}_p ! For $g = 4$, an easy modification of this argument can be made to work.

It is worth remarking that many of our arguments, including formula (3.2), are valid in the context of abstract F -crystals. Consequently they can also be expected to apply to diverse situations, such as families of $K3$ surfaces, hypersurfaces, etc. For the sake of concreteness we have chosen to treat here the case of curves exclusively, leaving the generalizations to the future, as need arises.

Many related questions remain to be investigated. Perhaps the most obvious is the problem of identifying the subsets of \mathcal{M}_g/k corresponding to curves which have canonical liftings mod higher powers of p . The question of constructing nonhyperelliptic curves of high genus with CM Jacobians remains quite mysterious; most intriguing is R. Coleman’s suggestion that, for large g , there should be only finitely many such curves. We also think the moduli of curves with a level p structure, which maps naturally to the Hilbert scheme of \mathbb{P}^{g-1} , warrants further investigation. We had intended to mention the problem of constructing “explicit” examples of curves with no canonical lifting, but in fact Oort and Sekiguchi have, completely independently, constructed many such examples [13]. They were able to conclude that the generic curve of genus g

has no canonical lifting, at least when $g \geq 2(p - 1) \geq 8$, providing a completely different proof of our main result in most cases. We should also point out that the first explicit example seems to be due to D. Mumford (letter to Dwork, 1972).

Acknowledgements

We would like to thank R. Coleman for rekindling our interest in the question of canonical liftings of Jacobians and for several valuable discussions. We also want to thank S. Bloch and his IBM PC for some enlightening computational examples, as well as G. Washnitzer, W. Messing, and Y. Ihara for their interest in this work.

§1. Canonical liftings of abelian varieties and Jacobians

Recall that an abelian variety A over an algebraically closed field k of characteristic p is said to be “ordinary” iff it satisfies the following equivalent conditions:

$$\text{The étale part of the } p\text{-divisible group of } A \text{ is isomorphic to } (\mathbb{Q}_p/\mathbb{Z}_p)^g. \tag{1.1.1}$$

$$\text{The } p\text{-rank of } A \text{ is } g, \text{ i.e. } A_p(k) \cong (\mathbb{Z}/p\mathbb{Z})^g. \tag{1.1.1}^{\text{bis}}$$

$$\text{The } p\text{-linear endomorphism of } H^1(A, \mathcal{O}_A) \text{ induced by the absolute Frobenius endomorphism } F_A \text{ of } A \text{ is bijective.} \tag{1.1.2}$$

The crystalline cohomology $H_{\text{cris}}^1(A/W)$ splits into a direct sum:

$$H_{\text{cris}}^1(A/W) \cong U \oplus T \tag{1.1.3}$$

invariant under the map F_A^* induced by F_A , where $F_A^*|_U: U \rightarrow U$ is an isomorphism and $F_A^*|_T: T \rightarrow T$ is p times an isomorphism.

(1.2) REMARK: There is a canonical isomorphism:

$$H_{\text{cris}}^1(A/W) \otimes k \xrightarrow{\sim} H_{\text{DR}}^1(A/k), \tag{1.2.1}$$

which we permit ourselves to view as an identification. The Hodge filtration: $\text{Fil} =: F_{\text{Hodge}}^1 \subseteq H_{\text{DR}}^1(A/k)$ is well-known to be the kernel of the endomorphism $H_{\text{DR}}^1(F_A)$ of $H_{\text{DR}}^1(A/k)$ induced by F_A ; in particular, this map always has rank g . Its image, often called the “conjugate filtration”, is denoted $F_{\text{con}}^1 \subseteq H_{\text{DR}}^1(A/k)$. It is clear that A is ordinary iff the natural map

$$F_{\text{Hodge}}^1 \oplus F_{\text{con}}^1 \rightarrow H_{\text{DR}}^1(A/k)$$

is an isomorphism.

Let $\Phi_k: F_k^* H_{\text{DR}}^1(A/k) \rightarrow H_{\text{DR}}^1(A/k)$ denote the k -linear map corresponding to $H_{\text{DR}}^1(F_A)$. It factors through the natural projection $F_k^* H_{\text{DR}}^1(A/k) \rightarrow F_k^* H^1(A, \mathcal{O}_A)$, fitting into a commutative diagram

$$\begin{array}{ccccc}
 F_k^* H_{\text{DR}}^1(A/k) & \rightarrow & F_k^* H^1(A, \mathcal{O}_A) & \xrightarrow{\Phi_k^0 \cong} & F_{\text{con}}^1 \\
 & \searrow \Phi_k & & \searrow h & \downarrow \pi := \text{proj} \circ \text{inc} \\
 & & H_{\text{DR}}^1(A/k) & \rightarrow & H^1(A, \mathcal{O}_A)
 \end{array} \tag{1.2.2}$$

Thus, A is ordinary iff h is bijective iff π is bijective. When this is the case, the U (resp. the T) in (1.1.3) is the unique $H_{\text{cris}}^1(F_A)$ -invariant direct summand of $H_{\text{cris}}^1(A/W)$ lifting F_{con}^1 (resp. F_{Hodge}^1). The letter U stands for “unit root”; we will also use the notation Fil_{can} for the subspace T . When the context permits, we shall also use these notations for the reductions of the corresponding spaces modulo various powers of p .

Let $W_n =: W/p^n W$ and let $S =: \text{Spec } W_n$ or $\text{Spf } W$. Recall that if X/S is any smooth proper formal scheme and X/k is its closed fiber, there is a canonical isomorphism

$$\sigma_{\text{cris}}: H_{\text{DR}}^1(X/S) \xrightarrow{\sim} H_{\text{cris}}^1(X/S) \tag{1.3.1}$$

compatible with the isomorphism (1.2.1). We set

$$\text{Fil}_X =: \sigma_{\text{cris}}(F_{\text{Hodge}}^1) \hookrightarrow H_{\text{cris}}^1(X/S). \tag{1.3.2}$$

In the absence of p -torsion in the Hodge groups, we see that Fil_X is a lifting of $\text{Fil}_X =: F_{\text{Hodge}}^1 \hookrightarrow H_{\text{DR}}^1(X/k)$ to a direct summand of $H_{\text{cris}}^1(X/S)$. For abelian varieties we have, since p is odd, the following “crystalline local Torelli theorem” [11, §5], [2, (3.23)].

(1.3) THEOREM: *If A/k is an abelian variety, the above correspondence*

$$\{\text{formal liftings of } A/K \text{ to } S\} \rightarrow \{\text{liftings of } \text{Fil}_A \text{ to a direct summand of } H_{\text{cris}}^1(A/S)\}$$

is bijective.

In particular, the lifting $T = \text{Fil}_{\text{can}}$ of (1.1) and (1.2) defines a formal lifting A_{can} of A to S , called the “canonical lifting”. This is the crystalline description of a famous result of Serre and Tate, who also proved that A_{can} is in fact algebraizable.

(1.4) THEOREM (Serre-Tate): *If A/k is an ordinary abelian variety, there is a lifting A_{can}/W of A/k to an abelian scheme over W , uniquely determined by any of the following properties:*

The p -divisible group of A_{can} splits as a direct sum of its étale and connected pieces. (1.4.1)

There is an F_W -morphism $F_{\text{can}}: A_{\text{can}} \rightarrow A_{\text{can}}$ lifting the absolute Frobenius endomorphism F_A of A . (1.4.2)

The action F_A^ of F_A on $H_{\text{cris}}^1(A/W)$ leaves $\text{Fil}_{A_{\text{can}}}$ invariant.* (1.4.3)

Moreover if B/k is another ordinary abelian variety the natural map

$$\text{Hom}_W(A_{\text{can}}, B_{\text{can}}) \rightarrow \text{Hom}_k(A, B)$$

is bijective. \square

Perhaps the best current reference for this result is Messing’s thesis, especially [11, V, §3 and Appendix]. Although (1.4.3) is not mentioned explicitly there, it follows immediately from the methods of proof. One can also refer to [2, 3.4] for an explicit discussion of (1.4.3). Note that only the implication (1.4.3) \Rightarrow (1.4.1) and (1.4.2) requires the hypothesis that p be odd. Moreover, if k is finite, the canonical lifting A_{can}/W is also characterized by: $\text{End}(A_{\text{can}}) \rightarrow \text{End}(A)$ is bijective [11, Appendix, (1.3)].

Note that if A is ordinary, so is its dual. One sees easily that the canonical lifting of the dual of A is the dual of its canonical lifting. The last part of (1.4) therefore implies that every polarization of A lifts uniquely to a polarization of A_{can} ; this can also be deduced from [2, (3.15)]. In particular,

A_{can} is algebraizable.

If A/S is any formal lifting of an ordinary A/k , a natural measure of the “distance” between A and A_{can} is the composite

$$\text{Fil}_A \xrightarrow{\text{inc}} H_{\text{cris}}^1(A/S) \xrightarrow{\pi_{\text{can}}} U$$

where π_{can} is the projection associated to the direct sum decomposition $H_{\text{cris}}^1 = \text{Fil}_{\text{can}} \oplus U$. Since Fil_A and Fil_{can} are equal modulo p , we in fact have a map:

$$\tau_A: \text{Fil}_A \rightarrow pU. \tag{1.5}$$

This map and its variants will be the key to our analysis. Note that $\tau_A = 0$ iff $A \simeq A_{\text{can}}/S$.

(1.6) REMARK: Although we shall not need it for what follows, it is interesting to relate τ_A to the theory of canonical coordinates [5], by means of explicit formulas. Recall from [5] that the universal formal deformation space T of A over W is formally smooth of dimension g^2 , and admits “canonical coordinates” $\{t_{i,j} = q_{i,j} - 1, 1 \leq i, j \leq g\}$, with respect to which the Gauss-Manin connection looks especially nice. Thus, if B/T is the universal formation deformation of A/k , $H^1_{\text{DR}}(B/T)$ admits a basis: $\{\omega_1, \dots, \omega_j, \eta_1 \dots \eta_g\}$ compatible with the Hodge filtration, and the Gauss-Manin connection ∇ is given by:

$$\begin{aligned} \nabla \eta_i &= 0 \\ \nabla \omega_i &= \sum_j d \log q_{i,j} \otimes \eta_j. \end{aligned}$$

Moreover, the canonical lifting corresponds to the W -section: $\{t_{i,j} = 0\}$. Now any formal lifting A/S corresponds to an S -valued point of T , i.e. to a choice of g^2 elements $\{t_{i,j}(A)\}$ of the maximal ideal of \mathcal{O}_S . The canonical isomorphisms $H^1_{\text{DR}}(A/S) \xrightarrow{\sim} H^1_{\text{cris}}(A/S) \leftarrow H^1_{\text{DR}}(A_{\text{can}}/S)$ can be calculated by well known formulas (cf. [1]) in terms of the Gauss-Manin connection. We find that

$$\tau_A(\omega_i(A)) = \sum_j \log(q_{i,j}(A)) \eta_j(A_{\text{can}})$$

where of course $q_{i,j}(A) = t_{i,j}(A) + 1$. □

Suppose now that X/k is a smooth projective curve of genus $g \geq 1$ and let $J = (A, \Theta)$ denote the Jacobian of A together with its principal polarization. There is a canonical commutative diagram:

$$\begin{array}{ccc} H^1_{\text{cris}}(A/W) & \xrightarrow{\sim} & H^1_{\text{cris}}(X/W) \\ \downarrow & & \downarrow \\ H^1_{\text{DR}}(A/k) & \xrightarrow{\sim} & H^1_{\text{DR}}(X/k) \\ \downarrow & & \downarrow \\ H^1(A, \mathcal{O}_A) & \xrightarrow{\sim} & H^1(X, \mathcal{O}_X) \end{array}$$

compatible with the actions of absolute Frobenius. In particular, A is ordinary iff F_X induces a bijective endomorphism of $H^1(X, \mathcal{O}_X)$; in this case one says that X is ordinary. It is clear that this condition depends only on the isomorphism class of X/k and hence only on the image of X in the coarse moduli space \mathcal{M}_g/k of curves of genus g over k . It is quite easy to see that the set $\mathcal{M}_{g, \text{ord}}(k)$ of points corresponding to ordinary

curves is Zariski open in $\mathcal{M}_g(k)$; L. Miller was the first to prove that for every g , it is also dense [12].

If X/k is ordinary, let Θ_{can} be the unique lifting of Θ to the canonical lifting A_{can} of A , and let $J_{\text{can}}(X) =: (A_{\text{can}}, \Theta_{\text{can}})$, a principally polarized abelian scheme over W . If R is any W -algebra, we let J_{can}/R denote the principally polarized abelian scheme over R obtained from J_{can} by base change.

(1.7) DEFINITION: *An ordinary curve X/k is “pre- R -canonical” iff there is a smooth curve Y/R whose Jacobian is isomorphic, as a principally polarized abelian R -scheme, to $J_{\text{can}}(X)/R$.*

Note that if $g \leq 3$, every principally polarized abelian variety of genus g over an algebraically closed field is the Jacobian of a curve, and that the curve is smooth if the Θ -divisor is irreducible. Thus, if $g \leq 3$, every ordinary curve over k is pre- \bar{K} -canonical.

It ought to be clear that the pre- R -canonicity of X/k depends only on the isomorphism class of X/k . Thus the class of all pre- R -canonical curves defines a subset of $\mathcal{M}_{g \text{ ord}}(k)$ which we denote by Σ_R .

(1.8) THEOREM: *If $g \geq 4$, the set $\Sigma_{\bar{K}}$ of pre- \bar{K} -canonical curves in $\mathcal{M}_{g \text{ ord}}(k)$ is nowhere dense in $\mathcal{M}_{g \text{ ord}}(k)$.*

Let $W_2 =: W/p^2W$. Theorem (1.8) will follow easily from:

(1.9) THEOREM: *The set Σ_{W_2} of pre- W_2 -canonical curves in $\mathcal{M}_{g \text{ ord}}(k)$ is constructible. Its intersection with the nonhyperelliptic locus is closed. If $g \geq 4$, it is nowhere dense.*

Let us explain how (1.9) implies (1.8). Clearly it suffices to prove that if $x \in \mathcal{M}_{g \text{ ord}}(k)$ represents the class of a nonhyperelliptic and pre- \bar{K} -canonical curve X , then X is also pre- W_2 -canonical. Suppose Y/\bar{K} is a smooth curve whose Jacobian is $J(X)_{\text{can}}/\bar{K}$. Necessarily there is a finite extension K' of K to which Y descends; write Y'/K' for some descent of Y/\bar{K} . For K' large enough, we still have $J(Y')/K' \cong J(X)_{\text{can}}/K'$. Since $J(X)_{\text{can}}/K'$ has stable reduction, so does Y'/K' : there is a stable curve Y' over the ring of integers V' of K' whose generic fiber is Y' . Then by [6, (2.5)ff], the Jacobian of the special fiber of Y' is the special fiber of $J(X)_{\text{can}}$; it follows that this special fiber is just X . In other words, Y'/V' is a lifting of X . Let \mathcal{T} denote the universal formal deformation of X and let \mathcal{A} denote the universal formal deformation of $J(X)$. Each of these is a formally smooth formal scheme over W , and since X is not hyperelliptic, the natural map $\mathcal{T} \rightarrow \mathcal{A}$ is a closed immersion. The lifting Y' defines a V' -valued point y' of \mathcal{T} , and its image in \mathcal{A} factors through a W -valued point. It follows that y' also factors through a

W -valued point, i.e. Y' descends to W . Then if Y/W is such a descent, $J(Y/W_2) \cong J(X)_{\text{can}}/W_2$, so X is pre- W_2 -canonical.

We shall explain the proof of Theorem (1.9) in the next sections.

§2. Obstructions mod p^2

Suppose X/k is a smooth projective curve of genus g . Poincaré duality in crystalline cohomology furnishes us with a perfect alternating pairing:

$$\langle \ , \ \rangle: H_{\text{cris}}^1(X/W) \times H_{\text{cris}}^1(X/W) \rightarrow W. \tag{2.1.1}$$

If ξ and ξ' are elements of $H_{\text{cris}}^1(X/W)$, one has the compatibility relation

$$\langle F_X^* \xi, F_X^* \xi' \rangle = p F_W \langle \xi, \xi' \rangle. \tag{2.1.2}$$

One deduces immediately that, when X is ordinary, the direct summands U and Fil_{can} (1.1.3) of $H_{\text{cris}}^1(X/W)$ are totally isotropic and mutually dual.

If $W_n =: W/p^n W$ and if X is a lifting of X to W_n , we obtain a totally isotropic lifting Fil_X of $\text{Fil}_X = F_{\text{Hodge}}^1$ to a direct summand of $H_{\text{cris}}^1(X/W_n)$. By (1.2), when X is ordinary, the natural map

$$\text{Fil}_X \oplus U \rightarrow H_{\text{cris}}^1(X/W_n) \tag{2.1.3}$$

is an isomorphism, since it is modulo p .

(2.2) PROPOSITION: *If X/k is an ordinary curve of genus g , the following are equivalent:*

$$X/k \text{ is pre-} W_n\text{-canonical (1.7)}. \tag{2.2.1}$$

$$\text{There is a lifting } X/W_n \text{ of } X/k \text{ such that } \text{Fil}_X = \text{Fil}_{\text{can}} \hookrightarrow H_{\text{cris}}^1(X/W_n). \tag{2.2.2}$$

$$\text{There is a lifting } X/W_n \text{ of } X/k \text{ such that } \text{Fil}_X \text{ is invariant under } H_{\text{cris}}^1(F_X, W_n). \tag{2.2.3}$$

PROOF: If X/k is pre- W_n -canonical, there is a smooth proper Y/W_n such that $J(Y/W_n) \cong J_{\text{can}}/W_n$ as principally polarized abelian schemes. In particular, the reduction of $J(Y/W_n)$ to k is isomorphic to the Jacobian of X/k , and by the Torelli theorem, $Y/k \cong X/k$, i.e. Y/W_n is a lifting of X/k . Thus, (2.2.1) implies (2.2.2). The remaining equivalences are clear.

□

Now if X/W_2 is any lifting of X/k , we obtain as in (1.5) a map:

$$\tau_X: \text{Fil}_X \rightarrow pU \xrightarrow{\sim} F_{\text{con}} \quad (2.3.1)$$

by composing the inclusion of Fil_X in $H_{\text{cris}}^1(X/W_2)$ with the projection associated with the direct sum decomposition

$$\text{Fil}_{\text{can}} \oplus U \xrightarrow{\sim} H_{\text{cris}}^1(X/W_2). \quad (2.3.2)$$

Of course, $pF_{\text{con}}^1 = 0$, so τ_X factors through $\text{Fil}_X \rightarrow \text{Fil}_X$. In other words, there is a unique map:

$$\delta_X: \text{Fil}_X \rightarrow F_{\text{con}} \quad (2.3.3)$$

such that for any $\omega \in \text{Fil}_X$, we have

$$\omega = \omega' + "p" \delta_X(\omega) \quad (2.3.4)$$

where $\omega \in \text{Fil}_X$ and $\omega' \in \text{Fil}_{\text{can}}$ are liftings of ω . It is clear that X/k is pre- W_2 -canonical iff we can choose X/W_2 such that $\delta_X = 0$.

Recall that the cup-product pairing on $H_{\text{DR}}^1(X/k)$ induces a perfect pairing $\langle , \rangle: \text{Fil}_X \times F_{\text{con}} \rightarrow k$. Thus, we can identify δ_X with a bilinear map:

$$\beta_X: \text{Fil}_X \times \text{Fil}_X \rightarrow k \quad (2.3.5)$$

given by: $\beta_X(\omega, \omega') = \langle \omega, \delta_X \omega' \rangle$. It follows immediately from the facts that \langle , \rangle is alternating and that Fil_X and Fil_{can} are totally isotropic that β_X is symmetric. Thus we can identify β_X with a map:

$$\beta_X: \text{Sym}^2 \text{Fil}_X \rightarrow k. \quad (2.3.6)$$

Of course, $\text{Fil}_X \cong \Gamma(X, \Omega_{X/k})$, so there is a natural map

$$\mu_X: \text{Sym}^2 \text{Fil}_X \rightarrow \Gamma(X, \Omega_{X/k}^{\otimes 2}). \quad (2.3.7)$$

(2.4) PROPOSITION: *The restriction β_X of β_X to $\ker(\mu_X) \subseteq \text{Sym}^2 \text{Fil}_X$ is independent of the lifting X/W_2 . The curve X/k is pre- W_2 -canonical iff $\beta_X = 0$.*

PROOF: Standard deformation theory tells us that $\text{Def}(X/W_2)$, the set of isomorphism classes of lifting of X/k to W_2 , is a torsor under the group $H^1(X, \Theta_{X/k})$, where of course $\Theta_{X/k}$ is the tangent bundle. Moreover, the set of liftings $L_{\text{cris}}(\text{Fil}_X/W_2)$ of Fil_X to totally isotropic direct summands

of $H_{\text{cris}}^1(X/W_2)$ is a torsor under $(\text{Sym}^2 \text{Fil}_X)^\vee$ (in the manner described above). We have a natural map of groups:

$$H^1(X, \Theta_{X/k}) \xrightarrow{\cong} \Gamma(X, \Omega_{X/k}^{\otimes 2}) \xrightarrow{\mu_X^\vee} (\text{Sym}^2 \text{Fil}_X)^\vee$$

and of sets:

$$\text{Def}(X/W_2) \rightarrow L_{\text{cris}}(\text{Fil}_X/W_2).$$

When $p \neq 2$, it is proved in [2, (3.21)] that the second of these maps is a morphism over the first one. (For $p = 2$, a correction term is required [4].) In particular, if $D \in H^1(X, \Theta_{X/k})$ and if X'/W_2 is the lifting obtained from X/W_2 by “adding” D , then $\text{Fil}_{X'}$ is obtained from Fil_X by “adding” $\mu_X^\vee(D)$. It follows that we have:

$$\beta_{X'} = \beta_X + \mu_X^\vee(D). \quad (2.4.2)$$

If we identify $\text{Cok}(\mu_X^\vee)$ with $\text{Ker}(\mu_X)^\vee$, we find that the image β_X of β_X in this space depends only on X , and that β_X vanishes iff X' can be chosen so that $\beta_{X'} = 0$. \square

(2.4.3) **REMARK:** The restriction β_X of β_X to $\text{Ker}(\mu_X)$ is an element of the dual $\text{Ker}(\mu_X)^\vee$ of $\text{Ker}(\mu_X)$, which we can identify with $Q_X =: \text{Cok}(\mu_X^\vee)$. This space has a natural geometric interpretation: it is the normal bundle to the deformation space of X in the deformation space to its Jacobian $J(X)$ as a principally polarized abelian variety. That is, we have an exact sequence

$$\begin{array}{ccccccc} \text{Def}(X/k[\epsilon]) & \rightarrow & \text{Def}(J(X)/k[\epsilon]) & \rightarrow & N & \rightarrow & 0 \\ & & \parallel \wr & & \parallel \wr & & \parallel \wr \\ \Gamma(X, \Omega_{X/k}^{\otimes 2})^\vee & \rightarrow & (\text{Sym}^2 \text{Fil}_X)^\vee & \rightarrow & Q_X & & \square \end{array}$$

(2.5) **VARIANT:** We can also use characterization (1.4.3) to measure how far X/W_2 is from defining the canonical lifting. Namely, let

$$\Phi: F_W^* H_{\text{cris}}^1(X/W_2) \rightarrow H_{\text{cris}}^1(X/W_2)$$

denote the W -linear map corresponding to F_X^* . If $\tilde{\omega} \in F_W^* \text{Fil}_X$, $\Phi(\tilde{\omega}) \in p H_{\text{cris}}^1(X/W_2) \cong H_{\text{DR}}^1(X/k)$. Thus, there are unique maps:

$$\begin{array}{l} \gamma_X: F_k^* \text{Fil}_X \rightarrow \text{Fil}_X, \quad \alpha_X: F_k^* \text{Fil}_X \rightarrow F_{\text{con}} \quad \text{such} \\ \text{that } \Phi(\omega) = “p” \gamma_X(\omega) + “p” \alpha_X(\omega) \quad \text{for } \omega \in \\ F^* \text{Fil}_X \text{ a lifting of } \omega. \end{array} \quad (2.5.1)$$

One verifies immediately that γ_X is in fact the inverse transpose of the map $\Phi: F_k^* F_{\text{con}} \rightarrow F_{\text{con}}$ induced by Φ (and hence does not depend on the

lifting X/W_2). It is well known that this is the inverse Cartier operator C^{-1} . It is also immediate to verify that

$$\alpha_X = \phi \circ F_k^*(\delta_X). \tag{2.5.2}$$

If we define $\tilde{\alpha}_X =: \phi^{-1} \circ \alpha_X: F_k^* \text{Fil}_X \rightarrow F_k^* F_{\text{con}}$ and $\tilde{\beta}_X: F_k^* \text{Sym}^2 \text{Fil}_X \rightarrow k$ by $\tilde{\beta}_X(\omega, \omega') = \langle \omega, \alpha_X(\omega') \rangle$, it follows that

$$\tilde{\beta}_X = F_k^*(\beta_X). \tag{2.5.3} \quad \square$$

In order to prove (1.9), we first show that the obstructions $\tau_X =: F_k^*(\tau_X)$ vary nicely in a family. To this end, let Y/T be a smooth proper family of curves of genus g , where T/W_2 is smooth, and let Y/T be the reduction of $Y/T \bmod p$. The yoga of crystalline cohomology then provides us with an F -crystal on T/W_2 . For our purposes, we may as well assume that T is affine, so that the absolute Frobenius endomorphism F_T of T admits (many) liftings to endomorphisms F_T of T . Our F -crystal then becomes the following set of data:

- a coherent sheaf of \mathcal{O}_T -modules with integrable connection (H_T, ∇) . (2.6.1)
- a horizontal map: $\Phi_{F_T}: F_T^*(H_T, \nabla) \rightarrow (H_T, \nabla)$.

Poincaré duality provides us with a horizontal alternating form:

$$\langle , \rangle_{H_T} \times H_T \rightarrow \mathcal{O}_T, \tag{2.6.2}$$

and one has all the expected compatibilities, e.g. with specialization. Note that these data depend only on Y/T and F_T , not on Y/T . On the other hand, there is a canonical isomorphism:

$$H_{\text{DR}}^1(Y/T) \xrightarrow{\sim} H_T, \tag{2.6.3}$$

under which ∇ becomes identified with the Gauss-Manin connection. Thus the lifting Y/T of Y/T does provide us with a filtration $\text{Fil}_Y \hookrightarrow H_T$, lifting the Hodge filtration on $H_T \cong H_{\text{DR}}(Y/T)$.

Assume from now on that each fiber of Y/T is ordinary. Then as explained in [9] there is a unique horizontal lifting U_T of $F_{\text{con}T} \hookrightarrow H_T$ to a local direct summand of H_T which is stable under Φ ; Φ in fact induces a (horizontal) isomorphism $\phi_{F_T}: F_T^* U_T \rightarrow U_T$.

On the other hand, there is *no* Φ -invariant lifting of Fil_Y , in general. The lifting Fil_Y provided by Y/T will not usually be Φ -invariant, but will define a direct sum decomposition:

$$U_T \oplus \text{Fil}_Y \xrightarrow{\sim} H_T. \tag{2.6.4}$$

Thus, there are unique maps

$$\begin{aligned} \gamma_{Y, F_T}: F_T^* \text{Fil}_Y &\rightarrow \text{Fil}_Y, \quad \alpha_{Y, F_T}: F_T^* \text{Fil}_Y \rightarrow U_T = \\ &F_{\text{con}T}, \text{ such that } \Phi_{F_T}(\omega) = "p" \gamma_{Y, F_T}(\omega) \text{ for } \omega \in \\ &F_T^* \text{Fil}_Y \text{ lifting } \omega. \end{aligned} \quad (2.6.5)$$

It is easy to verify that γ_{Y, F_T} is just the inverse transpose of $\Phi: F_T^* U_T \rightarrow U_T$, and is therefore independent of the liftings Y/T and F_T of Y/T and F_T . We shall denote it simply by $\gamma_{Y/T}$.

We shall find it convenient to work with the maps

$$\begin{aligned} \tilde{\alpha}_{Y, T} &=: \phi^{-1} \circ \alpha_{Y, T}: F^* \text{Fil}_Y \rightarrow F_T^* U_T \quad \text{and} \\ \tilde{\beta}_{Y, F_T} &: \text{Sym}^2 F_T^* \text{Fil}_Y \rightarrow \mathcal{O}_T \end{aligned} \quad (2.6.6)$$

$$\text{given by } \tilde{\beta}_{Y, F_T}(\omega, \omega') = \langle \omega, \tilde{\alpha}_{Y, F_T} \omega' \rangle.$$

Our next result expresses the relationship between $\tilde{\beta}_{Y, F_T}$ and the obstruction β_X of (2.4).

(2.7) PROPOSITION: *Let $Q_{Y/T}$ be the cokernel of the dual of the natural map*

$$\mu_{Y/T}: S^2 \text{Fil}_Y \rightarrow \Gamma(Y, \Omega_{Y/T}^{\otimes 2}).$$

Then the image $\tilde{\beta}_{Y/T}$ of $\tilde{\beta}_{Y, F_T}$ in $F_T^ Q_{Y/T}$ is independent of the liftings Y/T and F_T of Y/T and F_T . For any closed point $t \in T$, the image $\tilde{\beta}_{Y/T}(t)$ of $\tilde{\beta}_{Y/T}$ in*

$$F_T^* Q_{Y/T}(t) \cong F_T^* \text{Cok}(\Gamma(Y/T), \Omega_{Y(t)/k}^{\otimes 2})^\vee \rightarrow S^2 \text{Fil}_{Y(t)}^\vee$$

is precisely $F_k^(\beta_{Y(t)})$.*

PROOF: Associated to each closed point t of T is its Teichmüller lifting $\mathfrak{t} \in \mathfrak{T}(W_2)$, uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} \text{Spf } W_2 & \rightarrow & \mathfrak{T} \\ \text{Spf } F_{W_2} \downarrow & & \downarrow F_T \\ \text{Spf } W_2 & \rightarrow & \mathfrak{T} \end{array} \quad (2.7.1)$$

Via the canonical isomorphism: $\mathfrak{t}^* H_T \cong H_{\text{cris}}^1(Y(t)/W_2)$, $\mathfrak{t}^*(\Phi_{F_T})$ becomes the map $\Phi: F_W^* H_{\text{cris}}^1(Y(t)/W_2) \rightarrow H_{\text{cris}}^1(Y(t)/W_2)$ used in (2.5). Moreover, $Y(\mathfrak{t}) =: \mathfrak{t}^*(Y)$ is a lifting of $Y(t)/k$ to W_2 , and, via our isomorphism, $\mathfrak{t}^* \text{Fil}_Y$ becomes $\text{Fil}_{Y(t)} \hookrightarrow H_{\text{cris}}^1(Y(t)/W_2)$. It now follows from the definitions that

$$\begin{aligned} \tilde{\alpha}_{Y(t)} &= \mathfrak{t}^*(\tilde{\alpha}_{Y, F_T}), \quad \text{and} \\ \tilde{\beta}_{Y(t)} &= \mathfrak{t}^*(\tilde{\beta}_{Y, F_T}) \in F_k^*(\text{Sym}^2 \text{Fil}_{Y(t)})^\vee. \end{aligned} \quad (2.7.2)$$

The \mathcal{O}_T -modules Fil_Y and $\Gamma(Y, \omega_Y^{\otimes 2})$ are locally free, and their formation commutes with base change. It follows that the natural map from $Q_{Y/T}(t) =: t^*Q_{Y/T}$ to $\text{Cok}(\mu_{Y(t)}^v)$ is an isomorphism. By (2.5.3) and the definitions, we find that the image of $t^*(\tilde{\beta}_{Y,F_T})$ in $F_k^*(Q_{Y/T}(t))$ is precisely $F_k^*(\beta_{Y(t)})$, proving the last part of our proposition.

In particular, we see that $\beta_{Y,F_T}(t)$ is independent of the choices Y/T and F_T , for every closed point t of T . On the open subset of T where $Q_{Y/T}$ is locally free, at least, this is enough to show that $\tilde{\beta}_{Y,F_T}$ is also independent of the choices. It is more revealing, however, to give a ‘‘crystalline’’ proof.

First let us investigate the effect of changing the lifting Y/T . This involves an argument just like the proof of (2.4). The set of all such liftings is a torsor under $H^1(Y, \Theta_{Y/T}) \cong \Gamma(Y, \Omega_{Y/T}^{\otimes 2})^\vee$. If two liftings Y and Y' differ by an element θ of $H^1(Y, \Theta_{Y/T})$, the corresponding filtrations differ by an element β of $\text{Sym}^2(\text{Fil}_Y)^\vee$; in fact, $\beta = \mu_{Y/T}^v(\theta)$. Since F_T is the identity map (set-theoretically), there is a natural map of abelian sheaves:

$$\zeta: (\text{Sym}^2 \text{Fil}_Y)^\vee \rightarrow F_T^*(\text{Sym}^2 \text{Fil}_Y)^\vee.$$

One checks immediately that

$$\tilde{\beta}_{Y',F_T} = \tilde{\beta}_{Y,F_T} + \zeta(\mu_{Y/T}^v(\theta)), \tag{2.7.3}$$

and hence that the images of $\tilde{\beta}_{Y',F_T}$ and $\tilde{\beta}_{Y,F_T}$ in $F_T^*(Q_{Y/T})$ are equal.

On the other hand, the set of all liftings F_T of F_T is a torsor under $F_T^*(\Theta_{T/k})$. If F_T and F'_T are two liftings, differing by an element $\theta \in F_T^*(\Theta_{T/k})$, the yoga of crystalline cohomology furnishes us with a canonical isomorphism $\epsilon: F_T^*H_T \xrightarrow{\sim} F'^*_T H_T$ [1]. This isomorphism can be computed from the connection ∇ on the \mathcal{O}_T -module H_T . Using the fact that the divided powers $p^{[i]} = 0$ in \mathcal{O}_T for $i \geq 2$, one finds that, for $\omega \in H_T$ lifting $\omega \in H_T$,

$$\epsilon F_T^*(\omega) = F_T^*(\omega) + ‘‘p’’ \langle \theta, F_T^* \nabla \omega \rangle \tag{2.7.4}$$

Since Φ is a morphism of crystals, $\Phi_{F'_T} \circ \epsilon = \Phi_{F_T}$, so that

$$\Phi_{F_T}(F_T^*(\omega)) = \Phi_{F_T} F_T^*(\omega) + ‘‘p’’ \Phi_{F_T} \langle \theta, F_T^* \nabla \omega \rangle. \tag{2.7.5}$$

Suppose now that $\omega \in \text{Fil}_Y$, so that $\omega \in \text{Fil}_Y$. Write

$$\nabla \omega = \sigma \omega + \rho \omega \in (\Omega_{T/k} \otimes \text{Fil}_Y) \oplus (\Omega_{T/k} \otimes F_{\text{con}}). \tag{2.7.6}$$

Via the natural isomorphism: $F_{\text{con}} \rightarrow H_T/\text{Fil}_Y$, the map ρ becomes identified with the usual Kodaira-Spencer mapping [8]. Now let $\tilde{\omega} =:$

$F_T^*(\omega) \in F_T^* \text{Fil}_Y$. Using the obvious abbreviations and the definitions, as well as the fact that $\Phi_{F_T}: F_T^* \text{Fil}_Y \rightarrow H_T$ is zero, we find from (2.7.5) and (2.7.6):

$$\gamma(\tilde{\omega}) + \alpha(\tilde{\omega}) = \gamma(\tilde{\omega}) + \alpha'(\tilde{\omega}) + \phi\langle\theta, F_T^*(\rho)\tilde{\omega}\rangle. \quad (2.7.7)$$

Hence

$$\tilde{\alpha} = \tilde{\alpha}' + \langle\theta, F_T^*(\rho)\rangle. \quad (2.7.8)$$

Of course, the element

$$\rho \in \Omega_{T/k} \otimes \text{Hom}(\text{Fil}_Y, F_{\text{con}}) \cong \text{Hom}(\Theta_{T/k}, \text{Hom}(\text{Fil}_Y, F_{\text{con}}))$$

is given by the composite of Kodaira-Spencer $\xi: \Theta_{T/k} \rightarrow H^1(\Theta_{Y/T})$ with cup-product: $H^1(\Theta_{Y/T}) \rightarrow \text{Hom}(\text{Fil}_Y, F_{\text{con}})$ [8]. Using the notations and identifications above, we can write (2.7.8) as

$$\tilde{\beta} = \tilde{\beta}' + F^*(\mu_{Y/T}^\vee) \circ F^*(\xi)(\theta). \quad (2.7.9)$$

This implies that the class of β modulo the image of $F^*(\mu_{Y/T}^\vee)$ is independent of the choice of F_T , and completes the proof of (2.7). \square

(2.8) COROLLARY: *The set of all closed points t of T such that the fiber $Y(t)/k$ is pre- W_2 -canonical is equal to the set of closed points t such that $\tilde{\beta}_{Y/T}(t) = 0$ in $F^*Q_{Y/T}(t)$. In particular, it is constructible, and it is closed if $Q_{Y/T}$ is locally free.* \square

Suppose now that T/W_2 is the fine moduli space of curves of genus g with level structure and Y/T is the universal family. It is clear that $\Sigma_{W_2} \hookrightarrow \mathcal{M}_g$ is just the image, via the natural map $T \rightarrow \mathcal{M}_g$, of the constructible set in Corollary (2.8). This proves the first statement of Theorem (1.9). For the second statement, recall that if X/k is a nonhyperelliptic curve (or if $g = 2$), the map $\mu_{X/k}$ is surjective. Thus, in the situation of (2.7), if we assume that all the fibers of Y/T are nonhyperelliptic (or that $g = 2$), we find that $\mu_{Y/T}$ is a surjective map of locally free sheaves. In this case, $Q_{Y/T}$ is isomorphic to the dual of the kernel of $\mu_{Y/T}$, and in particular is locally free. This proves the second statement of Theorem (1.9). We shall prove the third statement in the next section.

(2.9) REMARK: Suppose that $Y/T/W_2$ is a versal family of curves, i.e. that the Kodaira-Spencer mapping:

$$\xi: \Theta_{T/k} \rightarrow H^1(Y, \Theta_{Y/k})$$

is an isomorphism. In this case, $F^*(Q_{Y/T})$ is the cokernel of the natural map

$$F_T^*(\mu_{Y/T}^\vee \circ \xi): F_T^*(\Theta_{T/k}) \rightarrow F_T^*(\text{Sym}^2 \text{Fil}_Y)^\vee. \quad (2.9.1)$$

We see from formula (2.7.9) that $\tilde{\beta}_{Y/T} = 0$ iff there exists a lifting F_T of F_T such that $\tilde{\beta}_{F_T} = 0$, i.e. such that Fil_Y is *invariant* under Φ_{F_T} . Such a lifting F_T is called an “excellent lifting of Frobenius” [7]. If all fibers of Y/T are nonhyperelliptic, or if $g = 2$, $\tilde{\beta}_{Y/T} = 0$ iff each $\tilde{\beta}_{Y/T}(t) = 0$, i.e. iff each $Y(t)$ is pre- W_2 -canonical. (In fact, if F_T exists, the Teichmüller lifting of t defines the canonical lifting.) If X is nonhyperelliptic and $g = 3$ (or if $g = 2$), the map (2.9.1) is in fact an isomorphism, so $Q_{Y/T}$ is zero. Thus, every nonhyperelliptic curve of genus 3 (and every curve of genus 2), is pre- W_2 -canonical, and there exists a unique excellent lifting F_T of Frobenius to T . Of course, this is no surprise, because, in these cases, the Torelli mapping from T to the moduli of principally polarized abelian varieties is a local isomorphism, and the theory of canonical coordinates applies.

§3. Differentiating the obstruction

We again let $Y/T/W_2$ denote an affine family of smooth curves of genus $g \geq 4$ and $Y/T/k$ its reduction mod p . In this section we shall prove that, if Y/T is versal, its general member is not pre- W_2 -canonical, i.e. $\beta_{Y(t)} \neq 0$ for general t . Now we have shown that $\{F_k^* \beta_{Y(t)} : t \in T\}$ can be “interpolated”: there is a section $\tilde{\beta}_{Y/T}$ of $F_T^*(Q_{Y/T})$ which specializes at each t to $F_k^* \beta_{Y(t)}$. We shall in fact see that $\{\beta_{Y(t)} : t \in T\}$ *cannot* be interpolated, i.e. the element $\tilde{\beta}_{Y/T}$ of $\Gamma(T, F_T^* Q_{Y/T})$ does not lie in the subspace $\Gamma(T, Q_{Y/T})$. Of course, this implies that it is not zero!

As is well known, for any quasicoherent sheaf Q on any smooth k -scheme T , the sheaf $F_T^* Q$ has a canonical integrable connection ∇ , characterized by $Q = \text{Ker}(F_T^* \nabla)$ [10]. Thus, what we must show is that for a versal family $Y/T/k$, the element $\nabla \tilde{\beta}_{Y/T}$ of $F_T^* Q_{Y/T} \otimes \Omega_{T/k}$ is not zero. As a first step, we present a simple formula for this element. It is important to note that the proof of our formula is purely formal, and in particular would make sense in the context of abstract F -crystals.

In order to write our formula in the most elegant form, we assume that the family Y/T is versal, i.e. that the Kodaira-Spencer mapping $\Theta_{T/k} \rightarrow H^1(Y, \Theta_{Y/k})$ is an isomorphism. We then identify $\Theta_{T/k}$ with $H^1(Y, \Theta_{Y/k})$ and $\Omega_{T/k}$ with $\Gamma(Y, \Omega_{Y/T}^{\otimes 2})$. Thus, we regard $\mu_{Y/T}$ as a map:

$$\mu_{Y/T}: S^2 \text{Fil}_Y \rightarrow \Omega_{T/k}. \tag{3.1.1}$$

Recall again that $\mu_{Y/T}$ can be computed from the Gauss-Manin connection ∇ and the bilinear form $\langle \cdot, \cdot \rangle$ on $H_{\text{DR}}^1(Y/T)$ by:

$$\mu_{Y/T}(\omega \cdot \omega') = \langle \omega, \nabla \omega' \rangle. \tag{3.1.2}$$

Let us also assume that all the fibers of Y/T are nonhyperelliptic, so that $\mu_{Y/T}$ is surjective, and let

$$K_{Y/T} =: \text{Ker}(\mu_{Y/T}). \tag{3.1.3}$$

Thus, our obstruction can be regarded as a \mathcal{O}_T -linear map

$$\tilde{\beta}_{Y/T}: F_T^* K_{Y/T} \rightarrow \mathcal{O}_T. \quad (3.1.4)$$

(3.2) **PROPOSITION:** *Let $\nabla: F_T^*(Q_{Y/T}) \rightarrow \Omega_{T/k} \otimes F_T^*(Q_{Y/T})$ be the canonical connection described above, let $\tilde{\beta}_{Y/T} \in F_T^*(Q_{Y/T})$ be the obstruction (2.7), and let $\gamma: F_T^* \text{Fil}_Y \rightarrow \text{Fil}_Y$ be the inverse Cartier isomorphism (2.6.5) (the inverse transpose of Hasse-Witt). Then the following diagram is commutative:*

$$\begin{array}{ccc} F_T^* K & \xrightarrow{\text{inc}} & F_T^* \text{Sym}^2 \text{Fil}_Y \\ -\nabla \tilde{\beta}_{Y/T} \downarrow & & \downarrow \text{Sym}^2(\gamma) \\ \Omega_{Y/k} & \xleftarrow{\mu_{Y/T}} & \text{Sym}^2 \text{Fil}_Y \end{array}$$

PROOF: Let $H =: H_T$, together with its integrable connection ∇ , and let $U =: U_T \rightarrow H$, $\mathbf{Fil} =: \text{Fil}_Y \rightarrow H$. Since $U \hookrightarrow H$ is invariant under the connection ∇ , the quotient module H/U inherits a connection $\bar{\nabla}$. Using the natural isomorphism: $\mathbf{Fil} \rightarrow H/U$, we can regard $\bar{\nabla}$ as a connection on \mathbf{Fil} , which we denote by ∂ . Note that $\mathbf{Fil} \rightarrow H$ is *not* invariant under ∇ ; in fact, (with the obvious notations),

$$\rho =: (\text{id} \otimes \pi_U) \circ \nabla \circ i_{\mathbf{Fil}}: \mathbf{Fil} \rightarrow \Omega_{T/W_2} \otimes U \quad (3.2.2)$$

“is” the Kodaira-Spencer mapping. Thus,

$$\nabla \circ i_{\mathbf{Fil}} = i_{\mathbf{Fil}} \circ \partial + (\text{id} \otimes i_U) \circ \rho. \quad (3.2.3)$$

We can view $i_{\mathbf{Fil}}: \mathbf{Fil} \hookrightarrow H$ and $\pi_U: H \hookrightarrow U$ as maps of modules with integrable connection. Using the usual conventions for the induced ∇ on \mathbf{Hom} 's, we find:

$$\nabla(i_{\mathbf{Fil}}) = (\text{id} \otimes i_U) \circ \rho \quad \text{and} \quad \nabla(\pi_U) = -\rho \circ \pi_{\mathbf{Fil}}. \quad (3.2.4)$$

Now let $F_T: \bar{T} \rightarrow \bar{T}$ be a lifting of F_T and $(\tilde{H}, \tilde{\nabla}, \tilde{U}, \tilde{\mathbf{Fil}}) =: F_T^*(H, \nabla, U, \mathbf{Fil})$. We have a horizontal map $\Phi =: \Phi_{F_T}: (\tilde{H}, \tilde{\nabla}, \tilde{U}) \rightarrow (H, \nabla, U)$, and the composite $\pi_U \circ \Phi \circ i_{\tilde{\mathbf{Fil}}}$, “divided by p ” is our obstruction α_{F_T} . We calculate:

$$\begin{aligned} \nabla(\pi_U \circ \Phi \circ i_{\tilde{\mathbf{Fil}}}) &= \nabla(\pi_U) \circ \Phi \circ i_{\tilde{\mathbf{Fil}}} + (\text{id} \otimes \pi_U) \circ (\nabla \Phi) \circ i_{\tilde{\mathbf{Fil}}} \\ &\quad + (\text{id} \otimes \pi_U \circ \Phi) \circ (\nabla i_{\tilde{\mathbf{Fil}}}) \\ &= -\rho \pi_{\mathbf{Fil}} \Phi i_{\tilde{\mathbf{Fil}}} + 0 + (\text{id} \otimes \pi_U \Phi i_U) \tilde{\rho}. \end{aligned} \quad (3.2.5)$$

Here $\tilde{\rho}: \widetilde{\text{Fil}} \rightarrow \Omega_{T/W_2} \otimes \tilde{U}$ is the map defined as in (3.2.2); one checks easily that $\tilde{\rho} = dF_T \circ F_T^*(\rho)$ and hence is divisible by p . Recall that $\phi: \tilde{U} \rightarrow U$ is the isomorphism induced by Φ , and that $\tilde{\alpha}_{Y, F_T} = \phi^{-1} \circ \alpha_{Y, F_T}$. Thus, if we let $\eta_{F_T}: F_T^* \Omega_T \rightarrow \Omega_T$ denote the map induced by $p^{-1} dF_T$, we find the following formula:

$$\nabla(\tilde{\alpha}_{Y, F_T}) = -(\text{id} \otimes \phi^{-1}) \circ \rho \circ \gamma + \eta_{F_T} \circ F_T^*(\rho). \quad (3.2.6)$$

Since our pairing $\text{Fil}_Y \times F_{\text{con}} \rightarrow \mathcal{O}_T$ is horizontal, we have the formula:

$$\begin{aligned} \nabla \tilde{\beta}_{Y, F_T}(\omega, \omega') &= \langle \omega, (\nabla \tilde{\alpha}_{Y, F_T})(\omega') \rangle \\ &= -\langle \omega, \text{id} \otimes \phi^{-1} \circ \rho \circ \gamma \omega' \rangle \\ &\quad + \langle \omega, \eta_{F_T} F_T^*(\rho) \omega' \rangle. \end{aligned} \quad (3.2.7)$$

But ϕ is the inverse transpose of γ , and (via the identification (3.1.1)), the bilinear form induced by ρ is just $\mu_{Y/T}$. Thus we have proved:

$$(\nabla \tilde{\beta}_{Y, F_T})(\omega, \omega') = -\mu_{Y/T}(\gamma\omega, \gamma\omega') + \eta_{F_T} \circ F_T^*(\mu_{Y/T})(\omega, \omega'). \quad (3.2.8)$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} F_T^* \text{Sym}^2 \text{Fil}_Y & \xrightarrow{(\text{Sym}^2 \gamma, -F_T^*(\mu_{Y/T}))} & \text{Sym}^2 \text{Fil}_Y \oplus F_T^* \Omega_{T/k} \\ \downarrow -\nabla \tilde{\beta}_{Y, F_T} & & \downarrow \mu_{Y/T} \oplus \eta_{F_T} \\ \Omega_{T/k} & \xleftarrow{\text{sum}} & \Omega_{T/k} \oplus \Omega_{T/k} \end{array} \quad (3.2.9)$$

We should perhaps remark that η_{F_T} depends on the lifting F_T of F_T , but that it factors through the *closed* one-forms, and its projection to $\mathcal{H}^1(\Omega_{T/k})$ is just the inverse Cartier operator [1, §8]. At any rate, this term is no immediate concern to us, because of course $F_T^*(\mu_{Y/T})$ vanishes on $F_T^*(K_{Y/T})$. Thus, the diagram (3.2.9) specializes to our desired formula (3.2.1) when we restrict to $F_T^*(K_{Y/T})$. \square

The following result completes our proof of Theorem 1.9.

(3.3) THEOREM: *Let Y/T be a versal family of nonhyperelliptic curves of genus $g \geq 4$. Then $\nabla \tilde{\beta}_{Y/T}$ is not identically zero. In other words, $\tilde{\beta}_{Y/T}$ does not descend to a section of $\mathcal{Q}_{Y/T}$.*

PROOF: Recall that we have an exact sequence

$$0 \rightarrow K_{Y/T} \rightarrow \text{Sym}^2 \text{Fil}_Y \rightarrow \Omega_{T/k} \rightarrow 0.$$

By (3.2), we see that $-\nabla\tilde{\beta}_{Y/T}$ can be viewed as the composite map:

$$F^*K_{Y/T} \xrightarrow{\text{inc}} F^* \text{Sym}^2 \text{Fil}_Y \xrightarrow{\text{Sym}^2\gamma} \text{Sym}^2 \text{Fil}_Y \xrightarrow{\text{proj}} \text{Sym}^2 \text{Fil}_Y/K_{Y/T}.$$

Here γ is the inverse Cartier operator. Thus, if $\nabla\tilde{\beta}_{B/Y} = 0$, $K_{Y/T}$ is *stable* under γ . We shall see that this is impossible. Indeed, for any curve X of genus $g \geq 1$, the linear system $\text{Fil}_X \cong \Gamma(X, \omega_X)$ is base point free, and there is a canonical map

$$\psi_X: X \rightarrow \mathbb{P}(\text{Fil}_X).$$

When X is nonhyperelliptic, this map is a closed immersion. If we let $\mathbb{P}\text{Fil}_Y$ denote the projective bundle over T associated to the locally free \mathcal{O}_T -module Fil_Y , we get a closed immersion of T -schemes

$$\psi_{Y/T}: Y \hookrightarrow \mathbb{P}(\text{Fil}_Y).$$

Now Fil_Y has no canonical basis, but it comes close to having a canonical \mathbb{F}_p -lattice, provided by the isomorphism $\gamma: F_T^*\text{Fil}_Y \rightarrow \text{Fil}_Y$. Namely, after T is replaced by a finite étale covering, the sheaf of \mathbb{F}_p -vector spaces $\text{Fil}_\gamma = \{ \omega \in \text{Fil}_Y: \gamma(1 \otimes \omega) = \omega \}$ spans Fil_Y , and in fact the canonical map

$$\mathcal{O}_T \otimes \text{Fil}_\gamma \rightarrow \text{Fil}_Y$$

is an isomorphism. (This is just Lang's theorem.) Thus, if $\mathbb{P}(\text{Fil}_\gamma)$ is the projective space over \mathbb{F}_p associated to Fil_γ , we have a canonical isomorphism: $T \times \mathbb{P}(\text{Fil}_\gamma) \xrightarrow{\sim} \mathbb{P}(\text{Fil}_Y)$. (We could even choose a basis of Fil_γ and write $\mathbb{P}(\text{Fil}) \cong \mathbb{P}^{g-1}$; such an isomorphism would be unique up to the action of the *finite* group $\text{PGL}(g, \mathbb{F}_p)$. Our map $\psi_{Y/T}$ then becomes a map of T -schemes:

$$\psi_{Y/T}: Y \hookrightarrow T \times \mathbb{P}(\text{Fil}_\gamma).$$

Recall that $\mu_{Y/T}: \text{Sym}^2 \text{Fil}_Y \rightarrow \Omega_{T/k}$ is identified with multiplication $\text{Sym}^2 \text{Fil}_Y \rightarrow \Gamma(Y, \omega_Y^{\otimes 2})$. Thus, its kernel $K_{Y/T}$ is just the set of quadratic forms which vanish on Y , and the corresponding divisors are precisely the quadrics in $T \times \mathbb{P}(\text{Fil}_\gamma)$ which contain Y . If $K_{Y/T}$ is invariant under γ , the subspace $K_\gamma = \{ \omega \in K_{Y/T}: \gamma(1 \otimes \omega) = \omega \} = K \cap \text{Sym}^2(\text{Fil}_\gamma)$ defines an \mathbb{F}_p -form of $K_{Y/T}$, i.e., $\mathcal{O}_T \otimes_{\mathbb{F}_p} K_\gamma \cong K_{Y/T}$. Then the subscheme

$Z(K_\gamma) \hookrightarrow \mathbb{P}(\text{Fil}_\gamma)$ corresponding to the homogeneous ideal spanned by K_γ is a descent of $Z(K_{Y/T})$: $Z(K_{Y/T}) \cong T \times Z(K_\gamma)$. Set-theoretically, $Z(K_{Y/T})$ corresponds to the intersection of all the quadrics containing Y , and we have $Y \hookrightarrow Z(K_{Y/T}) \cong T \times Z(K_\gamma)$. If $g \geq 5$, this is clearly absurd: Petri's theorem [16] tells us that the generic fiber Y_τ of Y/T is equal to the intersection of all quadrics containing it. Thus, $Y_\tau = Z(K_{Y/T})_\tau$ and of course does *not* descend to a curve over \mathbb{F}_p ! This contradiction completes the proof when $g \geq 5$.

In genus 4, a slightly different argument is needed. Since our problem is local for the étale topology, it will suffice to prove that $\nabla \tilde{\beta}_{Y/T} \neq 0$, for *one* versal family of curves of genus 4. In fact, it will suffice to prove that for *some* family Y/T of ordinary nonhyperelliptic curves of genus 4, the kernel $K_{Y/T}$ of the natural map $\text{Sym}^2 \Gamma(Y, \omega_Y) \cong \Gamma(Y, \omega_Y^{\otimes 2})$ is not preserved by the inverse Cartier operator γ . As is well-known, this kernel is one-dimensional, corresponding to the unique quadric surface Z^2 in $\mathbb{P}\Gamma(Y, \omega_Y)$ containing Y . Moreover, Y is the complete intersection of Z^2 with a cubic surface Z^3 , unique mod I_{Z^2} .

Suppose again that $K_{Y/T}$ is invariant under γ . Then, as above, we can, after making an étale base extension, find an \mathcal{O}_T -basis for $\Gamma(Y, \omega_Y)$ with respect to which $K_{Y/T}$ is constant. Thus $\mathbb{P}\Gamma(Y, \omega_Y) \cong \mathbb{P}^3 \times T$ and $Z^2 \cong Z_0^2 \times T$, where $Z^2 \hookrightarrow \mathbb{P}^3$ is a quadric (over \mathbb{F}_p). It is not hard to construct a family of curves for which this is impossible. Start with a fixed smooth cubic $Z^3 \subseteq \mathbb{P}^3$, and let $Z_T \hookrightarrow T \times \mathbb{P}^3$ be a family of quadrics with the property that Z_0 is a cone with vertex not on Z_0^3 , and such that the generic member of Z_T is smooth. Let Y_T be the family of curves $Z_T \cap Z^3$. It is clear that *no* recoordinitization of \mathbb{P}_T^3 can make Z_T constant!

All we have to do, then, is be certain that we can find a family of quadrics such that the fibers of Y_T are ordinary. Since this is an open condition, it suffices that the curve Y_0 corresponding to the quadric cone be ordinary. In other words, it is enough to show that for some quadric cone Z^2 and some cubic surface Z^3 , the Hasse-Witt matrix of their intersection is invertible. This ordinariness remains an open condition even if one allows further degeneration, e.g. to two planes meeting three planes in general position, which is easily seen to give an "ordinary" curve. \square

(3.4) REMARK: If X/k is an ordinary curve of genus g , there is a canonical isomorphism between the set of all holomorphic differentials on X fixed by Cartier and the set of p -torsion points on the Jacobian of X [14]. In particular, a choice of \mathbb{F}_p basis for $\Gamma(X, \omega_X)^\gamma$ corresponds precisely to a level p structure on X (i.e., on $\text{Jac}(X)$). This gives us a canonical map from the moduli of ordinary curves with level p structure to the Hilbert scheme of \mathbb{P}^{g-1} . It seems to us that this map, used implicitly in the previous proof, warrants further study. \square

Formula (3.2) can also be used to calculate the tangent space of Σ_{W_2} (in the nonhyperelliptic locus). If Y/T is again a versal family of nonhyperelliptic ordinary curves, we have seen that Σ_{W_2} is the closed subscheme of T defined by the vanishing of a certain section $\tilde{\beta}_{Y/T}$ of $\text{Hom}(F_T^* K_{Y/T}, \mathcal{O}_T)$. Here $K_{Y/T}$ is locally free of rank $(g-2)(g-3)/2$, and its formation commutes with base change. Thus, the ideal I_Σ of Σ_{W_2} is just the image of $\tilde{\beta}_{Y/T}$. We have a natural map:

$$\delta: I_\Sigma/I_\Sigma^2 \rightarrow \Omega_T^1|_\Sigma \xleftarrow{\bar{\mu}_{Y/T}} \text{Sym}^2 \text{Fil}_Y/K_{Y/T}.$$

(3.5) PROPOSITION: *The image of I_Σ/I_Σ^2 via the above map is $(\text{Sym}^2 \gamma(F_T^* K_{Y/T}) + K_{Y/T})/K_{Y/T}$. In particular, if X/k is nonhyperelliptic, ordinary, and belongs to Σ_{W_2} , then the conormal bundle of Σ_{W_2} in \mathcal{M}_g at X/k is canonically isomorphic to*

$$(\text{Sym}^2 \gamma(F^* K_X) + K_X)/K_X \subseteq \text{Sym}^2 \text{Fil}_X.$$

PROOF: Choose a local basis $\{\eta_i\}$ for $K_{Y/T}$; then $\{1 \otimes \eta_i\}$ is a local horizontal basis for $F_T^* K_{Y/T}$, and $\{\tilde{\beta}_{Y/T}(1 \otimes \eta_i)\}$ is a set of generators for I_Σ . Then $d\tilde{\beta}_{Y/T}(1 \otimes \eta_i) = \nabla(\tilde{\beta}_{Y/T})(1 \otimes \eta_i) = -\mu_{Y/T}(\text{Sym}^2 \gamma(1 \otimes \eta_i)) \in \Omega_{T/k}$, by (3.2). Projecting to $\Omega_{T/k}|_\Sigma$, we find the desired formula. \square

We have calculated some examples of these conormal spaces, one by hand, more with the aid of Spencer Bloch and his IBM personal computer. For example, the Fermat curve X of degree 5 is nonhyperelliptic of genus 6, and is ordinary and pre- W -canonical if $p \equiv 1 \pmod{5}$. The moduli space \mathcal{M}_g has dimension 15, and K_s has rank 6. For all those p which fit into the computer, we found that the conormal bundle also had rank 6. Thus Σ_{W_2} is smooth of codimension 6 at the point X . It would be interesting to study the “deeper” subschemes Σ_{W_n} for higher n , and of course to pursue higher genera.

References

- [1] BERTHELOT, P. and OGUS, A.: *Notes on Crystalline Cohomology*. Math. Notes Vol. 21, Princeton University Press (1978).
- [2] BERTHELOT, P. and OGUS, A.: F -isocrystals and De Rham cohomology, I. *Invent. Math.* 72 (1983) 150–199.
- [3] DELIGNE, P.: *Equations Differentielles à Points Singuliers Reguliers. Lecture Notes in Math., Vol. 163*, Springer-Verlag, 1970).
- [4] DELIGNE, P.: letter to Shafarevich (1976).
- [5] DELIGNE, P. and ILUSIE, L.: Cristaux ordinaires et coordonnées canoniques. In *Surfaces Algébriques, Lecture Notes in Math., Vol. 868*, Springer-Verlag (1981).
- [6] DELIGNE, P.: MUMFORD, D.: The irreducibility of the space of curves of a given genus. *Publ. Math. I.H.E.S.* 36 (1969) 75–110.
- [7] DWORK, B.: Normalized period matrices, II. *Annals of Math.* 98 (1973) 1–57.

- [8] KATZ, N.: Algebraic solutions of differential equations: p -curvature and the Hodge filtration. *Invent. Math.* 18 (1972) 1–118.
- [9] KATZ, N.: Slopes filtration of F -crystals. *Asterisque* 63 (1979) 113–164.
- [10] KATZ, N.: Nilpotent connections and the monodromy theorem: applications of a result of Turrittin. *Publ. Math. I.H.E.S.* 39 (1971) 175–232.
- [11] MESSING, W.: The Crystals Associated to Barsotti Tate Groups. *Lecture Notes in Math., Vol. 264*, Springer-Verlag (1972).
- [12] MILLER, L.: Curves over finite fields with invertible Hasse-Witte matrices. *Math. Annalen* 197 (1972).
- [13] OORT, F. and SEKIGUCHI, T.: The canonical lifting of an ordinary Jacobian variety need not be a Jacobian variety (preprint).
- [14] SERRE, J-P.: Sur la topologie des varietes algebriques en caracteristique p . *Symp. Int. de Topologia Algebraica, Mexico* (1958) 24–53.
- [15] SERRE, J-P. and TATE, J.: mimeographed notes from the 1964 AMS Summer Institute at Woods Hole.
- [16] ST. DONAT, B.: On Petri's analysis of the linear system of quadrics through a curve. *Math. Annalen* 206 (1973) 157–175.

(Oblatum 6-IX-1984)

B. Dwork
Department of Mathematics
Princeton University
Princeton, NJ 08544
USA

A. Ogus
Department of Mathematics
University of California
Berkeley, CA 94720
USA