

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 56, n° 2 (1985), p. 233-236

http://www.numdam.org/item?id=CM_1985__56_2_233_0

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RELATIVE EXTREME SUBSETS

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Generalizing the notion of extreme point of a set in the real linear space L , Klee [2] introduced the following definition of *relative extreme point*. Let $B \subset L$ and $C \subset L$. If a point of B does not belong to any open segment $(b, c) = \{(1 - \lambda)b + \lambda c; 0 < \lambda < 1\}$ determined by distinct points $b \in B$ and $c \in C$, then it is called an *extreme point in B relative to C* . Observe that the known notion of extreme subset can be generalized analogously:

DEFINITION: Let $A \subset B \subset L$ and $C \subset L$. We say that A is an *extreme subset of B relative to C* if, together with any point $a \in A$, the set A contains every point $b \in B$ such that $a \in (b, c)$ for some $c \in C$.

Let us note that the definition can be expressed more geometrically using the notion

$$P_C(A) = \{(1 - \mu)c + \mu a; \mu \geq 1, c \in C, a \in A\}$$

of the penumbra ([5], p. 22) of A with respect to C . Namely, a subset A of $B \subset L$ is an extreme subset of B relative to a non-empty set $C \subset L$ if and only if

$$P_C(A) \cap B = A.$$

Obviously in the case $A = \{a\}$ of our definition we get the notion of extreme point a in B relative to C and in the case $B = C$ we obtain the usual notion of extreme subset A of B . On the other hand, the above definition is a special case of the notion (presented as Remark in [3]) of Φ -extreme subset, where $\Phi: \mathcal{D} \rightarrow 2^L$ is a function such that \mathcal{D} consists of all one-point subsets of L and $\Phi(\{b\}) = \bigcup_{c \in C} (b, c)$. Let us observe also a connection of our definition with the notion of semi-extreme subset. Remember that a subset A of a convex set $B \subset L$ is called a *semi-extreme* subset of B if $B \setminus A$ is convex (comp. [1], p. 32). As in [6], pp. 186–187, this notion of semi-extreme subset can be extended to arbitrary (i.e. not necessary convex) set B : if $A \subset B$ and $A \cap \text{conv}(B \setminus A) = \emptyset$, then we call A a *semi-extreme* subset of B . The above mentioned connection is expressed by the following easily provable:

PROPOSITION: *If A is a semi-extreme subset of B , then A is an extreme subset of B relative to $B \setminus A$. When B is convex, the inverse implication also holds.*

The reader can without difficulty verify six properties of relative extreme subsets presented in Theorem 1, the first five of which generalize well-known properties of extreme subsets in the usual sense.

THEOREM 1: *Relative extreme subsets have the following properties*

- (a) *Any intersection of extreme subsets of B relative to C is an extreme subset of B relative to C .*
- (b) *Any union of extreme subsets of B relative to C is an extreme subset of B relative to C .*
- (c) *If A is an extreme subset of B relative to C and if A_1 is an extreme subset of A relative to C , then A_1 is an extreme subset of B relative to C .*
- (d) *If $A \subset B_1 \subset B_2$ and if A is an extreme subset of B_2 relative to C , then A is an extreme subset of B_1 relative to C .*
- (e) *Sets B and \emptyset are extreme subsets of B relative to any set C .*
- (f) *If $C_1 \subset C_2$ and if A is an extreme subset of B relative to C_2 , then A is an extreme subset of B relative to C_1 . Any subset of B is extreme in B relative to empty set.*

The notion of the usual extreme subset of a set B is considered mainly in the case when B is convex. Also the notion of extreme point of B relative to C plays an important part in the case when B is convex and $C \subset B$ (comp. [2] and [4]). This is why we now consider extreme subsets of a convex set B relative to a subset of B .

THEOREM 2: *Let B be a convex set of a real linear space L and let $A \subset B$, $C \subset B$. The set A is an extreme subset of B relative to C if and only if A is an extreme subset of B relative to the convex hull $\text{conv } C$.*

PROOF: Suppose that A is an extreme subset of B relative to C . To verify if A is an extreme subset of B relative to $\text{conv } C$ we shall show that for any $a \in A$, $b \in B$ and $c \in \text{conv } C$ such that $a \in (b, c)$ we have $b \in A$.

As an element of $\text{conv } C$, the point c belongs to the convex hull of a finite number of points of C . Consequently, there exists a minimal finite collection of points $c_1, \dots, c_k \in C$ such that

$$c \in \text{conv}\{b, c_1, \dots, c_k\}.$$

In other words

$$c = \alpha_0 b + \alpha_1 c_1 + \dots + \alpha_k c_k,$$

where $\alpha_0 \geq 0, \alpha_1 > 0, \dots, \alpha_k > 0$ and $\alpha_0 + \alpha_1 + \dots + \alpha_k = 1$. Since $a = \beta b + \gamma c$ for some $\beta > 0$ and $\gamma > 0$ such that $\beta + \gamma = 1$, we have

$$a = (1 - \delta_1 - \dots - \delta_k)b + \delta_1 c_1 + \dots + \delta_k c_k,$$

where $\delta_1 = \gamma \alpha_1 > 0, \dots, \delta_k = \gamma \alpha_k > 0$ and $1 - \delta_1 - \dots - \delta_k = 1 - \gamma(\alpha_1 + \dots + \alpha_k) = 1 - \gamma(1 - \alpha_0) = \beta + \gamma \alpha_0 > 0$.

Now, we recurrently define points b_k, b_{k-1}, \dots, b_1 as follows

$$b_k = b,$$

$$b_i = \frac{\delta_{i+1}}{1 - \delta_1 - \dots - \delta_i} c_{i+1} + \frac{1 - \delta_1 - \dots - \delta_{i+1}}{1 - \delta_1 - \dots - \delta_i} b_{i+1}, \quad i = k - 1, \dots, 1.$$

Since the coefficients

$$\delta_{i+1}/(1 - \delta_1 - \dots - \delta_i), (1 - \delta_1 - \dots - \delta_{i+1})/(1 - \delta_1 - \dots - \delta_i)$$

are positive and since the sum of them is equal to 1, the definition of b_i implies that

$$b_i \in (c_{i+1}, b_{i+1}), \quad i = 1, \dots, k - 1. \tag{1}$$

By the definition of b_i , the equality

$$\delta_{i+1} c_{i+1} + (1 - \delta_1 - \dots - \delta_{i+1}) b_{i+1} = (1 - \delta_1 - \dots - \delta_i) b_i$$

holds for $i = k - 1, \dots, 1$ and consequently

$$\begin{aligned} a &= \delta_1 c_1 + \dots + \delta_k c_k + (1 - \delta_1 - \dots - \delta_k) b_k \\ &= \delta_1 c_1 + \dots + \delta_{k-1} c_{k-1} + [\delta_k c_k + (1 - \delta_1 - \dots - \delta_k) b_k] \\ &= \delta_1 c_1 + \dots + \delta_{k-1} c_{k-1} + (1 - \delta_1 - \dots - \delta_{k-1}) b_{k-1} \\ &= \dots = \delta_1 c_1 + (1 - \delta_1) b_1. \end{aligned}$$

Thus in virtue of $\delta_1 > 0$ and $1 - \delta_1 > 0$ we have

$$a \in (c_1, b_1). \tag{2}$$

Since B is convex, from $b_k \in B$ and $c_k, \dots, c_1 \in B$ and also from $b_i \in (c_{i+1}, b_{i+1})$ for $i = k - 1, \dots, 1$ we get in turn that $b_i \in B$ for $i = k - 1, \dots, 1$.

Since A is an extreme subset of B relative to C and since $a \in A, b_i \in B$ and $c_i \in C$ for $i = 1, \dots, k$, we first obtain from (2) that $b_1 \in A$ and next (if $k \geq 2$), applying $(k - 1)$ -times (1) we get in turn that

$b_2 \in A, \dots, b_k \in A$. Thus $b = b_k \in A$. Hence A is an extreme subset of B relative to C .

The inverse implication of our theorem results immediately from the inclusion $C \subset \text{conv } C$ and from property (f) of Theorem 1.

References

- [1] R.B. HOLMES: *Geometric Functional Analysis and its Applications*. Springer-Verlag, New York (1975).
- [2] V. KLEE: Relative extreme points. *Proceedings of the International Symposium on Linear Spaces*, pp. 282–289. Jerusalem Academic Press, Jerusalem (1961).
- [3] M. LASSAK: Terminal subsets of convex sets in finite-dimensional real normed spaces. *Colloq. Math.* (to appear).
- [4] R.R. PHELPS: Support cones and their generalizations. *Proceedings of Symposia in Pure Mathematics* 7 (Convexity), pp. 393–401. American Mathematical Society, Providence (1963).
- [5] R.T. ROCKAFELLAR: *Convex Analysis*. Princeton, New York (1970).
- [6] G. SIERKSMA: Generalizations of Helly's theorem: Open problems. *Proceedings of the Second University Oklahoma Conference (Convexity and related combinatorial geometry)*, pp. 173–192. Marcel Dekker, Norman (1980).

(Oblatum 28-X-1983)

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