COMPOSITIO MATHEMATICA

TOSHIAKI ADACHI
TOSHIKAZU SUNADA

Energy spectrum of certain harmonic mappings

ENERGY SPECTRUM OF CERTAIN HARMONIC MAPPINGS

Toshiaki Adachi and Toshikazu Sunada

Let (M^m, g) and (N^n, h) be two compact Riemannian manifolds. Given a C^∞-mapping φ of M into N, we define the energy density $e(\varphi)$ to be the trace of the induced tensor φ^*h with respect to the metric g, which is occasionally written as $\|d\varphi\|^2$ by using the natural metric $\|\cdot\|$ on the vector bundle $T^*M \otimes \varphi^{-1}TN$. Define the energy functional $E(\varphi)$ by

$$E(\varphi) = \int_M e(\varphi)\,dv_g,$$

where dv_g is the volume element of (M, g). If the functional E is stationary at φ, then φ is called harmonic. We denote by $\mathscr{H}(M, N)$ the set of all harmonic mappings of M into N. The purpose of this paper is to study the energy spectrum $\{E(\varphi) \in \mathbb{R}; \varphi \in \mathscr{H}(M, N)\}$ from a quantitative view point.

In the special case that $M = S^1$ (the circle), harmonic mappings are nothing but closed geodesics in N, and the so-called Palais-Smale condition for the functional E guarantees that, for generic metrics on N, the energy spectrum is a discrete set in $\mathbb{R}$. Actually, several methods have been developed to get quantitative behavior of the energy spectrum (see for instance N. Hingston [9] and R. Gangolli [6]).

When dim $M > 1$, not much is known on the structure of $\mathscr{H}(M, N)$, and the aspect of the energy spectrum might be possibly complicated because we may no longer expect the P − S condition. Even the question whether the spectrum $E(\varphi)$ is bounded from above or not is rather hard to decide. But confining ourselves to manifolds N with non-positive sectional curvature, we find that "a weak compactness property" holds, if not the P − S condition, which enables us to prove discreteness of the energy spectrum, and to get some information about asymptotics of $E(\varphi)$ at infinity.

From now on we suppose that N is non-positively curved. As was shown by Eells-Sampson [3] and P. Hartman [8], each connected component of $C^0(M, N)$, the space of continuous mappings, contains one and only one component of $\mathscr{H}(M, N)$, and the functional E assumes constant value on each component. Thus we may set, for each component $\mathscr{C} \subset \mathscr{H}(M, N)$

$$E(\mathscr{C}) = E(\varphi), \quad \varphi \in \mathscr{C}.$$

One of main results in this paper is embodied in

THEOREM 1: *There are positive constants c_1, c_2, depending only upon the diameters D_M, D_N, the volumes V_M, V_N and the lower limits ρ_M, ρ_N of the Ricci curvature of M, N such that for any $x > 0$,*

$$\# \left\{ \mathscr{C} \subset \mathscr{H}(M, N);\ E(\mathscr{C}) \leqslant x^2 \right\} \leqslant c_1 \exp(c_2 x).$$

In particular, the energy spectrum constitutes a discrete set in $\mathbb{R}$.

We will observe in §1 that the constants c_1 and c_2 can be explicitly calculated in terms of D, V, ρ. Further, we will see in the course of proof that if N is flat, the left hand side in the above inequality has polynomial growth as x tends to infinity. In the case N is a flat torus, the exact growth rate can be obtained. To state this, we denote by $b_1 = b_1(M)$ the first Betti number, $A(M)$ the Albanese torus, that is, the torus

$$A(M) = H^1(M, \mathbb{R})/H^1(M, \mathbb{Z}).$$

which carries the flat metric derived from the global inner product of harmonic 1-forms on M. We will henceforth write $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$ as $x \uparrow \infty$.

THEOREM 2: *Let N be a flat torus. If we set*

$$c = V_N^{-b_1} V_{A(M)}^{-n} \pi^{b_1 n/2} / \Gamma\!\left(\frac{b_1 n}{2} + 1 \right),$$

then

$$\# \left\{ \mathscr{C} \subset \mathscr{H}(M, N);\ E(\mathscr{C}) \leqslant x^2 \right\} \sim c x^{b_1 n} \qquad \text{as} \quad x \uparrow \infty.$$

Another case which allows us to get precise information on the growth rate is the following, essentially due to G.A. Margulis [18] and Parry-Pollicott [29].

THEOREM 3: *Suppose that the sectional curvature of N is strictly negative. Then there exist a positive constant h_N such that*

$$\# \left\{ \mathscr{C} \subset \mathscr{H}(S^1, N);\ E(\mathscr{C}) \leqslant x^2 \right\} \sim \exp(h_N x)/h_N x \qquad \text{as} \quad x \uparrow \infty.$$

It turns out that the constant h_N is the topological entropy of the geodesic flow on the tangent unit sphere bundle of N. The following estimate is known ([17]):

$$(n - 1)A \leqslant h_N \leqslant (n - 1)B,$$

where $-A^2$ and $-B^2$ are the upper and lower limits of the sectional curvature respectively. If N is a rank-one symmetric space, then one may conclude that

$$h_N = \sum_{i=1}^{n-1} \mu_i^{1/2},$$

where $\mu_1 \geqslant \cdots \geqslant \mu_{n-1} > \mu_n = 0$ are the eigenvalues of the curvature operator: $X \to {}^N R(X, v)v$, v being arbitrary unit tangent vector (see [6],[25]).

It is known ([3]) that if the Ricci curvature of M is semi-positive definite, then any harmonic mapping of M into N is totally geodesic. In §4, we will see that if a component $\mathscr{C} \subset \mathscr{H}(M, N)$ contains a totally geodesic mapping, then all the mapping in $\mathscr{C}$ are totally geodesic.

As a byproduct of the proof of Theorem 1, we obtain a partial generalization of Theorem 3.

THEOREM 4: *Let $\mathscr{T}(M, N)$ be the space of totally geodesic mappings of M into N. Suppose that the fundamental group $\pi_1(M, p)$ is generated by homotopy classes of geodesic loops $\gamma_1, \gamma_2, \cdots, \gamma_t$ at p with length of $\gamma_i \leqslant l$. Then*

$$\limsup_{x \to \infty} \frac{1}{x} \log \# \left\{ \mathscr{C} \subset \mathscr{T}(M, N); E(\mathscr{C}) \leqslant x^2 \right\} \leqslant lth_N V_M^{-1/2}.$$

Especially,

$$\limsup_{x \to \infty} \frac{1}{x} \log \# \left\{ \mathscr{C} \subset \mathscr{H}(\mathbb{R}^m/\mathbb{Z}^m, N); E(\mathscr{C}) \leqslant x^2 \right\} \leqslant mh_N.$$

We should point out that Theorem 1 is, in its nature, regarded as a generalization of the results by H. Huber [10], H.C. Im Hof [11] and M. Maeda [16] about geometric estimates of the order of the isometry group acting on a non-positively curved manifold. In fact, combining Theorem 1 with the rigidity theorem in §4, we have the following which immediately applies to the isometries of N since any isometry φ is harmonic and $E(\varphi) = nV_N$.

THEOREM 5: *Suppose that N is non-positively curved and the Ricci curvature is negative at some point. Then for any $x > 0$, the set of surjective harmonic mappings $\varphi: M \to N$ with $E(\varphi) \leqslant x$ is finite, and the cardinarity is estimated as in Theorem* 1.

COROLLARY (see also §5): *The number of isometries acting on the above N has a bound expressed explicitly in terms of D_N, V_N and ρ_N.*

We give here several notations which will be used later. For a Riemannian manifold (M, g), let $(\tilde{M}, \tilde{g})$ denote the universal covering manifold. The distance function, the sectional curvature and the injectivity radius of M are denoted by d_M, $^M\mathrm{Riem}$ and i_M respectively. We further set

$$\rho_M = \min \mathrm{Ricc}(v, v)/(m-1), \quad v \in TM, \ \|v\| = 1.$$

As usual, $B_r(p)$ will denote the closed ball of radius r with center p, and D/dt will denote the covariant differentiation along a curve.

For information about the basic properties of harmonic mappings, see Eells-Lemaire [4].

Acknowledgement

§1. Proof of Theorem 1

Throughout this section, (M, g) is a compact Riemannian manifold and (N, h) is a non-positively curved compact manifold. We denote by A_x the set of components of $\mathscr{H}(M, N)$ containing some φ such that $\sup e(\varphi)^{1/2} \leqslant x$ (it is actually true that if φ and ψ are harmonic and homotopic, then $e(\varphi) \equiv e(\psi)$). We first estimate the cardinarity $\#A_x$. For this we recall Lemaire's idea in [14] which was originally applied to the finiteness theorem for harmonic mappings of uniform bounded dilatation. Fix points $p_0 \in \tilde{M}$ and $q_0 \in \tilde{N}$, and also fix a fundamental domain $\mathscr{D}$ in $\tilde{N}$ for the $\pi(N)$-action on $\tilde{N}$ such that $q_0 \in \mathscr{D} \subset B_{D_N}(q_0)$. For each $\mathscr{C} \in A_x$, we take $\varphi_\mathscr{C} \in \mathscr{C}$ with $\|d\varphi_\mathscr{C}\| \leqslant x$ and a (unique) lifting $\tilde{\varphi}_\mathscr{C} : \tilde{M} \to \tilde{N}$ of $\varphi_\mathscr{C}$ with $\tilde{\varphi}_\mathscr{C}(p_0) \in \mathscr{D}$. We also take a homomorphism $\delta_\mathscr{C}$: $\pi_1(M) \to \pi_1(N)$ satisfying $\tilde{\varphi}_\mathscr{C}(\gamma p) = \delta_\mathscr{C}(\gamma)\tilde{\varphi}_\mathscr{C}(p)$, which turns out to depend only upon $\mathscr{C}$. Since the natural bijection of the set of components of $C^0(M, N)$ onto the set of conjugacy classes of homomorphisms of $\pi_1(M)$ into $\pi_1(N)$ coincides with the mapping: $\mathscr{C} \mapsto$ the conjugacy class of $\delta_\mathscr{C}$, we find that the mapping

$$\begin{array}{ccc} A_x & \to & \mathrm{Hom}\left(\pi_1(M), \pi_1(N)\right) \\ \cup\!\!\!| & & \cup\!\!\!| \\ \mathscr{C} & \mapsto & \delta_\mathscr{C} \end{array}$$

is injective.

LEMMA 1.1: *Let ϵ be arbitrary positive number. The fundamental group $\pi_1(M)$ is generated by some elements $\tau_1, \cdots, \tau_t$ such that $d_{\tilde{M}}(p_0, \tau_t p_0) \leqslant 2(D_M + \epsilon)$.*

A proof of this easy lemma is found in [2] (Prop. 2.5.6).
We set

$$T = \{\tau_1, \cdots, \tau_t\}$$

$$K_x = \bigcup_{\mathscr{C} \in A_x} \tilde{\varphi}_{\mathscr{C}}\big(B_{2(D_M + \epsilon)}(p_0)\big).$$

Since $\|d\tilde{\varphi}_{\mathscr{C}}\| \leqslant x$, the set $\overline{K}_x$ is compact, and the set

$$S_x = \{\sigma \in \pi_1(N); \ \sigma\mathscr{D} \cap K_x \neq \phi\}$$

is finite. Moreover, from the definition, it follows that $\delta_{\mathscr{C}}(T) \subset S_x$ for any $\mathscr{C} \in A_x$, so that, noting $\delta_{\mathscr{C}}$ is characterized by the restriction $\delta_{\mathscr{C}} | T$, we have

$$\#A_x \leqslant (\#S_x)^{\#T}.$$

To estimate $\#S_x$ and $\#T$, we need the following general lemma.

LEMMA 1.2: *Let X^k be a compact Riemannian manifold, and let $x_0 \in \tilde{X}$. Then, for any $R > 0$,*

$$\#\{\gamma \in \pi_1(X); \ d_{\tilde{X}}(x_0, \gamma x_0) \leqslant R\}$$

$$\leqslant \frac{\omega_{k-1} \exp\big((k-1)|\rho_X|^{1/2}(R + D_X)\big)}{(k-1)V_X |\rho_X|^{k/2} 2^{k-1}}$$

provided that $\rho_X < 0$. If $\rho_X \geqslant 0$, then the right hand side can be replaced by

$$\frac{\omega_{k-1}(R + D_X)^k}{kV_X}.$$

In these inequalities, ω_{k-1} designates the volume of the unit sphere in the k-dimensional Euclidean space.

PROOF: Let $\mathscr{D}$ be a fundamental domain in $\tilde{X}$ such that $x_0 \in \mathscr{D} \subset B_{D_X}(x_0)$. If γ runs over elements in $\pi_1(X)$ with $d_{\tilde{X}}(x_0, \gamma x_0) \leqslant R$, then

$$\bigcup_{\gamma} \gamma D \subset B_{R + D_X}(x_0),$$

whence we have

$$\#\{\gamma; \ d_{\tilde{X}}(x_0, \gamma x_0) \leqslant R\} \leqslant V_X^{-1} \text{ volume } \big(B_{R + D_X}(x_0)\big).$$

According to the volume comparison theorem due to Bishop-Crittenden [1] and Gromov [7],

$$\text{Volume }\left(B_r(x_0)\right) \quad \leqslant \omega_{k-1}\int_0^r\left(\frac{\sinh|\rho_X|^{1/2}t}{|\rho_X|^{1/2}}\right)^{k-1}dt, \quad (\rho_X < 0)$$

or

$$\leqslant \frac{1}{k}\omega_{k-1}r^k, \qquad\qquad (\rho_X \geqslant 0),$$

whence the lemma.

Note that if $\rho_M > 0$, then any harmonic mapping is constant, so we suppose hereafter $\rho_M \leqslant 0$. Applying the above lemma to the case $X = M$, $R = 2(D_M + \epsilon)$, we get

$$\#T \leqslant \frac{\omega_{m-1}\exp\big((m-1)|\rho_M|^{1/2}(3D_M+2\epsilon)\big)}{(m-1)2^{m-1}V_M|\rho_M|^{1/2}}, \quad (\rho_M < 0)$$

or   (1.1)

$$\leqslant \frac{\omega_{m-1}(3D_M+2\epsilon)^m}{mV_M}, \qquad (\rho_M = 0).$$

REMARK ([2]): If $^M\text{Riem} \geqslant -A^2$, $A \geqslant 0$, then the fundamental group $\pi_1(M)$ can be generated by

$$t \leqslant 2\big(3 + 2\cosh 2A(D_M + \epsilon)\big)^{m/2}$$

elements.

In order to estimate $\#S_x$, we first notice that $K_x \subset B_{2(D_M+\epsilon)x+D_N}(q_0)$, hence for each $\sigma \in S$, taking some $q \in \mathscr{D}_x$ with $\sigma q \in K_x$, we find

$$d_{\tilde{N}}(q_0, \sigma q_0) \leqslant d_{\tilde{N}}(q_0, \sigma q) + d_{\tilde{N}}(\sigma q, \sigma q_0) \leqslant 2x(D_M + \epsilon) + 2D_N.$$

Using again Lemma 1.2, we have

$$\#S_x \leqslant \frac{\omega_{n-1}\exp\big((n-1)|\rho_N|^{1/2}(2x(D_M+\epsilon)+3D_N)\big)}{(n-1)2^{n-1}V_N|\rho_N|^{1/2}}, \quad (\rho_N < 0)$$

or

$$\leqslant \frac{\omega_{n-1}(2x(D_M+\epsilon)+3D_N)^n}{nV_N}, \qquad (\rho_N = 0).$$

$$(1.2)$$

Combining (1.1) and (1.2) and letting $\epsilon \downarrow 0$, we get

$$\#A_x \leqslant c_1(D_M, V_M, \rho_M, D_N, V_N, \rho_N)$$

$$\times \exp\left(c_2(D_M, V_M, \rho_M)|\rho_N|^{1/2}x\right)$$

provided that $\rho_N < 0$. If $\rho_N = 0$, or equivalently, if N is flat, then

$$\#A_x \leqslant c_1(D_M, V_M, \rho_M, V_N)(2xD_M + 3D_N)^{c_2(D_M, V_M, \rho_M)}.$$

Theorem 1 turns out to be a consequence of the following a-priori estimate.

PROPOSITION 1.3: *There exists a positive constant c depending only upon D_M, V_M and ρ_M such that, for every $\varphi \in \mathcal{H}(M, N)$*

$$\sup e(\varphi) \leqslant cE(\varphi).$$

REMARK: If we do not care for the dependence of the constant c on the geometric terms, then the above is just the result proved by Eells-Sampson [3] and K. Uhlenbeck [27].

The case $\rho_M \geqslant 0$ is easy because, this being the case, every harmonic mappings φ are totaly geodesic, and $e(\varphi)$ are constant functions on M. Hence $E(\varphi) = V_M e(\varphi)$.

We suppose $\rho_M < 0$. The proof will be carried out along the same line as Uhlenbeck [27], looking carefully at constants appearing in each step. Let $\varphi \in \mathcal{H}(M, N)$. We first observe

$$\Delta e(\varphi) = 2\|\nabla d\varphi\|^2 + 2\sum{}^M R^{ij}\partial_i\varphi^\alpha\partial_j\varphi^\beta h_{\alpha\beta}$$

$$-2\sum{}^N R_{\alpha\beta\gamma\delta}\partial_i\varphi^\alpha\partial_j\varphi^\beta\partial_k\varphi^\gamma\partial_l\varphi^\delta g^{ik}g^{jl}$$

$$\geqslant -2(m-1)|\rho_M|e(\varphi),$$

whence, for each $p \geqslant 1$

$$\int_M \|de(\varphi)^p\|^2 dv_g = \frac{p^2}{2p-1}\int_M \langle de(\varphi)^{2p-1}, \; de(\varphi)\rangle dv_g$$

$$= \frac{p^2}{2p-1}\int_M e(\varphi)^{2p-1}\Delta e(\varphi)dv_g$$

$$\leqslant \frac{2p^2}{2p-1}(m-1)|\rho_M|\int_M e(\varphi)^{2p}dv_g.$$

In view of [27], it remains only to estimate the Sobolev constant. But this has been recently done by S. Gallot [5] and P. Li [15], that is,

$$\left(\int_M f^{2\beta} dv_g \right)^{1/2\beta} \leqslant C_m C(M) \left(\int_M \| df \|^2 dv_g \right)^{1/2}$$

$$+ V_M^{-1/m} \left(\int_M f^2 dv_g \right)^{1/2}$$

for any C^1-function f on M, where in the case $m \geqslant 3$, we set

$$\beta = \frac{m}{m-2}, \quad C_m = \left\{ \frac{8(m-1)^2}{(m-2)^2} + 2^{3+|m-4|-2/m} \right\}^{1/2}$$

$$C(M) = V_M^{-1/m} D_M \left\{ \frac{1}{\sqrt{\Lambda}} \int_0^{\sqrt{\Lambda}} \left(\frac{\lambda \cosh u}{\sqrt{\Lambda}} + \frac{\sinh u}{m\sqrt{\Lambda}} \right)^{m-1} du \right\}^{1/m}$$

$$\Lambda = D_M^2 |\rho_M|$$

$$\lambda = \int_0^{\sqrt{\Lambda}/2} (\cosh u)^{m-1} du.$$

In the case $m = 2$, we set

$$\beta = 2, \quad c_2 = 4$$

$$C(M) = V_M^{-1/2} D_M \frac{1}{\sqrt{\Lambda}} \int_0^{\sqrt{\Lambda}} \left(\frac{\lambda \cosh u}{\sqrt{\Lambda}} + \frac{\sinh u}{2\sqrt{\Lambda}} \right) du.$$

Therefore, repeating the argument in [27], we finally have

$$e(\varphi) \leqslant \prod_{j=0}^{\infty} \left(c_m C(M) \beta^j \left(\frac{2(m-1)|\rho_M|}{2\beta^j - 1} \right)^{1/2} + V_M^{-1/m} \right)^{1/\beta^j}$$

$$\times \left(c_m C(M) (2(m-1)|\rho_M|)^{1/2} + V_M^{-1/m} \right)^{m/2} E(\varphi).$$

REMARK: One can use the argument in Eells-Sampson [3] to obtain a bound on the constant c in Proposition 1.3. In doing this, however, it seems difficult to eliminate the dependence on the injectivity radius i_M.

We now proceed to the proof of Theorem 4. In the proof of Theorem 1, we replace T, K_x by

$$T' = \{[\gamma_1], \cdots, [\gamma_t]\}, [\gamma] \text{ being the homotopy class of the loop } \gamma,$$

$$K_x' = \bigcup_{\mathscr{C} \in A_x} \tilde{\varphi}_{\mathscr{C}}(B_l(x_0)).$$

Noting that $E(\varphi) = V_M e(\varphi)$ for any $\varphi \in \mathscr{T}(M, N)$, we find

$$\limsup_{x \to \infty} \tfrac{1}{x} \log \# \{\mathscr{C} \subset \mathscr{T}(M, N); E(\mathscr{C}) \leqslant x^2\}$$

$$= \limsup \frac{1}{x} \log \# \{\mathscr{C} \subset \mathscr{T}(M, N); e(\varphi_{\mathscr{C}}) \leqslant x^2 V_M^{-1}\}$$

$$\leqslant \limsup \frac{t}{x} \log \# \{\sigma \in \pi_1(N); d_{\tilde{N}}(q_0, \sigma q_0) \leqslant l V_M^{-1/2} x\}$$

$$\leqslant \limsup \frac{t}{x} \log \text{Volume}\big(B_{l V_M^{-1/2} x}(q_0)\big)$$

$$= l t V_M^{-1/2} \limsup \frac{1}{x} \log \text{Volume}(B_x(q_0)).$$

We can make here use of the result by A. Manning [17] asserting that

$$\lim_{x \to \infty} \frac{1}{x} \log \text{Volume}(B_x(q_0)) = h_N,$$

whence the theorem.

So far we have considered the energy spectrum for the totality of harmonic mappings (or totally geodesic mappings). It is natural to ask what happens if we consider the spectrum for $\varphi \in \mathscr{H}(M, N)$ with a fixed mapping rank. But in general it seems to be difficult even to decide whether there are infinitely many harmonic mappings with high rank. As for totally geodesic mappings we can in fact show a finiteness theorem (see §4). Related with the problem is the following question: If N is a locally symmetric space of non-positive curvature whose rank is r ($\geqslant 1$), then is the following quantity positive?

$$\liminf_{x \to \infty} \frac{1}{x} \log \# \{\mathscr{C} \subset \mathscr{T}(\mathbb{R}^r/\mathbb{Z}^r, N);$$

$$\text{rank of } \varphi \text{ in } \mathscr{C} = r, E(\mathscr{C}) \leqslant x^2\}.$$

§2. Proof of Theorem 2

For a flat torus N, the harmonic maps are identified with n-tuples of harmonic 1-forms with integral periods [3]; therefore counting the components of $\mathscr{H}(M, N)$ reduces to counting the lattice points $\mathrm{Hom}(H^1(N, \mathbf{Z}), H^1(M, \mathbf{Z}))$ in the Euclidean space $\mathrm{Hom}(H^1(N, \mathbf{R}), H^1(M, \mathbf{R}))$. Thus we have

$$\# \{ \mathscr{C}; E(\mathscr{C}) \leqslant x^2 \}$$

$$= \# \{ l \in (H^1(N, \mathbf{Z}))^* \otimes H^1(M, \mathbf{Z}); \|l\|^2 \leqslant x^2 \}$$

$$\sim (V_N)^{-b_1} (V_{A(M)})^{-n} x^{b_1 n} \pi^{b_1 n/2} / \Gamma\left(\frac{b_1 n}{2} + 1 \right)$$

where we have used the facts.

LEMMA 2.1:
(i) For a lattice L in $\mathbf{R}^k$,

$$\# \{ l \in L; \|l\| \leqslant y \} \sim \mathrm{Vol}(R^k/L)^{-1} \pi^{k/2} y^k / \Gamma\left(\frac{k}{2} + 1 \right).$$

(ii) Let L_i be a lattice in $\mathbf{R}^{k_i}$ ($i = 1, 2$). Then

$$\mathrm{Volume}(\mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2}/L_1 \otimes L_2)$$

$$= \mathrm{Volume}(\mathbb{R}^{k_1}/L_1)^{k_2} \mathrm{Volume}(\mathbb{R}^{k_2}/L_2)^{k_1}.$$

PROOF OF (ii): Let $\{e_i\}$, $\{f_\alpha\}$ be bases for L_1 and L_2 respectively. If we form symmetric matrices

$$A_1 = (\langle e_i, e_j \rangle_{ij}), \quad A_2 = (\langle f_\alpha, f_\beta \rangle_{\alpha\beta}),$$

then $(\det A_i)^{1/2} = \mathrm{Volume}(\mathbb{R}^{k_i}/L_i)$, $(\det A_1 \otimes A_2)^{1/2} = \mathrm{Volume}(\mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2}/L_1 \otimes L_2)$. Hence it suffices to verify that $\det(A_1 \otimes A_2) = (\det A_1)^{k_2} (\det A_2)^{k_1}$. But this is immediate since for any orthogonal matrices U_1, U_2,

$$\det(A_1 \otimes A_2) = \det((U_1 \otimes U_2)(A_1 \otimes A_2)(U_1 \otimes U_2)^{-1})$$

$$= \det(U_1 A_1 U_1^{-1} \otimes U_2 A_2 U_2^{-1}),$$

thus the proof reduces to the case A_i are diagonal.

§3. Proof of Theorem 3

We let $\mathscr{P} = \{ p \}$ be the set of all periodic orbits of the geodesic flow on the tangent unit sphere bundle of N. If we identify $\mathscr{P}$ with the set of prime geodesic cycles ($=$ images of closed geodesics), then the period of p is the length of the corresponding geodescis. We set

$$\tilde{\omega}(x) = \#\{ p \in \mathscr{P};\ (\text{period of } p) \leqslant x \}.$$

It is known that if N has strictly negative curvature, then there is a natural one-to-one correspondence

$$\mathscr{P} \times \mathbb{N} \leftrightarrow \{ \mathscr{C} \subset \mathscr{H}(S^1,\, N);\ \text{non null homotopic components}\},$$

where $(p,\, k) \in \mathscr{P} \times \mathbb{N}$ corresponds to the component containing the k-fold covering of the prime geodesic p. According to Parry-Pollicott [29], one can find a positive constant h such that

$$\begin{aligned}
\tilde{\omega}(y) &= \exp(hy)/hy + \tilde{\omega}_1(y) \\
\tilde{\omega}_1(y)\, y/\exp(hy) &\to 0 \qquad \text{as} \quad y \uparrow \infty.
\end{aligned} \tag{3.1}$$

Note that if $\mathscr{C}$ corresponds to $(p,\, k)$, then $E(\mathscr{C}) = k^2(\text{period of } p)^2$. Therefore one has

$$\#\{ \mathscr{C};\ E(\mathscr{C}) \leqslant x^2 \} = 1 + \sum_{k=1}^{[x/\epsilon]} \tilde{\omega}(x/k) \tag{3.2}$$

where $[\cdot]$ is the Gauss symbol, and ϵ is a positive constant such that $\tilde{\omega}(\epsilon) = 0$. Substituting (3.1) into (3.2), we find

$$\#\{ \mathscr{C}:\ E(\mathscr{C}) \leqslant x^2 \}$$

$$= \exp(hx)/hx + (\exp(hx)/x)$$

$$\times \left\{ \exp(-hx/2) \sum_{k=2}^{[x/\epsilon]} k \exp(-hx(k-2)/2k) \right.$$

$$\times (1 + \tilde{\omega}_1(x/k)(x/k)\exp(-hx/k))$$

$$\left. + x(1 + \tilde{\omega}_1(x))\exp(-hx) \right\}$$

$$= \exp(hx)/hx + (\exp(hx)/x)$$

$$\times \{\exp(-hx/2)([x/\epsilon]-1)([x/\epsilon]+2)0(1)+0(1)\}$$

$$= \exp(hx)/hx + 0(1) \exp(hx)/x,$$

which completes the proof.

As a corollary of Theorem 3, we have

PROPOSITION 3.1: *If N is negatively curved, and if* dim $H^1(M, \mathbb{R}) > 0$, *then*

$$\liminf_{x \to \infty} \frac{1}{x} \log \# \{ \mathscr{C} \subset \mathscr{H}(M, N); E(\mathscr{C}) \le x^2 \} > 0.$$

PROOF: From the assumption, one can find a harmonic mapping φ: $M \to S^1$ such that the induced homomorphism φ_*: $\pi_1(M) \to \mathbb{Z}$ is surjective. Note that, for such a φ, the mapping

$$\begin{array}{ccc} C^0(S^1, N) & \to & C^0(M, N) \\ \underset{c}{\cup} & \mapsto & c \overset{\cup}{\circ} \varphi \end{array}$$

yields an injection of the set of components of $C^0(S^1, M)$ into that of $C^0(M, N)$. Further, for any geodesic c: $S^1 \to N$, the composition $c \circ \varphi$: $M \to N$ is harmonic and $E(c \circ \varphi) = E(c) \cdot E(\varphi)$, whence

$$\# \{ \mathscr{C} \subset \mathscr{H}(M, N); E(\mathscr{C}) \le x^2 \}$$

$$\ge \# \{ \mathscr{C} \subset \mathscr{H}(S^1, N); E(\mathscr{C}) \le E(\varphi)^{-1}x^2 \},$$

which, in view of Theorem 3, completes the proof.

§4. Rigidities and finiteness

Let N be a complete Riemannian manifold with non-positive sectional curvature, and let V be a subspace in $T_q N$. We set

$$V^0 = \{ X \in T_q N; \langle R(X, Y)X, Y \rangle = 0 \text{ for any } Y \in V \},$$

which, thanks to non-positivity of curvature, turns out to be a subspace in $T_q N$. We define integers $n(q)$ and $n(N)$ to be

$$n(q) = \max(\dim V; V^0 \ne (0)),$$

$$n(N) = \max_{q \in N} n(q),$$

respectively. Hence, if $\dim W > n(q)$, then $W^0 = (0)$. If N is negatively curved at $q \in N$, then $n(q) = 1$, and if the Ricci curvature is negative definite at q, then $n(q) \leqslant n - 1$.

PROPOSITION 4.1: *Let* $\varphi \colon M \to N$ *be a harmonic mapping. Suppose that there exists a point* $p \in M$ *such that* rank $d\varphi_p > n(\varphi(p))$. *Then* φ *is rigid in the sense that there is no other harmonic mapping homotopic to* φ. *In other words, the connected component of* $\mathscr{H}(M, N)$ *containing* φ *is the singleton* $\{\varphi\}$.

COROLLARY 4.2: *If the Ricci curvature of* N *is negative at some point, and if* $\varphi \colon M \to N$ *is surjective harmonic mapping, then* φ *is rigid.*

Indeed the Sard's theorem asserts that there exists a point $p \in M$ such that rank $d\varphi_p = n$ and $n(\varphi(p)) \leqslant n - 1$.

PROOF OF PROPOSITION 4.1: (Compare the argument in [22]). Given homotopic harmonic mappings φ, $\psi \in \mathscr{H}(M, N)$, we can find a unique section $X \in C^{\infty}(\varphi^{-1}TN)$ satisfying $\psi(p) = \mathrm{Exp}_{\varphi(p)} X(p)$. We set

$$\rho(p) = \| X(p) \|^2.$$

Fixing an orthonormal basis $\{e_1, \cdots, e_m\}$ in $T_p M$, we define surfaces α_j: $(-\epsilon, \epsilon) \times [0, 1] \to N$ by setting

$$\alpha_j(s, t) = \mathrm{Exp}\, tX(\mathrm{Exp}\, se_j), \quad j = 1, \cdots, m.$$

LEMMA 4.3:

$$\tfrac{1}{2}\Delta\rho(p) = \sum_{j=1}^{m} \int_0^1 \left\{ \| \frac{D}{\partial t} \frac{\partial \alpha_j}{\partial s} \|^2 - \left\langle R\left(\frac{\partial \alpha_j}{\partial s}, \frac{\partial \alpha_j}{\partial t} \right) \frac{\partial \alpha_j}{\partial s}, \frac{\partial \alpha_j}{\partial t} \right\rangle \right\}_{(0, t)} dt$$

PROOF. This is an easy consequence of the second variation formula

$$\frac{1}{2} \frac{d^2}{ds^2} \rho(\mathrm{Exp}\, se_j)$$

$$= \int_0^1 \left\{ \| \frac{D}{\partial t} \frac{\partial \alpha_j}{\partial s} \|^2 - \left\langle R\left(\frac{\partial \alpha_j}{\partial s} \frac{\partial \alpha_j}{\partial t} \right) \frac{\partial \alpha_j}{\partial s}, \frac{\partial \alpha_j}{\partial t} \right\rangle \right\}_{(0, t)} dt$$

$$+ \left\langle \frac{D}{\partial s} \frac{\partial \alpha_j}{\partial s}, \frac{\partial \alpha_j}{\partial t} \right\rangle \Big|_{(0,0)}^{(0,1)} - \int_0^1 \left\langle \frac{D}{\partial s} \frac{\partial \alpha_j}{\partial s}, \frac{D}{\partial t} \frac{\partial \alpha_j}{\partial t} \right\rangle_{(0, t)} dt$$

and the equalities

$$\frac{D}{\partial t}\frac{\partial \alpha_j}{\partial t} \equiv 0, \qquad \frac{\partial \alpha_j}{\partial t}\bigg|_{(0,0)} = X(p), \qquad \frac{\partial \alpha_j}{\partial t}\bigg|_{(0,1)} = \frac{d}{dt}\bigg|_{t=1} \mathrm{Exp}\, tX(p),$$

$$\sum_{j=1}^{m} \frac{D}{\partial s}\frac{\partial \alpha_j}{\partial s}\bigg|_{(0,0)} = \text{the trace of } \nabla d\varphi = 0$$

$$\sum_{j=1}^{m} \frac{D}{\partial s}\frac{\partial \alpha_j}{\partial s}\bigg|_{(0,1)} = \text{the trace of } \nabla d\psi = 0.$$

Since N is non-positively curved, the above lemma implies that $\Delta\rho \geqslant 0$, so that ρ is constant, and

$$\left\| \frac{D}{\partial t}\frac{\partial \alpha_j}{\partial s} \right\| \equiv 0,$$

$$\langle R(d\varphi_p(e_j), X(p))d\varphi_p(e_j), X(p)\rangle \equiv 0, \quad j = 1,\cdots, m.$$

From the assumption, there exists a point $p \in M$ such that dim $\mathrm{Im}(d\varphi_p) > n(\varphi(p))$, whence $0 = X(p) \in (\mathrm{Im}\, d\varphi_p)^0$, or $\rho(p) = 0$. This implies $\varphi \equiv \psi$.

Another consequence of the second variation formula is

PROPOSITION 4.4: *Let* $\varphi\colon M \to N$ *be a totally geodesic mapping and* ψ *be a harmonic mapping homotopic to* φ. *Then* ψ *is also totally geodesic.*

PROOF: Let $X \in C^\infty(\varphi^{-1}TN)$ be as in the proof of Proposition 4.1, and let $\gamma(s)$ be a geodesic in M with unit speed. We set

$$\alpha(s, t) = \mathrm{Exp}\, tX(\gamma(s)).$$

In view of the proof of Proposition 4.1, we have

$$\frac{D}{\partial t}\frac{\partial \alpha}{\partial s} \equiv 0, \quad R\left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial s}\right)\frac{\partial \alpha}{\partial t} \equiv 0, \quad (s, t) \in \mathbb{R} \times [0, 1].$$

(In fact, the curvature condition guarantees that if $\langle R(X, Y)X, Y\rangle = 0$, then $R(X, Y)X = 0$.) We wish to prove $D/\partial s\, (\partial \alpha/\partial s) \equiv 0$. Since $D/\partial s\, (\partial \alpha/\partial s)|_{(s,0)} = 0$, it is enough to show that $D/\partial s\, (\partial \alpha/\partial s)$ is parallel along the curve: $t \mapsto \alpha(s, t)$. But this is immediate because

$$\frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial \alpha}{\partial s} = \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial \alpha}{\partial s} - R\left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial s}\right)\frac{\partial \alpha}{\partial s} \equiv 0.$$

A mapping $\varphi: (M, g) \to (N, h)$ is called *weakly conformal* if there is a non-nagative function λ on M such that $\varphi^* h = \lambda g$. The following gives a version of the Franchis theorem (see for related results [12] [14] [20] [23]).

PROPOSITION 4.5: *The set of weakly conformal harmonic mappings φ with rank $\varphi > n(N)$ is finite.*

REMARK. For dim $M > 2$, any weakly conformal harmonic map is homothetic (i.e., has λ constaint).

PROOF. We are in the position to apply the argument in §1 and Proposition 4.1. It suffices to verify the uniform boundedness of the energy density $e(\varphi)$. Let φ be arbitrary weakly conformal harmonic mapping with rank $\varphi > n(N)$, and let $p_0 \in M$ be a point at which $e(\varphi)$ attains the maximum. We may suppose $\lambda(p_0)(\equiv (1/m) e(\varphi)(p_0)) > 0$. Using (1.3), we observe that

$$0 \geqslant \tfrac{1}{2} \Delta e(\varphi)(p_0)$$

$$\geqslant \Sigma \frac{S_M}{m} e(\varphi) - \sum_{ij} \langle R(d\varphi(e_i), d\varphi(e_j) d\varphi(e_i), d\rho(e_j) \rangle,$$

where $S_M = \Sigma^M R^{ij} g_{ij}$ is the scalar curvature, and $\{e_1, \cdots, e_m\}$ is an orthonormal basis of $T_{p_0} M$. If we set $f_i = \lambda^{-1/2} d\varphi(e_i)$, then $h(f_i, f_j) = \delta_{ij}$, and

$$-\frac{1}{m^2} \left(\sum_{ij} \langle R(f_i, f_j) f_i, f_j \rangle \right) e(\varphi)^2 \leqslant -\frac{S_M}{m} e(\varphi).$$

We note here that, in general, the quantity

$$R = \sum_{ij} \langle R(f_i, f_j) f_i, f_j \rangle, \quad (\langle f_i, f_j \rangle = \delta_{ij}).$$

depends only upon the subspace in $T.N$ spanned by $f_1, \cdots, f_m \in T.N$. Further, from the assumption $m > n(N)$, R is negative for any $\{f_1, \cdots, f_m\}$. Since R is continuous and attains the maximum $-c^2 (< 0)$ on the compact manifold consisting of m-dimensional subspaces in $T.N$, we obtain

$$\frac{c^2}{m^2} e(\varphi) \leqslant -\frac{S_M}{m}$$

at p_0, and hence at any point in M. This completes the proof.

COROLLARY 4.6: *Suppose that the universal covering $\tilde{M}$ is irreducible in the de Rham decomposition, and that* dim $M > n(N)$. *Then the set of totally geodesic mappings of M into N is finite.*

This is easy since every $\varphi \in \mathscr{T}(M, N)$ are conformal immersions (see J. Vilms [30]).

REMARK: In conjunction with Royden [21], the argument in this section leads to a finiteness theorem for holomorphic mappings (see [23]). Let M and N be compact Kähler manifolds. Suppose that the sectional curvature of N is non-positive and the holomorphic sectional curvature is bounded from above by a negative constant $-K$. Further assume $\rho_M \leqslant 0$. We denote by $\mathrm{Hol}(M, N)$ the set of holomorphic mappings of M into N. It is a standard fact that any $\varphi \in \mathrm{Hol}(M, N)$ is harmonic, and if a component $\mathscr{C}$ of $\mathscr{H}(M, N)$ contains a holomorphic mapping, then all the mappings in $\mathscr{C}$ are holomorphic. According to [21], any $\varphi \in \mathrm{Hol}(M, N)$ satisfies

$$\|\mathrm{d}\varphi\|^2 \leqslant \frac{2r}{r+1} \frac{|\rho_M|(n-1)}{K},$$

where r is the rank of φ over $\mathbf{C}$. Hence $\#\{\mathscr{C} \in \mathrm{Hol}(M, N)\}$ as well as $\#\{\varphi \in \mathrm{Hol}(M, N);\ \mathrm{rank}_{\mathbf{R}}\varphi > n(N)\}$ can be estimated by a geometric constant (see §1).

§5. Another approach

This section will give another method to count the components of $\mathscr{H}(M, N)$ which is more direct generalization of Maeda [16]. We retain the notations in §1.

Given a positive number x, we choose, a, $b > 0$ so that $0 < z < \min(i_N, i_N/2x)$ and $0 < b \leqslant i_N/2 - ax$. Take points $p_1, \cdots, p_t$ in M and $q_1, \cdots, q_s$ in N such that

$$M = \bigcup_{i=1}^{t} B_a(p_i), \quad N = \bigcup_{j=1}^{s} B_b(q_j).$$

Choosing an element $\varphi_{\mathscr{C}}$ with $\|\mathrm{d}\varphi_{\mathscr{C}}\| \leqslant x$ from each component $\mathscr{C} \subset A_x$, we set $L = \{\varphi_{\mathscr{C}}\}$. We define a mapping $\Phi\colon L \to \mathrm{Map}(T, S)$ as follows. For $\varphi \in L$ and $i \in T$, we define $j(i)$ to be the smallest j such that $\varphi(p_i) \in B_b(q_j)$. Then the mapping $i \to j(i)$ is the $\Phi(\varphi)$. We will show Φ is injective. For this assume $\Phi(\varphi) = \Phi(\psi) = j(\cdot)$. Take arbitrary point $p \in M$, and say $p \in B_a(p_i)$. Then

$$d_N(\varphi(p), \psi(p)) \leqslant d_N(\varphi(p), \varphi(p_i)) + d_N(\varphi(p_i), q_{j(i)})$$

$$+ d_N(q_{j(i)}, \psi(p_i)) + d_N(\psi(p_i), \psi(p))$$

$$\leqslant 2ax + 2b < i_N,$$

whence we can construct a homotopy between φ and ψ by using a unique geodesic joining $\varphi(p)$ and $\psi(p)$. This proves $\varphi \equiv \psi$.

The estimation of $\#A_x$ reduces, after all, to that of the cardinarity of a ball covering.

LEMMA 5.1: *Let X^k be a compact Riemannian manifold, and let $c < i_X$ be a positive number.*

(i) There exist finite number of points $p_1, \cdots, p_t \in X$ such that

$$\bigcup_{i=1}^{t} B_c(p_i) = X, \text{ and } t \leqslant \frac{\int_0^{D_X} \left(\sinh |\rho_X|^{1/2} u\right)^{k-1} \mathrm{d}u}{\int_0^{c/2} \left(\sinh |\rho_X|^{1/2} u\right)^{k-1} \mathrm{d}u}. \quad (5.1)$$

(ii) If the sectional curvature of X is non-positive, then the above (5.1) can be replaced by

$$t \leqslant \frac{k}{\omega_{k-1}} \left(\frac{2}{c}\right)^k V_x.$$

These are easy consequences of the volume comparison theorem and the fact that if $\{ \mathring{B}_{c/2}(p_i) \}_{i=1,\cdots,t}$ is a maximal family of open balls which are mutually disjoint, then $\bigcup B_c(p_i) = X$.

COROLLARY 5.2: *The number of components of isometry group acting on a non-positively curved manifold N is not greater than*

$$t^t, \quad t = \frac{n}{\omega_{n-1}} \left(\frac{8}{i_N}\right)^n V_N.$$

Indeed, in the above argument, we have only to set $x = n$, $a = i_N/4n = b$. See for related topic [28].

References

[1] R.L. BISHOP and R.J. CRITTENDEN: *Geometry of Manifold*, Academic Press: New York and London (1964).

[2] P. BUSER and H. KARCHER: Gromov's almost flat manifolds, *Asterisque* 81 (1981).

[3] J. EELLS and J. SAMPSON: Harmonic mappings of Riemannian manifold, *Amer. J. Math.* 86 (1964) 109–160.

[4] J. EELLS and L. LEMAIRE: A report on harmonic maps, Bull. London Math. *Soc.* 10 (1978) 1–68.

[5] S. GALLOT: Inégalité isoperimetrique sur les variétés compactes sans bord, to appear.

[6] R. GANGOLLI: On the length spectra of certain compact manifolds of negative curvature *J. Diff. Geom.* 12 (1977) 403–424.

[7] M. GROMOV: Structures metriques sur les variétés riemannienes (rédigé par J. Lafontaine et P. Pansu), Cedic, Paris (1982).

[8] P. HARTMAN: On homotopic harmonic maps, *Can. J. Math.* 19 (1967) 673–687.

[9] N. HINGSTON: Equivariant Morse theory and closed geodesics, preprint.