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*Compositio Mathematica*, tome 55, n° 3 (1985), p. 303-311

[http://www.numdam.org/item?id=CM\\_1985\\_\\_55\\_3\\_303\\_0](http://www.numdam.org/item?id=CM_1985__55_3_303_0)

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## SIZES OF QUOTIENT SPACES OF CERTAIN FUNCTION ALGEBRAS ON TOPOLOGICAL SEMIGROUPS

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### Introduction

Let  $S$  be a locally compact topological semigroup,  $C(S)$  the space of all bounded complex-valued continuous functions on  $S$ ,  $LWUC(S)$  the space of all left weakly uniformly continuous functions in  $C(S)$  and  $M_a(S)$  the convolution measure algebra of absolutely continuous bounded complex-valued Radon measures on  $S$ .

When  $S$  is a closed subsemigroup of a locally compact topological group such that  $S$  is neither compact nor discrete, we showed that the quotient space  $C(S)/LWUC(S)$  is nonseparable in [9]. In this paper, we will extend this result to a more general class of topological semigroups.

For a locally compact topological group  $G$ ,  $M_a(G)$  can be identified with the usual group algebra,  $L^1(G)$ , of  $G$ —see e.g. Hewitt and Ross [13]. When  $G$  is nondiscrete it is known that the quotient space  $L^\infty(G)/C(G)$  and the radical of the Banach algebra  $L^\infty(G)^*$  are nonseparable—see E.E. Granirer [10] and S.L. Gulick [12]. Motivated by these results, we will show that for a large class of nondiscrete topological semigroups  $S$  we have  $M_a(S)^*/C(S)$  and the radical of  $M_a(S)^{**}$  nonseparable; the actual setting of our results being more general.

### Definitions and notations

Let  $A$  and  $B$  be any subsets of a semigroup  $S$  and  $x$  any element of  $S$ . We take  $AB$ ,  $A^{-1}B$ ,  $x^{-1}B$  and  $A^{-1}x$  to denote  $\{ab: a \in A \text{ and } b \in B\}$ ,  $\{y \in S: ay \in B \text{ for some } a \in A\}$ ,  $\{x\}^{-1}B$  and  $A^{-1}\{x\}$  (respectively). By symmetry the definitions of  $BA^{-1}$ ,  $Bx^{-1}$  and  $xA^{-1}$  must be clear. By a *right cancellative semigroup* we mean a semigroup  $S$  such that whenever  $yx = zx$  then  $y = z$ , for all  $x$ ,  $y$  and  $z$  in  $S$ .

\* This research was partially supported by a Fulbright Research Fellowship.

Throughout this paper, a semigroup,  $S$ , endowed with a Hausdorff topology with respect to which the semigroup operation  $(x, y) \rightarrow xy$  is a jointly continuous mapping of  $S \times S$  into  $S$ , is called a *topological semigroup*.

Let  $S$  be a locally compact topological semigroup for the remainder of this section. For every function  $f$  in  $C(S)$  and  $x$  in  $S$ , we define the functions  ${}_x f$  and  $f_x$  in  $C(S)$  by

$${}_x f(y) := f(xy) \text{ and } f_x(y) := f(yx) (y \in S).$$

Let

$$LWUC(S) := \{f \in C(S) : \text{the map } x \rightarrow {}_x f \text{ of } S \text{ into } C(S) \text{ is weakly continuous}\},$$

$$WAP(S) := \{f \in C(S) : \text{the set } \{{}_x f : x \in S\} \text{ is relatively weakly compact}\},$$

$$AP(S) := \{f \in C(S) : \text{the set } \{{}_x f : x \in S\} \text{ is relatively norm compact}\}.$$

These spaces of functions have been studied widely - see e.g. [4] and [5]. If  $A$  is a subset of  $C(S)$  and  $E$  of  $S$  we write  $A|_E := \{f|_E : f \in A\}$  where  $f|_E$  denotes the restriction of a function  $f$  to  $E$ .

Let  $M(S)$  be the set of all bounded complex-valued Radon measures on  $S$ . It is well known that  $M(S)$  is a Banach algebra with respect to the usual total variation norm,  $\|\cdot\|$ , and convolution multiplication given by

$$\nu * \mu(E) := \int \mu(x^{-1}E) d\nu(x) = \int \nu(Ex^{-1}) d\mu(x),$$

for all Borel subsets  $E$  of  $S$  and measures  $\nu, \mu$  in  $M(S)$ . For each  $\mu$  in  $M(S)$  and  $x$  in  $S$  we take  $|\mu|$  to be the Radon measure arising from the total variation of  $\mu$  and  $\tilde{x}$  the point mass at  $x$ . Let  $M_a(S) := \{\mu \in M(S) : \text{the maps } x \rightarrow |\mu|(x^{-1}(C)) \text{ and } x \rightarrow |\nu|(Cx^{-1}) \text{ of } S \text{ into } \mathbb{R} \text{ are continuous, for every compact subset } C \text{ of } S\}$

The set  $M_a(S)$  has been studied in many publications; for  $S$ , it plays a role analogous to that of  $L^1(G)$  for a locally compact topological group  $G$  - see e.g. [1], [2], [15] and [16]. In particular, we have the following result proved in [1] and [2]

**THEOREM 1:** *We have  $M_a(S)$  an  $L$ -ideal of  $M(S)$  (–that is:  $M_a(S)$  is a norm-closed subalgebra of  $M(S)$  such that for all  $\mu \in M(S)$  and  $\nu \in M_a(S)$  we have  $\nu^*\mu, \mu^*\nu \in M_a(S)$  and if  $\mu \ll |\nu|$  then  $\mu \in M_a(S)$ ).*

For each  $\mu \in M(S)$ , let  $\text{supp}(\mu) := \{x \in S: \text{if } V \text{ is an open neighbourhood of } x \text{ then } |\mu|(V) > 0\}$ .

Following A.C. and J.W. Baker we say  $S$  is a *foundation semigroup* if  $S$  coincides with the closure of  $U\{\text{supp}(\mu): \mu \in M_a(S)\}$ .

For ease of reference we quote the following result proved by Sleijpen [15].

**THEOREM 2:** *Let  $S$  be a foundation semigroup with identity element 1 and  $S_1 := \{x \in S: 1 \in \text{int}(X^{-1}x \cap xX^{-1}) \text{ whenever } X \text{ is a neighbourhood of } x\}$ . Then  $S_1$  is dense in  $S$  and if  $V$  is an open neighbourhood in  $S$  then  $Vv^{-1}$  is a neighbourhood of 1, for all  $v \in V \cap S_1$ .*

**The main results**

Our next theorem is a generalization, to a larger class of semigroups, of a result we proved before—see [9, Theorem 2.5]. The proof employed contains a mixture of techniques we employed in [9] and those used in the proof of Baker and Butcher [3, Theorem 3]. The proof we give is also much simpler compared with that in [9].

**THEOREM 3:** *Let  $S$  be a normal, locally compact and right cancellative topological semigroup. Suppose  $S$  is neither countably compact nor discrete and  $C^{-1}D$  is compact for all compact subsets  $C$  and  $D$  of  $S$ . Then, for some closed subset  $\bar{X}$  of  $S$  we have that  $(C(S) \setminus LWUC(S))_{|\bar{X}}$  contains a linear isometric copy of  $l^\infty$  and so the quotient space  $C(S)/LWUC(S)$  is nonseparable.*

**PROOF.** Since  $S$  is nondiscrete, we can find a relatively compact infinite set  $\{s_n: n \in \mathbb{N}\}$  in  $S$ . As  $C := cl(\{s_n: n \in \mathbb{N}\})$  is compact, we can choose a sequence  $\{t_n\}$  in  $S$  with no cluster point and such that

$$t_{n+1} \notin \bigcup_{i=1}^n C^{-1}(Ct_i) \quad \text{for all } n \in \mathbb{N} \tag{1}$$

Choose infinite subsequences  $T_k := \{t_{k1}, t_{k2}, \dots\}$  of  $T := \{t_1, t_2, \dots\}$  such that

$$\bigcup_{k=1}^{\infty} T_k = T$$

and

$$T_k \cap T_{k'} = \emptyset$$

if and only if  $k \neq k'$ .

Let  $X_k := \{s_m t_{k_n} : m, n \in \mathbb{N}\}$ ,  $X := \{s_m t_n : m, n \in \mathbb{N}\}$  and note that our construction of the  $T_k$ 's and  $T$  imply

$$(a) \quad \bar{X}_k = \{ct_{k_n} : c \in C \text{ and } n \in \mathbb{N}\} \quad (\text{--see proof of [3, Theorem 3]}),$$

$$(b) \quad \bar{X}_k \cap \bar{X}_{k'} = \emptyset \quad \text{if and only if } k \neq k',$$

$$(c) \quad \bigcup_{k=1}^{\infty} \bar{X}_k = \bar{X}.$$

Next we define the functions  $f_k: \bar{X}_k \rightarrow \mathbb{R}$  by

$$f_k(s_m t_{k_n}) := \begin{cases} 1 & \text{if } m < n \\ -1 & \text{if } m \geq n \end{cases}$$

$$f_k(ct_{k_n}) := -1 \quad \text{if } c \in C \setminus \{s_m : m \in \mathbb{N}\}.$$

then (as similarly shown in [3, page 105],)  $f_k$  is continuous, for all  $k \in \mathbb{N}$ .

Corresponding to each element  $\{d_{k'}\}$  in  $l^\infty$ , let  $F_{(d_{k'})}$  be the function defined on  $\bar{X}$  by

$$F_{(d_{k'})}(x) := d_l f_l(x)$$

if and only if  $x \in \bar{X}_l$  for some  $l \in \mathbb{N}$ .

By items (b) and (c) we have  $F_{(d_{k'})}$  well-defined as a function. From items (b) and (c) we have that each  $\bar{X}_k$  is both closed and open in the space  $\bar{X}$ . Consequently  $F_{(d_{k'})}$  is continuous, by the continuity of the  $f_l$ 's.

Now noting that

$$(*) \quad F_{(d_{k'})}(s_m t_{k_n}) = \begin{cases} d_k & \text{if } m < n \\ -d_k & \text{if } m \geq n, \end{cases}$$

[3, Theorem 5] and Tietze's Extension Theorem imply the existence of a function  $\bar{F}_{(d_{k'})}$  in  $C(S) \setminus LWUC(S)$  such that

$$\bar{F}_{(d_{k'})}|_{\bar{X}} = F_{(d_{k'})} \quad \text{and} \quad \|\bar{F}_{(d_{k'})}\|_S = \|F_{(d_{k'})}\|_{\bar{X}} = \|\{d_{k'}\}\|_\infty.$$

Thus the (clearly) linear map  $\{d_{k'}\} \rightarrow \overline{F}_{(d_{k'})_1 X}$  of  $l^\infty$  into  $(C(S) \setminus LWUC(S))_{|\overline{x}}$  is isometric.

Since  $l^\infty$  is nonseparable, it follows that  $C(S) \setminus LWUC(S)$  and hence the quotient space  $C(S)/LWUC(S)$  is nonseparable.

For our next results, recall that the norm of  $M_a(S)^*$  is given by

$$\|h\|_{M_a(S)^*} := \sup\{|h(\nu)| : \nu \in M_a(S) \text{ with } \|\nu\| = 1\}.$$

For a locally compact topological group  $G$ ,  $M_a(G)^*$  is simply  $L^\infty(G)$ .

**THEOREM 4:** *Let  $S$  be a nondiscrete and right cancellative foundation semigroup with an identity element 1. Then the quotient spaces  $M_a(S)^*/C(S)$  and  $M_a(S)^*/LWUC(S)$  contain isometric linear copies of  $l^\infty$ .*

**PROOF.** Let  $W$  be a compact neighbourhood of 1 and corresponding to each function  $g$  in  $C(S)$  let  $G$  be the function in  $C(W \times W)$  given by

$$G(x, y) := g(xy) \quad \text{for all } x, y \in W.$$

Then a simple compactness argument shows that the set  $\{G(x, \cdot) : x \in W\}$  is relatively (norm and hence) weakly compact in  $C(W)$ . (Here each  $G(x, \cdot)$  is given by  $G(x, \cdot)(y) := G(x, y)$  for all  $x, y$  in  $W$  and  $C(X)$  denotes the space of all bounded complex-valued continuous functions on a topological space  $X$ .)

Since  $S$  is not discrete, 1 is not isolated and so we can find a sequence  $\{V_k\}$  of disjoint open neighbourhoods contained in  $W$ . Choose  $v_k \in V_k \cap S_1$  and note that  $V_k v_k^{-1}$  is a neighbourhood of 1, by Theorem 2. So there is a sequence  $\{U_k\}$  of open neighbourhoods of 1 such that

$$U_k^2 \subset V_k v_k^{-1} \quad \text{for all } k \text{ in } \mathbb{N}.$$

By [8, Lemma 4.2] we can choose sequences  $\{C_{k_1}, C_{k_2}, \dots\}$  and  $\{D_{k_1}, D_{k_2}, \dots\}$  of non- $M_a(S)$ -negligible compact subsets of  $U_k$  such that, for all  $n, m, i$  and  $j$  in  $\mathbb{N}$ ,

$$C_{k_n} D_{k_m} \cap C_{k_i} D_{k_j} = \emptyset \quad \text{whenever } n < m \text{ and } i > j.$$

By right cancellation we have

$$C_{k_n} D_{k_m} v_k \cap C_{k_i} D_{k_j} v_k = \emptyset \quad \text{whenever } n < m \text{ and } i > j. \tag{1}$$

In the notation of [8, page 166], take  $A = M_a(S)$  and set  $d_a := d_{M_a(S)}$ . We can choose sequences of points  $\{c_{k_n}\}$  and  $\{e_{k_n}\}$  such that

$$c_{k_n} \in d_a(C_{k_n}) \quad \text{and} \quad e_{k_n} \in d_a(D_{k_n}v_k). \tag{2}$$

Let

$$E_k := \bigcup_{i=1}^{\infty} \bigcup_{i < j} C_{k_i} D_{k_j} v_k \quad \text{and} \quad F_k := \bigcup_{j=1}^{\infty} \bigcup_{i > j} C_{k_i} D_{k_j} v_k.$$

We define the function  $h_k$  on  $S$  by

$$h_k := X_{E_k} - X_{F_k},$$

where  $X_A$  denotes the characteristic function of a subset  $A$  of  $S$ . Since  $E_k$  and  $F_k$  are disjoint  $\sigma$ -compact subsets of  $S$ , we also have that  $h_k$  is a functional in  $M_a(S)^*$  (where  $h_k(\nu) := \int h_k(x) d\nu(x)$ , for all  $\nu$  in  $M_a(S)$ ).

We claim that, in the norm of  $M_a(S)^*$ ,

$$\|h_k + g\|_{M_a(S)} \geq 1 \quad \text{for all } g \text{ in } C(S) \tag{3}$$

If not, then for some (real-valued) function  $g$  in  $C(S)$  we can find  $\epsilon > 0$  such that

$$\|h_k + g\|_{M_a(S)} \leq 1 - \epsilon.$$

In particular, for positive measures  $\nu_\alpha, \mu_\beta$  in  $M_a(S)$  such that  $\|\nu_\alpha\| = \|\mu_\beta\| = 1$ ,  $\text{supp}(\nu_\alpha) \subseteq C_{k_n}$  and  $\text{supp}(\mu_\beta) \subseteq D_{k_n}v_k$ , we have

$$|h_k(\nu_\alpha^* \mu_\beta) + g(\nu_\alpha^* \mu_\beta)| \leq 1 - \epsilon. \tag{4}$$

Recalling our definition of  $h_k$ , (4) implies that

$$\begin{cases} \text{if } n < m \text{ then } |1 + g(\nu_\alpha^* \mu_\beta)| \leq 1 - \epsilon \\ \text{if } n > m \text{ then } |-1 + g(\nu_\alpha^* \mu_\beta)| \leq 1 - \epsilon. \end{cases}$$

Letting the net  $(\nu_\alpha)$  converge in the weak\*-topology to  $\bar{c}_{k_n}$  and  $(\mu_\beta)$  to  $\bar{e}_{k_m}$  we thus get that (, since  $g$  is continuous on  $W$ ),

$$\begin{cases} \text{if } n < m \text{ then } |1 + g(c_{k_n} e_{k_m})| \leq 1 - \epsilon \\ \text{if } n > m \text{ then } |-1 + g(c_{k_n} e_{k_m})| \leq 1 - \epsilon. \end{cases}$$

It follows that

$$G(c_{k_n}, e_{k_m}) = g(c_{k_n}e_{k_m}) \begin{cases} < -\epsilon & \text{if } n < m \\ > \epsilon & \text{if } n > m \end{cases}$$

and so  $G(x, \cdot): x \in W$  is not relatively weakly compact, by Grothendieck's Theorem [11]. This contradicts the observation at the beginning of our proof. By this conflict, claim (3) holds.

Since the  $V_k$ 's are pairwise disjoint and  $C_{k_n}D_{k_m}v_k \subset V_k$  for all  $n, m$  and  $k$  in  $\mathbb{N}$ , we have that

$$\{t_k\} \rightarrow \sum_{k=1}^{\infty} t_k h_k + C(S)$$

defines a linear mapping of  $l^\infty$  into  $M_a(S)^*/C(S)$ . Noting that  $\|h_k\|_{M_a(S)} = 1$ , item (3) implies that

$$\|\{t_k\}\|_\infty = \|\sum_{k=1}^{\infty} t_k h_k + C(S)\|_{M_a(S)^*/C(S)}$$

and so the mapping  $\{t_k\} \rightarrow \sum_{k=1}^{\infty} t_k h_k + C(S)$  is isometric.

Similarly the mapping  $\{t_k\} \rightarrow \sum_{k=1}^{\infty} t_k h_k + LWUC(S)$  of  $l^\infty$  into  $M_a(S)^*/LWUC(S)$  is linear and isometric. This completes our proof. (The idea of embedding  $l^\infty$  used here is inspired by [6] and [9].)

The second dual of  $M_a(S)$ , namely  $M_a(S)^{**}$ , can be turned into a Banach algebra with Arens product  $\circ$  defined as follows: For  $\phi \in M_a(S)^{**}$ ,  $h \in M_a(S)^*$  and  $\nu \in M_a(S)$  we define  $\nu \circ h$ ,  $h \circ \nu$  and  $h^\phi$  in  $M_a(S)^*$  by

$$\nu \circ h(\mu) := h(\nu * \mu), \quad h \circ \nu(\mu) := h(\mu * \nu) \quad \text{and}$$

$$h^\phi(\mu) := \phi(\mu \circ h) \quad \text{for all } \mu \text{ in } M_a(S).$$

For all  $\phi, \psi$  in  $M_a(S)^{**}$  we have  $\phi \circ \psi$  given by

$$\phi \circ \psi(h) := \phi(h^\psi).$$

$R_a(S)$ , the radical of  $M_a(S)^{**}$ , is the intersection of all maximal modular left (or right) ideals (See [14] page 55).

Let  $G$  be a locally compact topological group. When  $G$  is nondiscrete and abelian, Civin and Yood [7] showed that  $R_a(G)$  is infinite dimensional and later S. Gulick [12] showed that  $R_a(G)$  is even nonseparable.



In [10], Granirer showed that  $R_a(G)$  is nonseparable whenever  $G$  is nondiscrete or  $G$  is discrete and amenable. We generalize the former result to the semigroup situation.

**THEOREM.** *Let  $S$  be a nondiscrete and right cancellative foundation semigroup with an identity element. Then there exists a subspace  $P$  of  $M_a(S)^*$  such that  $P^*$  is a linear isometric copy of  $(l^\infty)^*$  and the restriction of the radical of  $M_a(S)^{**}$  to  $P$  is  $P^*$ . In particular the radical of  $M_a(S)^{**}$  is nonseparable.*

**PROOF.** (c.f. [10] for a related proof in the group case.) Let

$$A := \{ \phi \in M_a(S)^{**} : \phi(f) = 0 \text{ for all } f \text{ in } LWUC(S) \}.$$

For all  $\psi \in M_a(S)^{**}$ ,  $\nu \in M_a(S)$  and  $h \in M_a(S)^*$ , we have that  $\nu \circ h \in LWUC(S)$ , by the left handed version of [9, Lemma 4.1]; consequently  $h^\phi(\nu) := \phi(\nu \circ h) = 0(\phi \in A)$  and so

$$\phi \circ \psi(h) := \psi(h^\phi) = 0(\phi \in A).$$

Thus  $A$  is a right ideal of  $M_a(S)^{**}$  such that

$$A \circ M_a(S)^{**} = \{0\}.$$

Hence  $A \subset R_a(S)$ —see e.g. Richart [14, Theorem 2.3.5(ii)].

Now by Theorem 4, there exists an isometric linear map  $\Pi$  of  $l^\infty M_a(S)^*/LWUC(S)$ . So for some closed subspace  $P$  of  $M_a(S)^*$ , we have  $\Pi(l^\infty)$  dense in  $P/LWUC(S)$ . The inverse map  $\Pi^{-1}$  therefore extends to a unique isometric linear map  $\tau: P/LWUC(S) \rightarrow l^\infty$ . Hence the dual  $\tau^*: (l^\infty)^* \rightarrow (P/LWUC(S))^*$  is an isometric linear map that is onto. But  $A = LWUC(S)^\perp \subset M_a(S)^{**}$  can be identified with  $(M_a(S)^*/LWUC(S))^*$ . Hence each element of  $(P/LWUC(S))^*$  can be identified with the restriction of some element of  $A$  to  $P$ . This completes our proof.

The case for a cancellative discrete topological semigroup  $S$  that is amenable can be similarly handled as in the equivalent group case -see [10, page 323]. To what extent one can drop the right cancellation requirement on  $S$ , in Theorems 4 and 5, remains an open problem.

We proved related results for other spaces of functions in [9]. In particular we showed that if  $S$  is a  $C$ -distinguished topological semigroup such that  $M_a(S)$  is nonzero and  $S$  is not relatively neo-compact, then  $WUC(S)/WAP(S)$  contains an isometric linear copy of  $l^\infty$ . (See [9] for definition of terms.) There seems to be some relationship between sizes of quotient spaces and the existence of continuous projections. We have the following conjecture.

CONJECTURE: *Let  $S$  be as stated in the preceding paragraph. Then there does not exist bounded linear projections from  $WUC(S)$  onto  $WAP(S)$  or from  $WUC(S)$  onto  $AP(S)$ .*

When  $S = \mathbb{R}$ —the usual additive group of reals with usual topology then this conjecture is true—see e.g. [17], we are indebted to Professor W.G. Bade for drawing our attention to the reference [17].

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(Obaltum 11-VIII-1983 & 30-I-1984.)

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