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S. GLASNER

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ABSOLUTELY EXTREMAL POINTS IN MINIMAL FLOWS

S. Glasner

Abstract

A point of a minimal flow is called absolutely extremal if it is an extreme point in every affine embedding. We show that distal points are absolutely extremal and give an example of a weakly mixing minimal flow all of whose points are absolutely extremal.

§1. Introduction

Let (X, T) be a metric flow i.e. X is compact metric and T a homeomorphism of X onto itself. A flow (Q, T) is called *affine* if Q is a convex compact subset of a locally convex linear space and T is an affine homeomorphism of Q. We say that a map $\varphi:(X, T) \to (Q, T)$ is an *affine embedding* if φ is continuous, one to one, equivariant and $\overline{co}(\varphi(X)) = Q$. Thus for any embedding, the set $ex(Q)$ of extreme points of Q is contained in $\varphi(X)$. We often identify X with $\varphi(X)$. If (X, T) is a minimal flow (every orbit is dense), then $(\varphi(X), T)$ is minimal and we have $\overline{ex}(Q) = \varphi(X)$. Call a point $x_0 \in X$ *absolutely extremal* if for every embedding $\varphi: X \to Q$, $\varphi(x_0)$ is an extreme point of Q. (X, T) is an absolutely extremal flow if every point of X is absolutely extremal.

Using this terminology a theorem of *I.* Namioka [3] asserts that a distal minimal flow is absolutely extremal. We simplify the proof of this theorem and generalize it in showing that every distal point of a minimal flow is absolutely extremal. We give examples of almost automorphic absolutely extremal flows (X, T) and (Y, T) such that in the product flow $(X \times Y, T)$ which is minimal and almost automorphic, there are points which are not absolutely extremal. Finally using a result of del Junco and Keane we demonstrate the existence of a weakly mixing minimal absolutely extremal flow.

§2. Distal points are absolutely extremal

For a flow (X, T) denote by $\mathscr{P}(X)$ the space of probability measures on X equipped with the weak* topology. The homeomorphism T of X

induces an affine homeomorphism of $\mathscr{P}(X)$ under which $(\mathscr{P}(X), T)$ is an affine flow. For $x \in X$, let δ_x be the point mass at x. If $\varphi: X \to Q$ is an affine embedding then the barycenter map $\beta: \mathscr{P}(X) \to Q$ is defined by

$$\beta(\lambda) = \int_X \varphi(x) d\lambda(x) \quad (\lambda \in \mathscr{P}(X)).$$

The map β is a continuous affine homomorphism (i.e. an equivariant map) of $\mathscr{P}(X)$ onto Q. A point $q \in Q$ is extremal iff $\beta^{-1}(q) = \{\delta_x\}$ for some $x \in X$ (with $\varphi(x) = q$).

LEMMA 2.1: *Let (X, T) be a metric flow, ν a probability measure on X and assume that in $\mathscr{P}(X)$, $\lim T^{n_i}\nu = \delta_{x_0}$ for some sequence $\{n_i\}$ and a point $x_0 \in X$. Then there exist an F_σ subset A of X and a subsequence $\{n_i'\}$ such that $\nu(A) = 1$ and for every $x \in A$, $\lim T^{n_i'}x = x_0$. In particular every two points of A are proximal.*

PROOF: Let $n_i' \in \{n_j\}$ satisfy $T^{n_i'}\nu(B_{1/i}(x_0)) > 1 - 2^{-(i+1)}$ $(i = 1, 2, \dots)$, where $B_r(x_0)$ is the closed ball of radius r around x_0. Then

$$\nu\left(\bigcap_{i=k}^{\infty} T^{-n_i'}B_{1/i}(x_0)\right) > 1 - 2^{-k}$$

and thus

$$A = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} T^{-n_i'}B_{1/i}(x_0)$$

is the required set.

COROLLARY 2.2: *If (X, T) is a distal flow and $\lim T^{n_i}\nu = \delta_{x_0}$ for some $x_0 \in X$ and $\nu \in \mathscr{P}(X)$, then $\nu = \delta_x$ for some $x \in X$.*

THEOREM 2.3: *Let (X, T) be a metric minimal flow, $x_0 \in X$ a distal point then x_0 is an absolutely extremal point.*

PROOF: Let $\varphi: X \to Q$ be an affine embedding; we identify X with $\varphi(X)$. Let $\beta: \mathscr{P}(X) \to Q$ be the barycenter map. If x_0 is not an extreme point of Q then there exists a measure $\nu \in \mathscr{P}(X)$ with $\beta(\nu) = x_0$ and $\nu \neq \delta_{x_0}$. Taking $1/2(\delta_{x_0} + \nu)$ instead of ν we can assume ν has an atom at x_0. Choose $\bar{x} \in X$ which is extreme in Q; then by minimality there exists a sequence $\{n_i\}$ such that $\lim T^{n_i}x_0 = \bar{x}$. We can also assume that

$\lim T^{n_i}\nu = \bar{\nu}$ exists and then

$$\beta(\bar{\nu}) = \lim \beta(T^{n_i}\nu) = \lim T^{n_i}\beta(\nu) = \lim T^{n_i}x_0 = \bar{x}.$$

Thus $\bar{\nu} = \delta_{\bar{x}}$ and $\lim T^{n_i}\nu = \delta_{\bar{x}}$. By lemma 2.1. there exist an F_σ subset A of X with $\nu(A) = 1$ - hence $x_0 \in A$ - and a subsequence $\{n_l'\}$ such that $\lim T^{n_l'}x = x_0$ for every $x \in A$. In particular every point of A is proximal to x_0. Since x_0 is distal $A = \{x_0\}$ and $\nu = \delta_{x_0}$, a contradiction. Thus x_0 is extreme and the proof is completed.

§3. Almost antomorphic examples

Let $I = [0, 1)$, α an irrational number and (I, R_α) the rotation by α on I. Let $x_0(n) = \text{sgn} \cos(2\pi n\alpha)$; consider x_0 as an element of $\{1, -1\}^{\mathbb{Z}} = \Omega$ and let T be the shift on Ω. Set $X = \bar{\mathcal{O}}(x_0)$, the orbit closure of x_0 in Ω; then (X, T) is an almost automorphic minimal flow. There exists a homomorphism $\pi : (X, T) \to (I, R_\alpha)$ such that for $\xi \in I \setminus E$, where $E = \{1/4 + n\alpha,\ 3/4 + n\alpha : n \in \mathbb{Z}\}$, $\pi^{-1}(\xi) = \{\text{sgn} \cos(2\pi(n\alpha + \xi))\}$ and for $\xi = 1/4 + k\alpha$, $\pi^{-1}(\xi) = \{x^+, x^-\}$ where for $n \neq -k$, $x^+(n) = x^-(n) = \text{sgn} \cos(2\pi(n + k)\alpha)$ and $x^+(-k) = 1$, $x^-(-k) = -1$. Similar situation exists for $\pi^{-1}(\xi)$ where $\xi = 3/4 + n\alpha$. Since every point of $\pi^{-1}(I \setminus E)$ is distal we conclude, by theorem 2.3 that these points are absolutely extremal. However it is clear that the points of the form $x^\pm$ are also absolutely extremal. For if, say $\beta(\nu) = x^+$ in some affine embedding then by the proof of theorem 2.3. ν is supported by the proximal cell of x^+ (i.e. the set $\{y \in X$: y is proximal to $x^+\}$) which in our case is the set $\{x^+, x^-\}$. Clearly than $\nu = \delta_{x^+}$ and x is extremal. We have shown that *every point of X is* absolutely extremal. Next let Y be the orbit closure of y_0 in Ω where $y_0(n) = \text{sgn} \cos(2\pi n\gamma)$ and γ is an irrational number independent of α. Put $z_0(n) = x_0(n) + \frac{1}{10}y_0(n)$ and set $Z = \bar{\mathcal{O}}(x_0) \subset \mathbb{R}^{\mathbb{Z}}$. Then (X, T) and (Y, T) are disjoint minimal flows and the minimal flow $(X \times Y, T)$ is isomorphic to (Z, T). Now $\mathbb{R}^{\mathbb{Z}}$ with its product topology is a locally convex linear space and, denoting $Q = \overline{co}(Z) \subset \mathbb{R}^{\mathbb{Z}}$, we see that the inclusion map of Z into Q is an embedding of (Z, T) into the affine flow (Q, T). Let $x^\pm$, $y^\pm$ be the points in X and Y respectively, which lie over $1/4 \in I$. Then e.g.

$$x^+ + \tfrac{1}{10}y^- = \tfrac{10}{11}\left(x^+ + \tfrac{1}{10}y^+\right) + \tfrac{1}{11}\left(x^- + \tfrac{1}{10}y^-\right)$$

and we conclude that although x^+ and y^- are absolutely extremal, the point (x^+, y^-) of the minimal flow $X \times Y$ is not absolutely extremal. Thus *the property of being an absolutely extremal point is not preserved under products*.

Taking $x_0(n) = f(\cos 2\pi n\alpha)$ where f is continuous on $[-1, 1]$ except for at zero where it has say, $[0, 1]$ as a limit set, the embedding

$X = \bar{\mathcal{O}}(x_0) \to \overline{co}(X) \subset \mathbb{R}^{\mathbb{Z}}$ will yield a continuum of non absolutely extremal points of X.

If (X, T) is an almost automorphic minimal flow, there exist an almost periodic flow (Y, T) and an almost one to one homomorphism $\pi: X \to Y$. Clearly every pair $(x, x') \in X \times X$ with $\pi(x) = \pi(x')$ has the property that the only minimal set in $\mathcal{O}(x, x')$ is the diagonal Δ. For a general minimal flow (X, T) put

$$L = \{(x, x') \in X \times X \colon \Delta \text{ is the unique minimal set in } \bar{\mathcal{O}}(x, x')\}.$$

L is an invariant equivalence relation but not necessarily closed. We shall use the following lemma in the next section.

LEMMA 3.1: *Let (X, T) be a minimal flow and $\varphi: X \to Q$ an affine embedding. Suppose $\beta(\nu) = \varphi(x)$ for some $\nu \in \mathcal{P}(X)$ and $x \in X$. If y is an atom of ν then $(x, y) \in L$.*

PROOF: Since $\beta(1/2(\nu + \delta_x)) = \varphi(x)$ we can assume that x itself is an atom of ν. Write $\nu = a\delta_x + b\delta_y + (1 - (a + b))\theta$ where $0 < a,\ b < 1$ and $\theta \in \mathcal{P}(X)$. If $M \subset \bar{\mathcal{O}}(x, y)$ is a minimal set then, since X is minimal, there is a point $(z, w) \in M$ with $\varphi(z) \in ex(Q)$. Let $\{n_i\}$ be a sequence with $\lim T^{n_i}(x, y) = (z, w)$ and we can assume that $\lim T^{n_i}\theta = \tilde{\theta}$ and $\lim T^{n_i}\nu = \tilde{\nu}$ exist. Then

$$\beta(\tilde{\nu}) = a\beta(\delta_z) + b\beta(\delta_w) + (1 - (a + b)\beta(\tilde{\theta}) = \varphi(z),$$

whence $\delta_z = \delta_w = \tilde{\theta}$. Thus $M = \Delta$ and $(x, y) \in L$.

§4. A weakly mixing example

In this section we demonstrate the existence of a minimal absolutely extremal weakly mixing flow. Recall that a minimal flow (X, T) is P.O.D. if it is weakly mixing and for every $x,\ y \in X,\ x \neq y$ there exists some $n \neq 0$ with $T^n y$ and x proximal. Every P.O.D flow is prime [1]. Let (X, T) be P.O.D. and suppose $\varphi: X \to Q$ is an affine embedding of X and assume $X \subset Q$. Let $\bar{\nu} \in \mathcal{P}(X)$ and $x_0 \in X$ with $\beta(\bar{\nu}) = x_0$. If y is an atom of $\bar{\nu}$ then by lemma 3.1. $(x_0, y) \in L$. However it is easy to see that in a P.O.D. flow $L = \Delta$, so that $x_0 = y$ and $\bar{\nu}$ can have an atom only at x_0. If $\bar{\nu} \neq \delta_{x_0}$ then there exists a measure $\nu \in \mathcal{P}(X)$ which is continuous (has no atoms) and for which $\beta(\nu) = x_0$.

Suppose further now that (X, T) is also strictly ergodic with an invariant measure μ and that for every $x \in X$ the set,

$$F_x = \{y \in X \colon (x, y) \text{ is not generic for } \mu \times \mu\},$$

is countable. We show that every point of X is absolutely extremal.

That such a flow exists is a result of A. del Junco and M. Keane [4]. They showed that the Chacon transformation which is P.O.D. [2] and strictly ergodic, has also the latter property. Set

$$F = \{(x, y) : (x, y) \text{ is not generic for } \mu \times \mu\},$$

then

$$\nu \times \nu(F) = \int 1_F \mathrm{d}\nu \times \nu = \iint 1_{F_x}(y)\mathrm{d}\nu(y)\mathrm{d}\nu(x) = 0,$$

since ν is continuous and for every x, F_x is countable.

If $f(x, y)$ is a continuous function on $X \times X$ then for $(x, y) \notin F$ we have

$$\frac{1}{2N+1} \sum_{j=-N}^{N} f(T^j x, T^j y) \to \int f \mathrm{d}\mu \times \mu.$$

In particular this convergence holds $\nu \times \nu$ a.e. and integrating we get

$$\frac{1}{2N+1} \sum_{j=-N}^{N} \iint f(T^j x, T^j y)\mathrm{d}\nu(x)\mathrm{d}\nu(y) \to \int f \mathrm{d}\mu \times \mu.$$

Now let g be a continuous affine function on X (i.e. g is the restriction of an affine function on Q). Then for $f(x, y) = g(x)g(y)$ we have

$$\left(\int g d\mu\right)^2 \leftarrow \frac{1}{2N+1} \sum_{j=-N}^{N} \iint g(T^j x) g(T^j y)\mathrm{d}\nu(x)\mathrm{d}\nu(y)$$

$$= \frac{1}{2N+1} \sum_{j=-N}^{N} \left(\int g(T^j x)\mathrm{d}\nu(x)\right)^2.$$

Since g is affine so is $g \circ T^j$ and recalling our assumption that $\beta(\nu) = x_0$ we see that $\int g(T^j x)\mathrm{d}\nu(x) = g(T^j x_0)$. Thus the right hand side of the above equation equals $1/(2N+1)\sum_{j=-N}^{N}(g(T^j x_0))^2$. By strict ergodicity this tends to $\int g^2 \mathrm{d}\mu$ so that

$$\int g^2 \mathrm{d}\mu = \left(\int g \mathrm{d}\mu\right)^2.$$

Choosing $g \neq 0$ with $\int g \mathrm{d}\mu = 0$ we get a contradiction. Thus x_0 is extremal in Q and the proof is completed.

PROBLEMS:

(1) Is there a minimal flow no point of which is absolutely extremal?
(2) Is every minimal flow with $L = \Delta$, absolutely extremal?
(3) Is the homomorphic image of an absolutely extremal point an absolutely extremal point?

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School of Mathematics
Tel-Aviv University
Tel-Aviv
Israel

Added in proof: This problem has been solved by D. Maon and S. Glasner; see "On absolutely extremal points", to appear in this journal.