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SPHERICAL FUNCTIONS AND SPECTRAL SYNTHESIS

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Abstract

Let G be a noncompact connected semisimple real Lie group with finite centre and a maximal compact subgroup K. Suppose further that G/K has rank equal to one and dimension greater than two. Fix a polar decomposition $G = KAK$. We show that for every $a \in A$, $a \neq 1$, the double coset KaK is not a set of synthesis for the Fourier algebra of G. This is a consequence of a local regularity property of inverse Jacobi transforms, similar to the more familiar behaviour of Hankel transforms, and is a noncompact group version of a result of Franco Cazzaniga and myself concerning Jacobi polynomials.

Combining the above result with the rank-one reduction enables us to exhibit sets of nonsynthesis for some other noncompact semisimple Lie groups. A similar device applies to Cartan motion groups associated with Cartan decompositions of these groups.

Finally, using formulae of Koornwinder, Berezin and Karpelevič, we obtain a local regularity property for the bi-$S(U(n)\times U(n+k))$-invariant elements of the Fourier algebra of $SU(n, n+k)$.

1. The Fourier algebra and Gel'fand pairs

In this section we recall the notion of spectral synthesis for the Fourier algebra of a locally compact group and outline a general procedure which yields sets of nonsynthesis for certain groups.

Suppose that G is a unimodular locally compact group with a fixed Haar measure. Eymard [8] defined the *Fourier algebra* $A(G)$ to be equal to $L^2(G) * L^2(G)$. The norm of an element $f \in A(G)$ is the infimum of the products $\|\psi_1\|_2 \cdot \|\psi_2\|_2$, taken over all those ψ_1, $\psi_2 \in L^2(G)$ with $f = \psi_1 * \psi_2$. Every closed subset $E \subset G$ gives rise to two ideals in $A(G)$, namely, $I(E) := \{f \in A(G) : f(x) = 0, \ \forall x \in E\}$ and $J(E) := \{f \in A(G) : f$ is zero on a neighbourhood of $E\}$. The subset E is said to be a *set of synthesis* for $A(G)$ if $I(E)$ is equal to the closure of $J(E)$ in $A(G)$. For examples of sets of synthesis see [8]. Chapitre 4, and [22], Propositions 1 and 2.

From now on we limit our attention to *Gel'fand pairs*. That is, we assume that G has a compact subgroup K for which ${}^{K}L^1(G)^{K}$, the bi-K-invariant elements of $L^1(G)$, is a commutative algebra when equipped with convolution. Let m_K denote the normalized Haar measure on K and for each continuous function f on G set

$$Pf := m_K * f * m_K, \tag{1.1}$$

so that Pf is bi-K-invariant. In particular, from the definition of the $A(G)$-norm, we see that if $f \in A(G)$ then $Pf \in A(G)$ and

$$\| Pf \|_A \leqslant \| f \|_A. \tag{1.2}$$

We denote by ${}^K A(G)^K$ the subalgebra of bi-K-invariant elements of $A(G)$.

Mizony [28], Proposition 1.2.10, has described ${}^K A(G)^K$ in terms of the inverse spherical transform. See also [26], Lemma 3. If $\boldsymbol{Z}$ denotes the set of zonal spherical functions for the pair (G, K) and if $\boldsymbol{Z}$ is equipped with the Godement-Plancherel measure ν then there is an isometric isomorphism

$${}^K A(G)^K \cong L^1(\boldsymbol{Z}, \nu). \tag{1.3}$$

This isomorphism is given by the inverse spherical transform. If $f \in {}^K A(G)^K$ then there is a unique element $\hat{f} \in L^1(\boldsymbol{Z}, \nu)$ such that

$$f(x) = \int_{\boldsymbol{Z}} \hat{f}(\varphi)\varphi(x)\mathrm{d}\nu(\varphi), \quad \forall x \in G, \tag{1.4}$$

and

$$\| f \|_{A(G)} = \int_{\boldsymbol{Z}} |\hat{f}| \mathrm{d}\nu. \tag{1.5}$$

This also states that ${}^K A(G)^K$ can be identified with the Fourier algebra of the commutative hypergroup $K \backslash G/K$, see [4], section 2, and [26]. Chilana and Ross [4], section 4, have shown that several commutative hypergroups have the property that their Fourier algebras possess bounded point derivations and these lead to examples of sets of non-synthesis for the *hypergroups* involved. Here we consider how a similar strategy can be applied to $A(G)$.

Recall that a *bounded point derivation* on ${}^K A(G)^K$ *at a point* $x_0 \in G$ is a bounded linear functional δ on ${}^K A(G)^K$ such that

$$\delta(f \cdot g) = \delta(f)g(x_0) + f(x_0)\delta(g), \tag{1.6}$$

for all f, $g \in {}^K A(G)^K$.

1.7. PROPOSITION: *For each $x_0 \in G$ and a neighbourhood U of Kx_0K in G there exists a neighbourhood V of x_0 in G such that $KVK \subset U$.*

PROOF: The map $K \times G \times K \to G$, given by $(k, x, k') \mapsto kxk'$, is continuous and so the inverse image of U is a neighbourhood of $K \times \{x_0\} \times K$ in $K \times G \times K$. The compactness of K and the nature of the product topology imply the statement. Q.E.D.

Combining this with (1.1) we see that P preserves the ideal $J(Kx_0K)$.

1.8. COROLLARY: *If $x_0 \in G$ and $f \in J(Kx_0K)$ then $Pf \in J(Kx_0K)$.*

Now fix $x_0 \in G$, assume that G is not discrete, and suppose that there is a bounded point derivation δ for ${}^KA(G)^K$ at x_0. We wish to show

$$\delta(Pf) = 0, \quad \forall f \in J(Kx_0K). \tag{1.9}$$

On account of Proposition 1.7 we know that if $f \in J(Kx_0K)$ then there exists an open neighbourhood V of x_0 such that both f and Pf are zero on KVK. Let W be a neighbourhood of x_0 such that $\overline{W}$ is compact and $\overline{W} \subset V$. Since $A(G)$ is a regular tauberian algebra of functions on G [22] there exists $g \in A(G)$ such that $g(x) = 1$, $\forall x \in K\overline{W}K$, and $g(x) = 0$, $\forall x \notin KVK$. The same is true for Pg. In particular, $f \cdot Pg = 0$ and

$$\begin{aligned} \delta(P(f \cdot Pg)) &= \delta((Pf) \cdot (Pg)) \\ &= \delta(Pf) \cdot Pg(x_0) + Pf(x_0) \cdot \delta(Pg) \\ &= \delta(Pf). \end{aligned}$$

This proves (1.9).

If $I(Kx_0K) \cap {}^KA(G)^K$ is not contained in the kernel of δ then Kx_0K is not a set of synthesis for $A(G)$. To see this, note that the kernel of the composition $\delta \circ P$ is a closed linear subspace of $A(G)$ which contains $J(Kx_0K)$. We summarize.

1.10. PROPOSITION: *Let (G, K) be a Gel'fand pair such that G is not discrete. Suppose there exists a bounded point derivation δ for ${}^KA(G)^K$ at some point $x_0 \in G$. If there exists $f \in I(Kx_0K)$ for which $\delta(Pf) \neq 0$ then Kx_0K is not a set of synthesis for $A(G)$.*

This procedure has already been successful in several cases.

1.11. EXAMPLES: (i) Let $G = \mathbb{R}^n \rtimes SO(n)$ and $K = \{0\} \times SO(n)$, with $n \geqslant 3$. Then ${}^KA(G)^K$ is equal to the algebra of radial Fourier transforms.

From [31], page 822, we conclude that for every nonzero $\xi \in \mathbb{R}^n$ the coset $K(\xi, 1)K \cong S^{n-1} \times K$ is not a set of synthesis for the Fourier algebra of the Euclidean motion group (L. Schwartz' theorem).

(ii) Let U be a compact connected semisimple Lie group. For $G = U \times U$ and $K = \{(u, u) : u \in U\}$, ${}^K A(G)^K$ is the subalgebra of central functions in $A(U)$. See [27,30]. Similarly, one can demonstrate the failure of synthesis for the motion group $\mathfrak{u} \rtimes_{Ad} U$, where $\mathfrak{u}$ is the Lie algebra of U, see [27] and [5], section 7.

(iii) Let G/K be a compact rank one Riemannian symmetric space. Then ${}^K A(G)^K$ can be identified with a certain algebra of absolutely convergent series of Jacobi polynomials. If G/K is of dimension greater than two then [3] there exist bounded point derivations for ${}^K A(G)^K$.

In the next section we prove an analogue of [3], Theorem 4.8, for *noncompact* rank one Riemannian symmetric spaces.

1.12. Remark: The existence of a system of bounded derivations at one point can sometimes be used to produce chains of ideals between $I(Kx_0K)$ and $\overline{J(Kx_0K)}$, see [4], and [24].

2. A local property of inverse Jacobi transforms

In 1938 I.J. Schoenberg [31], page 822, proved that Hankel transforms have certain differentiability properties, depending on the order of the Bessel function involved. This property is the key to proving that S^{n-1} is not a set of synthesis for $A(\mathbb{R}^n)$, $n \geqslant 3$, and example 1.11(i). Subsequently, algebras of Hankel transforms were studied by M. Gatesoupe [15] and A. Schwartz [32,33]. R.J. Stanton and P.A. Tomas [34] have shown that the zonal spherical functions for noncompact rank one symmetric spaces (i.e. special cases of Jacobi functions) have asymptotic behaviour similar to Bessel functions, up to multiplication by a certain function. Hence we should expect an analogue of Schoenberg's result for inverse Jacobi transforms. In this section such a result is proved.

We begin by sketching some properties of Jacobi functions, due to M. Flensted-Jensen and T. Koornwinder. For details see [10,11,12,16,18,25,29].

Fix real numbers $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. For each $\lambda \geqslant 0$ the *Jacobi function* φ_λ of *order* (α, β) is defined by

$$\varphi_\lambda(t) := F\big((\alpha+\beta+1+i\lambda)/2,\quad (\alpha+\beta+1-i\lambda)/2;\ \alpha+1;\ -(\sinh t)^2\big)$$

for all $t \geqslant 0$. Here $F(,,;;)$ is the usual hypergeometric function, [36] Chapter 14. We also define

$$\Delta(t) := 2^{2(\alpha+\beta+1)}(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}, \quad \forall t \geqslant 0,$$

and

$$c(\lambda) := \frac{2^{\alpha+\beta+1-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma((\alpha+\beta+1+i\lambda)/2)\Gamma((\alpha-\beta+1+i\lambda)/2)},$$

$$\forall \lambda \in \mathbb{R}\setminus\{0\}.$$

These provide densities for two measures on $[0, \infty)$, namely,

$$d\mu(t) := \Delta(t)dt \text{ and } d\nu(\lambda) := (2\pi)^{-1}|c(\lambda)|^{-2}d\lambda.$$

Note that [10] lemma 11,

$$|\varphi_\lambda(t)| \leqslant 1 = \varphi_\lambda(0), \quad \forall t, \lambda \geqslant 0. \tag{2.1}$$

The *Jacobi transform* is the map of $L^1(\mu)$ into $C_0([0, \infty))$ defined by

$$\mathscr{F}f(\lambda) := \int_0^\infty f\varphi_\lambda d\mu,$$

for all $f \in L^1(\mu)$ and $\lambda \geqslant 0$. It is a fact that $\mathscr{F}$ extends from $L^1 \cap L^2(\mu)$ to provide an isometric isomorphism between $L^2(\mu)$ and $L^2(\nu)$, so that ν is the Plancherel measure for $L^2(\mu)$. The *inverse Jacobi transform* is

$$\mathscr{F}^{-1}g(t) := \int_0^\infty g(\lambda)\varphi_\lambda(t)d\nu(\lambda), \tag{2.2}$$

for all $g \in L^1(\nu)$ and $t \geqslant 0$. Note that the case $\alpha = \beta = -\frac{1}{2}$ is the usual cosine transform.

2.3. Definition: *For α, β, ν and $\mathscr{F}^{-1}$ as above we let*

$$A(\alpha, \beta) := \mathscr{F}^{-1}L^1(\nu).$$

In particular, $A(\alpha, \beta) \subset C_0([0, \infty))$. If $f = \mathscr{F}^{-1}g$ for some $g \in L^1(\nu)$ we define the norm of f to be

$$\|f\|_{(\alpha,\beta)} := \int_0^\infty |g| d\nu$$

and set $\mathscr{F}f$ to be equal to g.

From [10] Theorem 4, we see that if f is an *even* element of $C_c^\infty(\mathbb{R})$ then $f|_{[0,\infty)} \in A(\alpha, \beta)$. Flensted-Jensen and Koornwinder [12], Corollary 4.6, have shown that $A(\alpha, \beta)$ is a Banach algebra of continuous functions on $[0, \infty)$. In fact, $A(\alpha, \beta)$ is the Fourier algebra of the hypergroup $[0, \infty)$, when $L^1(\mu)$ is equipped with the convolution described in [11].

We wish to show that if $\alpha \geqslant \frac{1}{2}$ and $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ then elements of $A(\alpha, \beta)$ are differentiable on $(0, \infty)$. To prove this we employ the asymptotic properties of c and the description of φ_λ in terms of *Jacobi functions of the second kind*.

2.4. DEFINITION: *For $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ fixed, $\lambda \in \mathbb{R}$ and $t > 0$ set $\Phi_\lambda(t)$ to be equal to*

$$(e^t - e^{-t})^{i\lambda - \alpha - \beta - 1} F((\beta + 1 - \alpha - i\lambda)/2,$$

$$(\alpha + \beta + 1 - i\lambda)/2; 1 - i\lambda; -(\sinh t)^{-2}).$$

It is known [25], equation (2.5), that

$$\varphi_\lambda(t) = c(\lambda)\Phi_\lambda(t) + c(-\lambda)\Phi_{-\lambda}(t), \tag{2.5}$$

for all $\lambda > 0$ and $t > 0$. The following result is due to Flensted-Jensen, [10] Theorem 2. Here we continue with fixed $\alpha \geqslant \beta \geqslant -\frac{1}{2}$.

2.6. LEMMA:

(a) *For each $\epsilon > 0$ and $n \geqslant 0$ there exists a positive constant $K_n(\epsilon)$ such that*

$$\Phi_\lambda(t) = e^{(i\lambda - \alpha - \beta - 1)t}(1 + e^{-2t}\theta(\lambda, t))$$

and $|(\partial/\partial t)^n \theta(\lambda, t)| \leqslant K_n(\epsilon)$ uniformly in $t \in [\epsilon, \infty)$ and $\lambda \in \mathbb{R}$.

(b) *There exists a constant $k > 0$ such that*

$$|\lambda c(\lambda)| \leqslant k(1 + |\lambda|)^{(1/2) - \alpha}$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$.

Combining this with (2.5) we can estimate $\varphi_\lambda^{(n)}(t)$. Let us put

$$E_\lambda(t) = 1 + e^{-2t}\theta(\lambda, t)$$

for $\lambda \in \mathbb{R}$ and $t > 0$. From lemma 2.6(a) we see that for each $\epsilon > 0$ and $n \geqslant 0$,

$$\left| E_\lambda^{(n)}(t) \right| \leqslant \text{const.}_{\epsilon, n} \tag{2.7}$$

uniformly in $\epsilon \leqslant t < \infty$ and $\lambda \in \mathbb{R}$. Furthermore,

$$\varphi_\lambda^{(n)}(t) = \sum_{s=+1,-1} \sum_{l=0}^{n} c(s\lambda)(is\lambda - \alpha - \beta - 1)^l \times e^{(is\lambda-\alpha-\beta-1)t} E_{s\lambda}^{(n-l)}(t)\binom{n}{l},$$

and so for $\epsilon \leqslant t < \infty$ and $\lambda > 0$ we see that

$$\left|\varphi_\lambda^{(n)}(t)\right| \leqslant \text{const.}_{\epsilon,n} \lambda^{-1}(1+\lambda)^{n+(1/2)-\alpha} e^{-(\alpha+\beta+1)t}. \tag{2.8}$$

This is the *key estimate* in this paper.

For small values of λ we can use [10] Theorem 2(ia), so as to avoid the λ^{-1} term.

2.9. Lemma: *For each $n \geqslant 0$ there exists $K_n > 0$ such that*

$$\left|\varphi_\lambda^{(n)}(t)\right| \leqslant K_n(1+\lambda)^n(1+t)\, \mathrm{e}^{-(\alpha+\beta+1)t}$$

for all $\lambda \geqslant 0$ and $t \geqslant 0$.

Note that (2.8) is a refinement of this estimate when $\lambda \geqslant 1$. In particular, if $0 \leqslant n \leqslant [\alpha + \frac{1}{2}]$ and $t \neq 0$ then the function $\lambda \mapsto \varphi_\lambda^{(n)}(t)$ is ***uniformly bounded*** on $[0, \infty)$. This shows that if $\alpha \geqslant \frac{1}{2}$ then we can differentiate (2.2), as long as $t \neq 0$.

Fix $g \in L^1(\nu)$ and set $f = \mathscr{F}^{-1}g$. For $\epsilon > 0$ and $t_0 > \epsilon$ we see that

$$\begin{aligned}
&\lim_{t \to t_0} (f(t) - f(t_0))/(t - t_0) \\
&\quad = \lim_{t \to t_0} \int_0^\infty g(\lambda)(\varphi_\lambda(t) - \varphi_\lambda(t_0))/(t - t_0)\mathrm{d}\nu(\lambda) \\
&\quad = \int_0^\infty g(\lambda)\varphi'_\lambda(t_0)\mathrm{d}\nu(\lambda) \quad \text{(dominated convergence)}
\end{aligned}$$

$$\text{and so } |f'(t_0)| \leqslant \text{const. } t_0\, \mathrm{e}^{-(\alpha+\beta+1)t_0}\|g\|_1. \tag{2.10}$$

We can repeat this $[\alpha + \frac{1}{2}]$ times.

2.11. Theorem: *For $\alpha \geqslant \frac{1}{2}$, $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and $\epsilon > 0$ there is a constant $k > 0$ such that if $f \in A(\alpha, \beta)$ then $f|_{[\epsilon,\infty)} \in C^{[\alpha+1/2]}([\epsilon, \infty))$ and*

$$\sup_{t \geqslant \epsilon} |f^{(j)}(t)| \leqslant k\|f\|_{(\alpha,\beta)}, \quad 0 \leqslant j \leqslant [\alpha + \tfrac{1}{2}].$$

Compare this with [3], Theorem 2.9, and [15].

2.12. REMARK: An alternative method for obtaining (2.8) is to differentiate the integral (2.21) in [25].

2.13. COROLLARY: *If* $\alpha \geqslant \frac{1}{2}$, $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ *and* E *is a nonempty finite subset of* $(0, \infty)$ *or a sequence of positive numbers with no finite accumulation points then* E *is not a set of synthesis for* $A(\alpha, \beta)$.

PROOF: On account of Theorem 2.11 we see that if $x \in E$ then $f \mapsto f'(x)$ is a bounded point derivation for $A(\alpha, \beta)$. It remains then to observe that for each $x \in E$ there exists $f \in C_c^\infty((0, \infty))$ with $f|_E = 0$ and $f'(x) \neq 0$. Q.E.D.

Suppose that G/K is a noncompact rank one Riemannian symmetric space of dimension d. For $\alpha = (d-2)/2$ and a certain $-\frac{1}{2} \leqslant \beta \leqslant \alpha$, determined by the geometry of G/K, it is known [34] that the subset of zonal spherical functions for the Gel'fand pair (G, K) which form the support of the Godement-Plancherel measure can be identified with the Jacobi functions of order (α, β). Under this identification the Godement-Plancherel measure corresponds to $|c(\lambda)|^{-2} d\lambda$ on $[0, \infty)$, up to normalization, and so we have an isomorphism of Banach algebras

$$ {}^K A(G)^K \cong A(\alpha, \beta). \tag{2.14} $$

Theorem 2.11 then tells us when it is possible to equip ${}^K A(G)^K$ with bounded point derivations.

2.15. THEOREM: *Let* G *be a noncompact connected semisimple Lie group with finite centre and a fixed maximal compact subgroup* K. *Suppose that* G/K *is a* d*-dimensional rank one Riemannian symmetric space and let* G *have an Iwasawa decomposition* $G = KAN$. *Fix a nonzero element* H *in the Lie algebra of* A. *If* $d \geqslant 3$ *and* $\epsilon > 0$ *then there is a constant* $k > 0$ *such that for each* $f \in {}^K A(G)^K$ *the function*

$$ t \mapsto f(\exp(tH)) $$

is $[(d-1)/2]$*-times differentiable and*

$$ \sup_{t \geqslant \epsilon} \left| (d/dt)^n f(\exp(tH)) \right| \leqslant k \| f \|_{A(G)} $$

for $0 \leqslant n \leqslant [(d-1)/2]$.

2.16. COROLLARY: *For* G, K, A, *and* H *as above, if* $\dim(G/K) \geqslant 3$ *and* $t_0 > 0$ *then the double coset* $K \exp(t_0 H) K$ *is not a set of synthesis for* $A(G)$.

PROOF: On account of Proposition 1.10 we need only to remark that there exists $f \in {}^K C_c^\infty(G)^K \subset A(G)$ with $f(\exp(t_0 H)) = 0$ and $(\mathrm{d}/\mathrm{d}t) f(\exp(t_0 H)) \neq 0$.

2.17. REMARKS: (a) The referee has suggested the following alternative proof of Theorem 2.11, based on (3.7) and (3.13) in [25]. Fix $\alpha \geqslant \beta \geqslant -1/2$ and let $F_{\alpha,\beta}$ and $\mathscr{W}_\mu^\sigma$ be the operators defined on pages 152-3 of [25]. In addition, fix a compact interval $[a, b] \subset (0, \infty)$ and $\psi \in C_c^\infty((0, \infty))$ such that $\psi(t) = 1$ if $a < t < b$ and $\psi(t) = 0$ if $t > 2b$. For every $f \in A(\alpha, \beta)$ we know that $\|\psi \cdot f\|_{(\alpha,\beta)} \leqslant \mathrm{const} \cdot \|f\|_{(\alpha,\beta)}$ and that $F_{\alpha,\beta}(\psi \cdot f)$ is continuous and has support in [0, 2b]. Koornwinder has shown that

$$\mathscr{F}(\psi \cdot f)(\lambda) = \pi^{-1/2} \Gamma(\alpha + 1) \int_0^\infty F_{\alpha,\beta}(\psi \cdot f)(s) \cdot \cos(\lambda s) \mathrm{d}s$$

and so the cosine transform of $F_{\alpha,\beta}(\psi \cdot f)$ is integrable with respect to the measure $|c(\lambda)|^{-2} \mathrm{d}\lambda$. Lemma 2.6(b) tells us that if $\alpha \geqslant 1/2$ and $2 \leqslant \mu \leqslant 2\alpha + 1$ then

$$\int_0^\infty |\mathscr{F}(\psi \cdot f)(\lambda)| \cdot \lambda^\mu \mathrm{d}\lambda \leqslant \mathrm{const}. \|f\|_{(\alpha,\beta)}. \tag{2.18}$$

From this we conclude that $F_{\alpha,\beta}(\psi \cdot f)$ is of class $C^{[2\alpha+1]}$ on $[0, \infty)$ and for $0 \leqslant k \leqslant [2\alpha + 1]$,

$$\sup_{t \geqslant 0} \left| (\mathrm{d}/\mathrm{d}t)^k F_{\alpha,\beta}(\psi \cdot f)(t) \right| \leqslant \mathrm{const} \cdot \|f\|_{(\alpha,\beta)}. \tag{2.19}$$

Note that we cannot claim this if $\alpha < 1/2$.

It remains to apply equation (3.13) of [25]. For every $f \in A(\alpha, \beta)$ and $a \leqslant x \leqslant b$,

$$f(x) = 2^{-3\alpha - (3/2)} \mathscr{W}^2_{-\alpha-(1/2)} \circ \mathscr{W}^2_{\alpha-\beta} \circ \mathscr{W}^1_{\beta-\alpha} \circ F_{\alpha,\beta}(\psi \cdot f)(x).$$

Examining (3.10) and (3.11) in [25] and recalling (2.19) above shows that for every $0 \leqslant k \leqslant [\alpha + (1/2)]$,

$$\sup_{a \leqslant x \leqslant b} |f^{(k)}(x)| \leqslant \mathrm{const} \cdot \|f\|_{(\alpha,\beta)},$$

where the constant depends on a, b, α and β.

(b) The case $\alpha = \beta = 0$ corresponds to the case $G = SL(2, \mathbb{R})$ and $K = SO(2)$. In particular, $\dim(G/K) = 2$ and the zonal spherical functions are

$$\phi_\lambda(t) = F\big((1 + \mathrm{i}\lambda)/2, (1 - \mathrm{i}\lambda)/2; 1; -(\sinh t)^2\big), \quad \lambda \geqslant 0, t \geqslant 0.$$

Differentiating yields

$$\phi'_\lambda(t) = (\sinh(2t)) \cdot (1+\lambda^2) \cdot F((3+\mathrm{i}\lambda)/2,$$

$$(3-\mathrm{i}\lambda)/2;\, 2;\, -(\sinh t)^2)/4.$$

The hypergeometric function here corresponds to a Jacobi function with indices (1, 1). Fix t_0 in the interval (0, 1]. The methods of section 2 in [34] show that as $\lambda \to \infty$,

$$\phi'_\lambda(t_0) = \text{const.}(1+\lambda^2) J_1(\lambda t_0) \cdot (\lambda t_0)^{-1} + O(\lambda^{-1/2})$$

and it is known that $J_1(\lambda t_0)$ behaves like

$$(2/(\pi\lambda t_0))^{1/2} \cos(\lambda t_0 - (3\pi/4)) \text{ as } \lambda \to \infty, \text{ see [36], page 368.}$$

This shows that $\phi_\lambda(t_0)$ is not bounded and so elements of ${}^{SO(2)}A(SL(2, \mathbb{R}))^{SO(2)}$ need not be differentiable on A_+. The example of the circle in $\mathbb{R}^2$, see [20], suggests that the case $\dim(G/K) = 2$ should be different from higher dimensional cases.

(c) Yet another proof of the fact that $\lambda \to \phi_\lambda^{(n)}(t)$ is uniformly bounded when $t \neq 0$ and $0 \leqslant n \leqslant [\alpha + 1/2]$ is given by differentiating the right hand side of (2.16) in [25] and keeping track of the integrability of the functions

$$s \to (\partial/\partial t)^l A_{\alpha,\beta}(s, t)$$

for $0 \leqslant l \leqslant n$ and $0 < s < t$.

3. Semisimple Lie groups

Our principal tools in this section will be the rank one reduction [19], section IX.2, and the following result of Herz [21].

3.1. LEMMA: *Let G be a locally compact group with a closed subgroup H. Then $A(G)|_H \subset A(H)$ and $\|f|_H\|_{A(H)} \leqslant \|f\|_{A(G)}$, $\forall f \in A(G)$.*

Now let G denote a noncompact connected real semisimple Lie group with finite centre and a fixed maximal compact subgroup K. Equip $\mathfrak{g}$, the Lie algebra of G, with the Killing form $\langle , \rangle$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition determined by the choice of K. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and set $A = \exp(\mathfrak{a})$. The polar decomposition of G is $G = KAK$.

For each $\gamma \in \mathfrak{a}^*$ let $\mathfrak{g}_\gamma = \{X \in \mathfrak{g} : [H, X] = \gamma(H)X, \forall H \in \mathfrak{a}\}$ The set of *restricted roots* is

$$R := \{\gamma \in \mathfrak{a}^* \setminus \{0\} : \mathfrak{g}_\gamma \neq \{0\}\}.$$

For each $\gamma \in R$ let $m(\gamma) = \dim \mathfrak{g}_\gamma$. Now put $\mathfrak{a}' := \{H \in \mathfrak{a} : \gamma(H) \neq 0 \forall \gamma \in R\}$ and fix one component $\mathfrak{a}_+$ of $\mathfrak{a}'$. The corresponding set of positive roots is denoted by R^+. Furthermore, let

$$R_0^+ := \{\gamma \in R^+ : \tfrac{1}{2}\gamma \notin R^+\}.$$

Each root $\gamma \in R_0^+$ determines an element $H_\gamma \in \mathfrak{a}$ by setting $\gamma(H) = \langle H, H_\gamma \rangle \ \forall H \in \mathfrak{a}$. The elements of R_0^+ give rise to closed connected semisimple subgroups of G of real rank one.

Fix $\gamma \in R_0^+$ and let $\mathfrak{g}^\gamma$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{-\gamma}$. Then [19], Proposition IX.2.1, $\mathfrak{g}^\gamma$ is semisimple and it has a Cartan decomposition

$$\mathfrak{g}^\gamma = \mathfrak{k}^\gamma \oplus \mathfrak{p}^\gamma \tag{3.2}$$

where $\mathfrak{k}^\gamma := \mathfrak{k} \cap \mathfrak{g}^\gamma$ and $\mathfrak{p}^\gamma := \mathfrak{p} \cap \mathfrak{g}^\gamma$. The connected Lie subgroup of G with Lie algebra $\mathfrak{g}^\gamma$ is denoted by G^γ. This is a closed subgroup of G, [35], Lemma 1.1.5.7. Furthermore, $K^\gamma := G^\gamma \cap K$ is a maximal compact subgroup of G^γ. The subspace $\mathfrak{a}^\gamma := \mathbb{R} H_\gamma$ is maximal abelian in $\mathfrak{p}^\gamma$ and we set $A^\gamma := \exp(\mathfrak{a}^\gamma)$.

Now we have a rank *one* symmetric space G^γ / K^γ of dimension $d(\gamma) = 1 + m(\gamma) + m(2\gamma)$, that is, the dimension of $\mathfrak{p}^\gamma$.

3.3. THEOREM: *Let G, K, A and R_0^+ be as above. Suppose that there is a root $\gamma \in R_0^+$ with*

$$m(\gamma) + m(2\gamma) \geqslant 2.$$

Then for each $t > 0$ and $0 \leqslant n \leqslant [(m(\gamma) + m(2\gamma))/2]$ the map

$$f \mapsto (\mathrm{d}/\mathrm{d}t)^n (Pf)\big(\exp(tH_\gamma)\big)$$

is a bounded linear functional on $A(G)$.

PROOF: Firstly, we know that if $f \in A(G)$ then $Pf \in A(G)$. Furthermore from Lemma 3.1 it follows that $Pf|_{G^\gamma} \in A(G^\gamma)$ and

$$\| Pf|_{G^\gamma} \|_{A(G^\gamma)} \leqslant \| f \|_{A(G)}.$$

Since $K^\gamma = K \cap G^\gamma$ and $Pf \in {}^K A(G)^K$ we see that $Pf|_{G^\gamma} \in {}^{K^\gamma} A(G)^{K^\gamma}$. Now apply Theorem 2.15. Q.E.D.

3.4. COROLLARY: *Notation and hypothesis as in Theorem* 3.3. *For each* $t > 0$ *the double coset* $K \cdot \exp(tH_\gamma) \cdot K$ *is not a set of synthesis for* $A(G)$.

We can also apply this reasoning to the Cartan motion group $\mathfrak{p} \rtimes_{Ad} K$, since

$$\left({}^{K}A(\mathfrak{p} \rtimes K)^{K}\right)|_{\mathfrak{p}^\gamma \rtimes K^\gamma} \subset {}^{K^\gamma}A(\mathfrak{p}^\gamma \rtimes K^\gamma)^{K^\gamma}. \tag{3.5}$$

It is known [19], p. 535, that G^γ/K^γ is isotropic and so

$${}^{K^\gamma}A(\mathfrak{p}^\gamma \rtimes K^\gamma)^{K^\gamma}$$

can be identified with the *radial* elements of $A(\mathfrak{p}^\gamma)$. If $f \in {}^{K}A(\mathfrak{p} \rtimes K)^{K}$ then $t \to f(tH_\gamma, 1)$ is $[(m(\gamma) + m(2\gamma))/2]$-times differentiable on $(0, \infty)$ and for each $t > 0$ there is a constant $k > 0$ such that

$$\left|(\mathrm{d}/\mathrm{d}t)f(tH_\gamma, 1)\right| \leqslant k\|f\|_{A(\mathfrak{p} \rtimes K)}. \tag{3.6}$$

See [32].

3.7. THEOREM: *Let* G, K, A, $\mathfrak{p}$ *and* R_0^+ *be as above. If there is a root* $\gamma \in R_0^+$ *with* $m(\gamma) + m(2\gamma) \geqslant 2$ *then for each* $t > 0$ *the double coset* $K(tH_\gamma, 1)K$ *is not a set of synthesis for the Fourier algebra of* $\mathfrak{p} \rtimes K$.

Similarly, the orbit $Ad(K)(tH_\gamma)$ *is not a set of synthesis for* $A(\mathfrak{p})$, *the algebra of Fourier transforms of* $L^1(\mathfrak{p}^*)$.

The selection of a radial function f in $C_c^\infty(\mathfrak{p})$ with $f(tH_\gamma) = 0$ and $(\mathrm{d}/\mathrm{d}t)f(tH_\gamma) \neq 0$, completes the requirements of Proposition 1.10. Note that the dimension of $Ad(K)(tH_\gamma)$ is less than or equal to $\dim \mathfrak{p} - \dim \mathfrak{a}$, so that we have produced submanifolds in $\mathfrak{p}$ which are not sets of synthesis and which have codimension $\geqslant$ the rank of G/K, see [24].

3.8. EXAMPLES: (i) Groups G of real rank one which satisfy the hypotheses of Theorem 2.15.

(ii) A real semisimple Lie group G with the property that $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras, since then $m(\gamma)$ is even for all $\gamma \in R_0^+$, see [19], Theorem IX.6.1.

In particular, all connected semisimple complex Lie groups, in which case $m(\gamma) = 2$, $\forall \gamma \in R_0^+$.

(iii) Amongst the classical groups not covered in (i) and (ii) we can read off the following examples from Table VI in [19], pages 523-4 and section X.6.

	G	K
AIII:	$SU(p, q), (p \text{ or } q \geqslant 2)$;	$S(U_p \times U_q)$
BI & DI:	$SO_e(p, q),\ p > q > 1$;	$SO(p) \times SO(q)$
CII:	$Sp(l, l),\ l \geqslant 2$;	$Sp(l) \times Sp(l)$
DIII:	$SO^*(2n),\ n > 2$;	$U(n)$

4. Complex groups

When G is a connected semisimple *complex* Lie group we can refine Theorem 3.3 and demonstrate an analogue of Ricci's Theorem 1 in [30]. Maintain the notation set up in section 3 and assume that G is complex. Let W denote the Weyl group of (G, K). It is known [14] that the Godement-Plancherel measure is carried on $\mathfrak{a}^*/W$, so that we can view it as a W-invariant measure ν on $\mathfrak{a}^*$. Define

$$D(\exp H) := \prod_{\gamma \in R^+} (\mathrm{e}^{\gamma(H)} - \mathrm{e}^{-\gamma(H)}), \quad \forall H \in \mathfrak{a},$$

$$\pi(\lambda) := \prod_{\gamma \in R^+} \lambda(H_\gamma), \quad \forall \lambda \in \mathfrak{a}^*_{\mathbf{C}},$$

$$\text{and } \rho(H) := \sum_{\gamma \in R^+} \gamma(H), \quad \forall H \in \mathfrak{a}.$$

To each $\lambda \in \mathfrak{a}^*$ such that

$$\lambda(H_\gamma) \neq 0, \quad \forall \gamma \in R^+, \tag{4.1}$$

there is associated the zonal spherical function

$$\varphi_\lambda(\exp H) = \frac{\pi(\rho) D(\exp H)^{-1}}{\pi(\mathrm{i}\lambda)} \sum_{s \in W} \det(s)\, \mathrm{e}^{\mathrm{i}(s\lambda)(H)}, \tag{4.2}$$

for $H \in \mathfrak{a}$. See [17], page 304.

Fix $x_0 \in A_+$ and let V be a neighbourhood of x_0 in A with compact closure contained in A_+. There is a function $h \in {}^K C_c^\infty(G)^K$ such that

$$h(x) = D(x)^{-1}, \quad \forall x \in V.$$

Hence, for every $f \in {}^K A(G)^K$, we have

$$(D \cdot h) f \in {}^K A(G)^K$$

and

$$(Dhf)(\exp H) = \frac{\pi(\rho)}{\#(W)} \int_{\mathfrak{a}^*} \sum_{s \in W} \det(s)\, e^{i(s\lambda)(H)} (hf)\hat{}(\lambda)$$

$$\times \pi(i\lambda)^{-1} d\nu(\lambda). \tag{4.3}$$

Furthermore, there is a constant $k > 0$, depending on x_0 and V, such that

$$\int_{\mathfrak{a}^*} |(h \cdot f)\hat{}(\lambda)|\, d\nu(\lambda) \leqslant k \| f \|_{A(G)}. \tag{4.4}$$

It is known that ν is absolutely continuous with respect to Lebesgue measure on $\mathfrak{a}^*$. We can view $H_\gamma \in \mathfrak{a}$ as a translation invariant vector field on A. Then (4.3) and (4.4) show that the distribution

$$\Big(\prod_{\gamma \in R^+} H_\gamma \Big)(Dhf) \in A(A)$$

and

$$\left\| \Big(\prod_{\gamma \in R^+} H_\gamma \Big)(Dhf) \right\|_{A(A)} \leqslant k \| f \|_{A(G)}. \tag{4.5}$$

Note also that $f|_A \in A(A)$ and has

$$\| f \|_{A(A)} \leqslant \| f \|_{A(G)}.$$

Arguing as in [27], page 54, we see the following.

4.6. THEOREM: *Notation and hypothesis as above. If $S \subseteq R_+$ and $f \in {}^K A(G)^K$ then the distribution*

$$\Big(\prod_{\gamma \in S} H_\gamma \Big) f \Big|_V$$

is a continuous function and

$$\sup_{x \in V} \left| \Big(\prod_{\gamma \in S} H_\gamma \Big) f(x) \right| \leqslant k_{x_0, V, S} \| f \|_{A(G)}.$$

4.7. COROLLARY: *Suppose G is a complex semisimple Lie group with maximal compact subgroup K and polar decomposition KAK. If $x_0 \in A$ is a regular element then Kx_0K is not a set of synthesis for $A(G)$.*

The motion group result in this case is already contained in [27], Theorem 4.3.

4.8. REMARKS: If we lift the hypothesis that G is complex then we lose Harish-Chandra's formula (4.2). If we try to use the Gangolli expansion [14], [3.45], in its place we find we cannot get estimates for $(\prod_{\gamma \in R_+} H_\gamma)\varphi_\lambda$ which are uniform in $\lambda \in \mathfrak{a}^*$, on account of the singularities of the c-functions, see [7] Lemma 5. However, the Gangolli expansion does provide another means of proving (2.8) in the rank one case. For very detailed analysis of the asymptotics for zonal spherical functions on G and $\mathfrak{p} \rtimes K$, see [2,5,6,13].

5. Real semisimple Lie groups again

The referee's proof of Theorem 2.11 replaces direct estimates of derivatives of spherical functions with the properties of the Abel and Fourier transforms. We examine these transforms in the case $\operatorname{rank}(G/K) > 1$. In the final part of this section we present an analogue of Theorem 4.6 in the case $G = SU(n, n+k)$ and $K = S(U(n) \times U(n+k))$, with $n \geqslant 1$ and $k \geqslant 1$.

Assume that G is a connected noncompact semisimple real Lie group with finite centre and recall the notation of sections 1 and 3. Let $G = KAN$ be the Iwasawa decomposition corresponding to our choice of $\mathfrak{a}$ in $\mathfrak{p}$ and normalize the Haar measures on A and N as in [7]. Let W be the Weyl group and identify the Godement-Plancherel measure ν with the corresponding W-invariant measure on $\mathfrak{a}^*$, noting that it is absolutely continuous with respect to Lebesgue measure. The Fourier transform on $\mathfrak{a}$ is denoted by $\mathscr{F}_\mathfrak{a}$.

For every $f \in {}^K C_c(G)^K$ and $H \in \mathfrak{a}$ the Abel transform is

$$\mathscr{A}f(H) = e^{\rho(H)} \int_N f(\exp(H) \cdot n)\,dn \tag{5.1}$$

and the spherical transform satisfies

$$\hat{f}(\lambda) = \mathscr{F}_\mathfrak{a}(\mathscr{A}f)(\lambda), \quad \lambda \in \mathfrak{a}^*. \tag{5.2}$$

Now suppose that E is a compact subset of A and fix $\psi \in {}^K C_c^\infty(G)^K$ such that $\psi = 1$ on a neighbourhood of E. For every $f \in {}^K A(G)^K$ and $x \in E$,

$$f(x) = \int_{\mathfrak{a}^*} \mathscr{F}_\mathfrak{a}(\mathscr{A}(\psi \cdot f))(\lambda)\phi_\lambda(x)\,d\nu(\lambda) \tag{5.3}$$

and

$$\int_{\mathfrak{a}^*} |\mathscr{F}_\mathfrak{a}(\mathscr{A}(\psi \cdot f))(\lambda)|\,d\nu(\lambda) \leqslant \text{const} \cdot \|f\|_{A(G)}. \tag{5.4}$$

Lemma 5 in [7] shows that for all $\lambda \in \mathfrak{a}^*$,

$$\left|\frac{d\nu}{d\lambda}(\lambda)\right| \leqslant \prod_{\gamma \in R_0^+} \left|\lambda(H_\gamma)\right|^2 \cdot \left(1 + \left|\lambda(H_\gamma)\right|\right)^{m(\gamma)+m(2\gamma)-2}. \tag{5.5}$$

5.6. LEMMA: *Notation and hypotheses as above. Suppose that every $\gamma \in R_0^+$ satisfies $m(\gamma)+m(2\gamma) \geqslant 2$. Then for every $f \in {}^K\!A(G)^K$ the distributional derivative*

$$\left(\prod_{\gamma \in R_0^+} \partial_\gamma^{m(\gamma)+m(2\gamma)}\right) \mathscr{A}(\psi \cdot f)$$

is an element of $A(\mathfrak{a})$ and its norm is less than or equal to const $\cdot \|f\|_{A(G)}$. *The constant here depends on G and ψ.*

There seem to be very few cases where an explicit formula for $\mathscr{A}$ is known, see [1], apart from complex and rank-1 groups. This lemma would be useful in determining local regularity properties for elements of ${}^K\!A(G)^K$ *if* it were known that the "inverse" of $\mathscr{A}$ preserved some order of differentiability of functions. Recall Remarks 2.17(a).

When $G = SU(n, n+k)$ we can use a formula of Berezin and Karpelevič to prove an analogue of Theorem 4.6. From now on we fix $n \geqslant 1$ and $k \geqslant 1$ and we follow the notation of Hoogenboom's paper [23]. The symmetric space $SU(n, n+k)/S(U(n) \times U(n+k))$ has rank n and we identify $\mathfrak{a}$ with $\mathbb{R}^n$. Recall that if $\lambda \in \mathfrak{a}^*$ then the corresponding zonal spherical function is

$$\phi_\lambda(a_T) = \text{const} \cdot \det\left(\phi_{\lambda_i}^{(k,0)}(t_j)\right) \times \ldots$$

$$\left(\prod_{i<j}\left(\lambda_i^2 - \lambda_j^2\right) \cdot \left(\cosh(2t_i) - \cosh(2t_j)\right)\right)^{-1}. \tag{5.7}$$

Here a_T is a regular element of A_+ if its coordinates satisfy

$$t_1 > t_2 > \ldots > t_n > 0.$$

We now focus our attention on the function

$$\lambda \to \det\left(\phi_{\lambda_i}^{(k,0)}(t_j)\right),$$

for a fixed regular element a_T.

Koornwinder (see (2.21) in [25]) has shown that

$$\phi_{\lambda_i}^{(k,0)}(t) = \int_0^t \cos(\lambda_i s) \cdot A(s, t)\,ds \tag{5.8}$$

where

$$A(s,\, t) = \text{const}\cdot(\sinh t)^{-2k}\cdot(\cosh t)^{-k}$$
$$\cdot(\cosh(2t) - \cosh(2s))^{k-(1/2)} \times \ldots$$
$$\ldots \times F(k,\, k;\, k+(1/2);\, (\cosh(t) - \cosh(s))/(2\cosh t)). \tag{5.9}$$

From this it follows that

$$\det\left(\phi_{\lambda_i}^{(k,0)}(t_j)\right) = \int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_n} A(s_1,\, t_1)\ldots A(s_n,\, t_n) \times \ldots$$
$$\ldots \times \det\left(\cos(\lambda_i s_j)\right) \mathrm{d}s_n \ldots \mathrm{d}s_1. \tag{5.10}$$

Using a similar proof to that of Lemma 4.1 in [23], one can show that if $M > 0$ and $|s_j| < M$ for $1 \leqslant j \leqslant n$, then there is a constant $\text{const}_M > 0$ such that

$$\left|\det\left(\cos(\lambda_i s_j)\right)\right| \leqslant \text{const}_M \cdot \left|\prod_{i<j}\left(\lambda_i^2 - \lambda_j^2\right)\right|, \tag{5.11}$$

for all $\lambda \in \mathbb{R}^n$. That is to say, the function

$$F(s,\, \lambda) = \det\left(\cos(\lambda_i s_j)\right) \Big/ \prod_{i<j}\left(\lambda_i^2 - \lambda_j^2\right)$$

is smooth on $\mathbb{R}^n \times \mathbb{R}^n$ and if E is a compact subset of $\mathbb{R}^n$ then

$$\sup_{\lambda \in \mathbb{R}^n}\ \sup_{s \in E} |F(s,\, \lambda)| < \infty.$$

Combining (5.7), (5.10), and (5.11) we see that

$$\phi_\lambda(a_T) = \text{const}\cdot\int_0^{t_1}\ldots\int_0^{t_n} A(s_1,\, t_1)\ldots A(s_n,\, t_n)$$
$$\cdot F(s,\, \lambda)\mathrm{d}s_n\ldots\mathrm{d}s_1 \prod_{i<j}\left(\cosh(2t_i) - \cosh(2t_j)\right)^{-1},$$

for all regular a_T. Now observe that if $k \geqslant 1$ then the function

$$s \mapsto (\partial/\partial t)^l A(s,\, t)$$

is integrable on $(0, t)$ for each $0 \leqslant l \leqslant k$. See remark 2.17.(c).

5.12. THEOREM: *For $t_1 > t_2 > \ldots > t_n > 0$ and $0 \leqslant l_j \leqslant k$, $1 \leqslant j \leqslant n$, there is a constant $C_T > 0$ such that*

$$\left|(\partial/\partial t_1)^{l_1} \ldots (\partial/\partial t_n)^{l_n} \phi_\lambda(a_T)\right| \leqslant C_T \qquad \textit{for all} \quad \lambda \in \mathbb{R}^n.$$

In particular, for this element a_T of A the double coset $K.a_T.K$ is no a set of synthesis for the Fourier algebra of $SU(n, n+k)$.

This theorem leads to the following local regularity property. For every compact subset E properly contained in the set of regular elements of A and for each n-tuple l as in the statement of the theorem,

$$\sup_{a_T \in E} \left|(\partial/\partial t_1)^{l_1} \ldots (\partial/\partial t_n)^{l_n} f(a_T)\right| \leqslant \mathrm{const} \cdot \|f\|_{A(G)},$$

where f is a bi-K-invariant element of $A(G)$.

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