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COVERING SPACES OF AN ELLIPTIC SURFACE

R.V. Gurjar and A.R. Shastri

Introduction

Not enough is known about covering spaces of a projective, non-singular variety/$\mathbb{C}$ of dimension bigger than 1. In this connection, the following question remains unanswered.

"Is the universal covering space of a projective, nonsingular variety/$\mathbb{C}$ holomorphically convex?."

See [9, Chapter IX] for a discussion of this question. Recall that a complex manifold X is holomorphically convex if given any sequence of points $x_1,\dots,x_n,\dots$ without a limit point, there exists a holomorphic function f on X such that the sequence $f(x_n)$ is unbounded. A compact, complex manifold is clearly holomorphically convex. In this paper, we will prove the following.

THEOREM: *Let S be an irreducible, non-singular, projective surface/$\mathbb{C}$ with an elliptic fibration π: $S \to \Delta$. If π has at least one singular fibre which is not of the type mI_0 (see §1 for the notation), then any unramified covering of S is holomorphically convex. If all the singular fibres of π are of mI_0 type, then the universal covering space of S is holomorphically convex.*

We will give an example (cf. Morimoto [7], p. 262) of an abelian surface (which is actually a product of elliptic curves) having a regular, unramified cover which is not holomorphically convex. As a corollary of the theorem, we get the following:

Let S be a projective, irreducible, non-singular elliptic surface/$\mathbb{C}$ such that the elliptic fibration $S \to \Delta$ has at least one singular fibre which is not of the type mI_0. Suppose $C \subset S$ is an irreducible curve with $C^2 > 0$. Then the image of the fundamental group of the non-singular model of C has finite index in the fundamental group of S.

In particular if C is rational, then π must have at least one singular fibre not of mI_0 type and hence $\pi_1(S)$ is finite. This result has been conjectured by M. Nori for arbitrary projective, non-singular, irreducible surface. See [8] for some results about this question.

One result in this paper is that the image I, of the fundamental group of a good fibre of an elliptic fibration $S \to \Delta$ (having at least one singular fibre not of the type mI_0) in $\pi_1(S)$ is a cyclic group of odd order (I is trivial if $\Delta \approx \mathbb{P}^1$). This fact is crucial for the holomorphic convexity of coverings of S. S Iitaka has described the fundamental group of an Elliptic Surface in [3]. The extra information about $\pi_1(S)$ given in this paper supplements Iitaka's results.

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§1. Notation and preliminaries

For a compact, complex surface S, we will use the following notation.

$$P_g(S) = \dim H^2(S, \mathcal{O}) = \dim H^0(S, \Omega^2)$$

$$q(S) = \dim H^1(S, \mathcal{O}).$$

We will use the definitions of elliptic surface, multiple fibre, multiplicity of a singular fibre as in K. Kodaira's fundamental papers [4]. Kodaira has described the possible singular fibres of an elliptic fibration $S \to \Delta$ where S and Δ need not be compact. Only possible multiple fibres are of the type mI_b for $b = 0, 1, \ldots$. Here mI_0 stands for an elliptic curve occuring with multiplicity m.

First, let S be an irreducible, projective, non-singular surface/$\mathbb{C}$ and $S \xrightarrow{\pi} \Delta$ be an elliptic fibration with Δ a compact Riemann surface of genus g. Since π_1 is a birational invariant and the conclusions about holomorphic convexity of coverings of S are preserved after blowing up points on S, we will assume throughout that no fibre of π contains an exceptional curve of the 1^{st} kind.

We will recall some basic results about the neighbourhoods of singular fibres of π. For these, see [5,6]. Let $a \in \Delta$ be a point such that $\pi^*(a)$ is a singular fibre. Choose a small disc D around a in Δ and let $\delta = \partial D$ be the loop going around a once in the counter clock-wise direction. Choose a point $b \in \partial D$. Let $\pi^{-1}(D) = U$, $\pi^{-1}(b) = E$, $p \in E$, $U' = U - F$, i: $U' \to U$ be the inclusion map. Then $\pi_1(E)(\approx \mathbb{Z} \oplus \mathbb{Z})$ is a subgroup of $\pi_1(U')$.

LEMMA A [6]:

(1) *If F is not of the type mI_h $(h > 0)$, then $\pi_1(F) = (1)$, and hence $\pi_1(U) = (1)$, since F is a strong deformation retract of U.*

(2) *Let F be of the type mI_h. Then $\exists$ loops β, γ in E at p and a loop α in U' at p such that $\pi_{\#}(\alpha) = \delta$ and β, γ generate $\pi_1(E)$. $\pi_1(U')$ is given by*

$$\pi_1(U') = \langle \alpha, \beta, \gamma / [\alpha, \beta] = 1 = [\beta, \gamma], [\alpha, \gamma] = \beta^h \rangle.$$

Also $i_{\#}(\alpha^m) \in i_{\#}(\pi_1(E))$.

Further if $h \geqslant 1$, *then* $i_{\#}(\beta) = 1$, $\pi_1(F) \approx \mathbb{Z}$ *hence* $\pi_1(U) \approx \mathbb{Z}$ *is generated by* $i_{\#}(\alpha)$ *and* $i_{\#}(\gamma)$. *If* $m = 1$, *then* $i_{\#}(\alpha) = 1$.

If F *is of the type* mI_0, *then* $\pi_1(E)$ *injects into* $\pi_1(U)$ *which is isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}$.

LEMMA B [5]: *Let* F *be of the type* mI_h $(h > 0)$. *Then there is an elliptic fibration* $\tilde{S} \xrightarrow{\tilde{\pi}} \Delta$ *such that* $\tilde{F} = \tilde{\pi}^*(a)$ *is a singular fibre of type* ${}_1I_h$ (*so, for* $h = 0$, $\tilde{F}$ *is a good fibre of* $\tilde{S}$) *and* $\tilde{S} - \tilde{F}$ *is complex-analytically isomorphic to* $S - F$. *Furthermore, the kernels of the homomorphisms* $\pi_1(E) \to \pi_1(U)$ *and* $\pi_1(E) \to \pi_1(\tilde{U})$ *are the same, where* $\tilde{U} = \tilde{\pi}^{-1}(D)$.

In Kodaira's terminology, S is obtained from $\tilde{S}$ by performing a logarithmic transformation in U.

LEMMA C [1]: *Let* $S \xrightarrow{\pi} \Delta$ *be an elliptic fibration with* Δ *and* S *compact, as above. Assume* π *has at least one singular fibre which is not of the type* mI_0. *Then any torsion, analytic line bundle on* S *comes from a divisor supported on the fibres of* π. *Further, if* π *has no multiple fibres, then any torsion line bundle on* S *is the pull-back of a torsion line bundle on* Δ.

PROOF: This is essentially proved in Dolgacev's paper [1].

LEMMA D: *Let* $S \xrightarrow{\pi} \Delta$ *be as in Lemma C and assume that* π *has no multiple fibres. Then* $H_1(S, \mathbb{Z})$ *is torsion-free.*

PROOF: By Lemma C, any analytic, torsion line bundle L on S is of the form $\pi^*(\mathscr{L})$, where $\mathscr{L}$ is a torsion line bundle on Δ. If $H_1(S, \mathbb{Z})$ has torsion, then $H^2(S, \mathbb{Z})$ also has torsion. From the long exact cohomology sequence $\cdots \to H^1(S, \mathcal{O}) \xrightarrow{\lambda} H^1(S, \mathcal{O}^*) \to H^2(S, Z) \to H^2(S, \mathcal{O}) \to \dots$ any torsion-element z in $H^2(S, \mathbb{Z})$ is the 1st chern class of a line bundle L', $c_1(L') = z$. Suppose $nz = 0$. Then $\exists\ \omega \in H^1(S, \mathcal{O})$, with $\lambda(\omega) = nL'$. Let $L'' = \lambda(1/n\omega)$, then $nL' = nL''$ in Pic S. But then $n(L) = 0$ where $L = L' - L''$. Also $c_1(L) = c_1(L')$ since $c_1(L'') = 0$. But $L = \pi^*(\mathscr{L})$ where $\mathscr{L}$ is a torsion-line bundle on Δ. $c_1(L) = \pi^* c_1(\mathscr{L})$. But $H^2(\Delta, \mathbb{Z}) \approx \mathbb{Z}$ hence $c_1(\mathscr{L}) = 0$, so $c_1(L) = c_1(L') = 0$ i.e. $z = 0$.

§2. Description of $\pi_1(S)$

Let $a_1, \dots, a_r$ be all the points in Δ for which $\pi^*(a_i)$ is a singular fibre with multiplicity $m_i \geqslant 1$. Let $\Delta' = \Delta - \{a_1, \dots, a_r\}$ and $S = \pi^{-1}(\Delta')$, $S' \subset S$. For $i = 1, \dots r$ choose small open discs D_i in Δ around $a_i (D_i \cap D_j = \emptyset$ *for* $i \neq j)$. *Choose* p_i *in* $\pi^{-1}(D_i')$ *as a base point for* $U_i' = \pi^{-1}(D_i')$ and $U_i = \pi^{-1}(D_i)$. By choosing arcs from p_0 to p_i in S' and conjugating by them we obtain isomorphisms $\pi_1(S', p_i) \simeq \pi_1(S', p_0)$ under which we are going to identify $\pi_1(S', p_i)$ with $\pi_1(S', p_0)$. Then $\pi_1(E_i, p_i)$ gets identified with $\pi_1(E, p_0)$ where E_i is the fibre of π through p_i. If α_i, β_i, γ_i, are chosen in U_i' as in Lemma A, we shall continue to denote their images in $\pi_1(S', p_0)$ also by the same symbols. We also dispense away with writing down base points.

$S' \overset{\pi}{\to} \Delta'$ is a C^∞-fibration, so we have an exact sequence $1 \to \pi_1(E) \to \pi_1(S') \to \pi_1(\Delta') \to 1$. Clearly

$$\pi_1(\Delta') = \langle x_j', y_j', \delta_1, \dots, \delta_r,\ 1 \leqslant j \leqslant g / \prod_{j-1}^{r} [x_j', y_j']\, \delta_1 \dots \delta_r = 1 \rangle$$

is a free group and hence the above sequence splits. Choose lifts x_j, y_j, and α_i for x_j', y_j' and δ_i respectively in $\pi_1(S')$. We can then consider $\pi_1(\Delta')$ as a subgroup of $\pi_1(S')$ generated by x_j, y_j, $\alpha_1, \dots, \alpha_r$.

For a base $\{\beta, \gamma\}$ of $\pi_1(E)$ and any $x \in \pi_1(\Delta')$, write

$$\begin{aligned} x\beta x^{-1} &= a_x\beta + b_x\gamma \\ x\gamma x^{-1} &= c_x\beta + d_x\gamma \end{aligned} \qquad a_x d_x - b_x c_x = \pm 1. \tag{$**$}$$

For $1 \leqslant i \leqslant s$, let F_i be a fibre of type $m_i I_{h_i}$ with $h_i \geqslant 1$. For $s+1 \leqslant i \leqslant t$, let F_i be of type $m_i I_0$ and for $t+1 \leqslant i \leqslant r$ let F_i be a simply-connected singular fibre. By applying Van-Kampen's theorem finitely many times, we obtain $\pi_1(S)$ as the quotient of $\pi_1(S')$ by the following set of relations:

(i) $\beta_i = 1$, $\alpha_i^{m_i} = \gamma_i^{n_i}$, $[\alpha_i, \gamma_i] = 1$ for $1 \leqslant i \leqslant s$.
(ii) $[\alpha_i, \gamma_i] = 1$; $\alpha_i^{m_i} = \omega_i(\beta_i, \gamma_i)$ for $s+1 \leqslant i \leqslant t$, where $\omega_i(\beta_i, \gamma_i)$ are some words in β_i and γ_i.
(iii) $\alpha_i = \beta_i = \gamma_i = 1$ for $t+1 \leqslant i \leqslant r$.

We are now ready to prove:

THEOREM 1: *Let S be an irreducible, projective nonsingular surface/$\mathbb{C}$ with an elliptic fibration $S \overset{\pi}{\to} \Delta$ over a compact Riemann surface of genus $g \geqslant 0$. Let I denote the image of $\pi_1(E)$ in $\pi_1(S)$ where E is a nonsingular fibre. Then we have an exact sequence*

$$1 \to I \to \pi_1(S) \overset{\varphi}{\to} \Gamma \to 1 \tag{$*$}$$

where

$$\Gamma = \langle x_i, y_i,\ 1 \leqslant i \leqslant g,$$

$$\alpha_j,\ 1 \leqslant j \leqslant r / \prod_{i=1} [x_i, y_i] \prod_{j=1}^{r} \alpha_j = 1,\ \alpha_j^{m_j} = 1 \rangle$$

If π has at least one singular fibre not of the type mI_0, then I is a cyclic group of odd order; further if $g=0$, then $I=(e)$.

PROOF: That we have the exact sequence $(*)$ follows immediately from the considerations above. If π has at least one simply connected singular fibre F_i, $t+1 \leqslant i \leqslant r$, from relations (iii) it follows that $I=(e)$. Hence from now on we shall assume that π has no simply connected fibres, i.e. $r=t$.

Let B be the subgroup of $\pi_1(E)$ generated by $\beta_1, \ldots \beta_s$. By our assumption, $s \geqslant 1$. Now I is a quotient of $\pi_1(E)/B \approx \mathbb{Z}/(d)$ for some $d \geqslant 0$. Let $\{\beta, \gamma\}$ be a basis of $\pi_1(E)$ such that $\beta \in B$ and γ generates $\pi_1(E)/B$. From $(**)$ it follows that $\pi_1(S)$ is generated γ, x_j, y_j, $1 \leqslant j \leqslant g$ and α_i, $1 \leqslant i \leqslant r$ with the following set of relations:

(i) $\prod_{j=1}^{g}[x_j, y_j] \prod_{i=1}^{r} \alpha_i = 1$
(ii) $\gamma^d = 1$, $\gamma^b x = 1$, $x\gamma x^{-1} = \gamma^d x$ $\forall x \in \pi_1(\Delta')$
(iii) $[\alpha_i, \gamma] = 1$, $1 \leqslant i \leqslant r$.
(iv) $\alpha_i^{m_i} = \gamma^{m_i}$ with $n_i = 0$ if $m_i = 1$, $1 \leqslant i \leqslant r$.

Let $\tilde{S} \xrightarrow{\pi} \Delta$ be the elliptic fibration as in Lemma B (replacing all the multiple fibres of π by simple singular fibres). If $\tilde{B}$ denotes the subgroup of $\pi_1(E)$ generated by $\tilde{\beta}_i (1 \leqslant i \leqslant s)$, then from Lemma B, it follows that $B = \tilde{B}$. Since $\tilde{\pi}$ has no multiple fibres, in the presentation $\pi_1(\tilde{S})$, $\alpha_i = 1$ for $1 \leqslant i \leqslant s)$ Thus $\pi_1(\tilde{S})$ is the group

$$\langle \gamma, x_j, y_j, 1 \leqslant j \leqslant g / \prod_{j=1}^{g}[x_j, y_j] = 1, \gamma^d = \gamma^b x = 1,$$

$$x\gamma x^{-1} = \gamma^d x, \forall_x \in \pi_1(\Delta') \rangle$$

If H is the subgroup of integers generated by the integers d, b_x, $d_x - 1$ for $x \in \pi_1(\Delta)$, then it follows that

$$H_1(\tilde{S}, \mathbb{Z}) \approx \mathbb{Z}/_H \oplus H_1(\Delta, \mathbb{Z}).$$

Now $\tilde{\pi}$ has at least one singular fibre of type ${}_1I_h$, so the sheaf $\mathscr{G}$ on $\tilde{S}$ constructed by taking the 1st homology groups (co-efficients in $\mathbb{Z}$) of regular fibres of $\tilde{\pi}$ is nonconstant. By a result of Kodaira, this implies that $b_1(\tilde{S})$ is even; see [4, Theorem 11.8]. But by Lemma D, $H_1(\tilde{S}, \mathbb{Z})$ is torsion free. Thus $\mathbb{Z}/H = (0)$. If one of the integers b_x is odd, then clearly the image of $\pi_1(E)$ in $\pi_1(S)$ is a cyclic group of odd order. If b_x is even for all $x \in \pi_1(\Delta')$, then d_x is odd for all x since $a_x d_x - b_x c_x = \pm 1$. But then $d_x - 1$ is even for all x. In this case $\mathbb{Z}/_H$ cannot be trivial unless d is odd. Thus I is a cyclic group of odd order.

If $\Delta \approx \mathbb{P}^1$, then $b_x = 0 = d_x - 1$ for all $x \in \Gamma$, α_i commute with γ. Thus H is generated by d. But then $d = 1$ otherwise $H_1(\tilde{S}, \mathbb{Z})$ will have either odd rank or non-trivial torsion, which is not possible. This completes the proof of the theorem.

§3. Applications

THEOREM 2: *Let S be an irreducible non-singular, projective surface/$\mathbb{C}$ with an elliptic fibration $S \xrightarrow{\pi} \Delta$.*

(*i*) *If π has at least one singular fibre not of the type mI_0, then any unramified covering of S is holomorphically convex.*
(*ii*) *In general, the universal covering space $\tilde{S}$ of S is holomorphically convex.*

REMARK: We will give an example of an abelian surface $S \approx E_1 \times E_2$ with E_i elliptic curves such that S has an infinite sheeted unramified covering $\bar{S}$ with no non-constant holomorphic functions; in particular $\bar{S}$ is not holomorphically convex. We shall need the following:

LEMMA E: *Suppose $\Delta = \mathbb{P}^1$ and π has at most two singular fibres and these are of type mI_0. Then S is birationally a ruled surface and hence its universal covering $\tilde{S}$ is holomorphically convex.*

PROOF: Let $\pi^*(a_i)$ be a singular fibre of type $m_i I_0$, $i = 1, 2$. For the canonical bundle K_S of S, Kodaira has proved the formula

$$K_S \approx \pi^*\left(_{\mathbb{P}^1}(-2 + \chi(S, \mathcal{O}))\right) \otimes \left[P_{a_1}\right]^{\otimes(m_1-1)} \otimes \left[P_{a_2}\right]^{\otimes(m_2-1)}$$

Here P_{a_i} is the divisor such that $\pi^*(a_i) = m_i P_i$ and $[P_{a_i}]$ is the corresponding complex-analytic line bundle.

By Noether's formula $\chi(S, \mathcal{O}) = (K_S^2 + c_2(S))/12$. But $K_S^2 = 0$ for our elliptic surface and $c_2(S)$ is equal to the sum of the topological Euler-characteristics of all the singular fibres of π. Thus $c_2(S)$ is also 0 and $\chi(S, \mathcal{O}) = 0$.

Since any two points in $\mathbb{P}^1$ are rationally equivalent, we see that

$$\pi^*\left(_{\mathbb{P}^1}(1)\right) \approx \left[P_{a_1}\right]^{\otimes m_1} \approx \left[P_{a_2}\right]^{\otimes m_2}$$

We see easily that $K_S \approx [P_{a_1}]^{-1} \otimes [P_{a_2}]^{-1}$, thus forcing $|nK_S| = \emptyset$ for $n \geqslant 1$. This means that S is a birationally ruled surface.

Similar argument shows that when π has at most one singular fibre (and that too of type mI_0), S is a birationally ruled surface. To see that $\tilde{S}$, the universal cover of S is holomorphically convex we can assume that S is a relatively minimal model with a $\mathbb{P}^1$-bundle $S \xrightarrow{\Psi} \Delta$. Then $\Delta \not\approx \mathbb{P}^1$ since

$\chi(S, \mathcal{O}) = 0$. Pulling back the fibration Ψ to the universal cover $\tilde{\Delta}$ of Δ we see that $S \times_\Delta \tilde{\Delta}$ is complex analytically isomorphic to $\tilde{\Delta} \times \mathbb{P}^1$, which is holomorphically çonvex.

PROOF OF THEOREM 2: Let σ: $W \to \Delta$ be the ramified covering with Γ as the group of analytic automorphisms such that $W/\Gamma \approx \Delta$. (In particular W is simply connected). For any subgroup Γ_1 of Γ, the pull back fibration $W/\Gamma_1 \times_\Delta S$ over W/Γ_1, yields an elliptic fibration $X_{\Gamma_1} \to W/\Gamma_1$ after normalization, such that $X_{\Gamma_1} \to S$ is the unramified covering corresponding to the subgroup $\pi_1(X_{\Gamma_1}) = \varphi^{-1}(\Gamma_1)$ of $\pi_1(S)$.

To prove (i) let H be any subgroup of $\pi_1(S)$. Put $\Gamma_1 = \varphi(H)$. Then $IH = \varphi^{-1}(\Gamma_1)$. By theorem 1, I is finite and hence H is of finite index in IH. If $\tilde{S}_H$ is the covering of S with $\pi_1(\tilde{S}_H) = H$, then clearly $\tilde{S}_H \to X_{\Gamma_1}$ is a finite covering. Since $X_{\Gamma_1} \to W/\Gamma_1$ has compact fibres and W/Γ_1, being a Riemann surface, is holomorphically convex, it follows that $\tilde{S}_H$ is holomorphically convex.

To prove (ii), by (i) it suffices to consider the case when all singular fibres are of the type mI_0. By lemma E, we can further assume that either $g > 0$ or $g = 0$ and there are at least three singular fibres. But then one easily checks that $\text{ord}(\alpha_i) = m_i$ in Γ. This implies that the ramification index of σ at $Q \in \sigma^{-1}(a_i)$ is precisely m_i. Hence the fibration $X_{(I)} \to W$ is non-singular. If W is noncompact then it is contractible. Hence by a theorem of Grauert (see [2]) $X_{(I)} \approx W \times E$, and hence $\tilde{S} \approx W \times \mathbb{C}$. If W is compact then $W \approx \mathbb{P}^1$ and again by lemma E, $X_{(I)}$ is a ruled surface and hence in any case $\tilde{S}$ is holomorphically convex. This completes the proof of theorem 2.

AN EXAMPLE: For any irrational number λ let

$$v_1 = (1, 0), \quad v_2 = (0, 1)$$

$$v_3 = \left(\frac{\log 2}{2\pi i}, \frac{\lambda \log 2}{2\pi i}\right), \quad v_4 = \left(\frac{-\log 2}{2\pi i}, \frac{\lambda \log 2}{2\pi i}\right)$$

$$v_3' = \left(\frac{\log 2}{\pi i}, 0\right), \quad v_4' = \left(0, \frac{\lambda \log 2}{\pi i}\right)$$

be the vectors in $\mathbb{C} \times \mathbb{C}$. For any set A of elements in $\mathbb{C}$ (or $\mathbb{C} \times \mathbb{C}$) let $L[A]$ denote the additive subgroup of $\mathbb{C}$(or $\mathbb{C} \times \mathbb{C}$) generated by A. Then clearly

$$L[v_1, v_2, v_3', v_4'] = L[v_1, v_3'] \oplus L[v_2, v_4']$$

$$\simeq L\left[1, \frac{\log 2}{\pi i}\right] \times L\left[1, \frac{\lambda \log 2}{\pi i}\right]$$

If $E_1 = \mathbb{C}/L[1, \log 2/\pi i]$ and $E_2 = \mathbb{C}/L[1, \lambda \log 2/\pi i]$ are the elliptic curves then it follows that $S = E_1 \times E_2 \simeq \mathbb{C} \times \mathbb{C}/L[v_1, v_2, v_3', v_4']$. Since $L[v_1, v_2, v_3', v_4']$ is a subgroup of inxed 2 in $L[v_1, v_2, v_3, v_4]$ it follows that S is a double cover of $X = \mathbb{C} \times \mathbb{C}/L[v_1, v_2, v_3, v_4]$. Let $W = \mathbb{C} \times \mathbb{C}/L[v_1, v_2, v_3]$. Then $W \to X$ is an infinite cyclic cover. Pulling back this via the double covering $S \to X$ yields an infinite cyclic cover $\bar{S} \to S$.

We claim that $\bar{S}$ admits no nonconstant holomorphic function. Since $\bar{S}$ is a double cover of W it suffices to show that W does not admit any nonconstant holomorphic function. Since W is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*/\mathbb{Z}$ where $\mathbb{Z} = (g)$ acts on $\mathbb{C}^* \times \mathbb{C}^*$ via,

$$g(z_1, z_2) = (2z_1, 2\lambda z_2)$$

it follows that a holomorphic function f on W is given by a holomorphic function $\tilde{f}$ on $\mathbb{C}^* \times \mathbb{C}^*$ invariant under the $\mathbb{Z}$-action. Let

$$\tilde{f}(z_1, z_2) = \sum q_{ij} z_1^i z_2^j$$

be the Laurent series for $\tilde{f}$. It follows that $a_{ij} = a_{ij} 2^{i+\lambda j}$ for every (i, j). Thus if $a_{ij} \neq 0$ then $i + \lambda j = 0$; since λ is irrational this means $(i, j) = (0, 0)$. Thus $\tilde{f}$ and hence f is a constant.

As a corollary of Theorem 2 we prove the

PROPOSITION: *Let S be an irreducible, non-singular projective surface with an elliptic fibration $S \xrightarrow{\pi} \Delta$. Let $C \subset S$ be an irreducible, complete curve with $C^2 > 0$. Assume that $\chi(S, \mathcal{O}) > 0$. Then $[\pi_1(S)\colon \operatorname{Im} \pi_1(\bar{C})] < \infty$, where $\bar{C} \to C$ is the non-singular model of C.*

CROLLARY: *If C is rational, then $\pi_1(S)$ is finite.*

REMARKS:

(1) If C is rational, we will show later that $q(S) = 0$ and hence $\chi(S, \mathcal{O}) > 0$.

(2) M.V. Nori has given an example of an elliptic fibration $S \to \Delta$ with $\chi(S, \mathcal{O}) = 0$ and an irreducible curve $C \subset S$ with $C^2 > 0$ such that $[\pi_1(S)\colon \operatorname{Im} \pi_1(C)] = \infty$.

(3) The arguments in the proof of the proposition show that if we delete the condition $C^2 > 0$, then we can still conclude that $[\operatorname{Im} \pi_1(C); \operatorname{Im} \pi_1(\bar{C}] < \infty$ in $\pi_1(S)$. Further, if C is a connected chain of rational curves then we can conclude that $\operatorname{Im} \pi_1(C)$ is finite in $\pi_1(S)$.

PROOF OF THE PROPOSITION: By the argument at the end of Proof of Lemma E $\chi(S, \mathcal{O}) > 0$ implies that π has at least one singular fibre not of the type mI_0. Let $H = \text{Im}\ \pi_1(\bar{C}) \subset \pi_1(S)$. Consider the covering $\tilde{S} \xrightarrow{\varphi} S$ such that $\varphi_{\#}\pi_1(\tilde{S}) = H$. By Theorem 2, $\tilde{S}$ is holomorphically convex. Since $C^2 > 0$, it can be shown that $\pi_1(C) \to \pi_1(S)$ is surjective. See [8] for a proof. Hence $\varphi^{-1}(C)$ is a connected curve on $\tilde{S}$. Also, by construction, $\bar{C} \to S$ lifts to $\bar{C} \to \tilde{S}$. Let $\varphi^{-1}(C) = \bigcup_{i=1}^{r} C_i$ where $C_1, \ldots,$ are the irreducible components of $\varphi^{-1}(C)$. Corresponding to each C_i, $\exists$ a unique lift of the map $\bar{C} \to S$ to $\bar{C} \to \tilde{S}$ which has C_i as the image of $\bar{C}$. Hence each C_i is compact. Choose points $x_i \in C_i$. If the set $\{x_1, x_2, \ldots\}$ is infinite, it has no limit point. In this case $\exists$ a holomorphic function f such that $f(x_n)$ is unbounded as $n \to \infty$. But since each C_i is compact and $\cup C_i$ is connected, f has to be constant on $\cup C_i$. This means r is finite and the covering $\tilde{S} \to S$ is of finite degree i.e. $[\pi_1(S)\colon \text{Im}\ \pi_1(\bar{C})] < \infty$.

For the proof of the corollary, we need the following.

LEMMA: *If S is a non-singular, irreducible, projective surface and $C \subset S$ an irreducible, complete, rational curve with $C^2 > 0$, then $q(S) = 0$.*

PROOF: There are several proofs of this result. We will give one due to M.P. Murthy.

Let $S \xrightarrow{\Psi} \text{A}1b\ S$ be the morphism from S to its Albanese variety. Clearly $\Psi(C)$ is a point. If the image of S in $A1B\ S$ is a surface V, then by the negative definiteness of the intersection form on the inverse image of a point of V, C^2 will have to be negative, which is not true. If $\varphi(S)$ is a curve, then the intersection form on any fibre of the map $S \to \Psi(S)$ can be seen to be negative semi-definite. This shows that $A1b\ S$ is a point i.e. $q(S) = 0$.

REMARK:

(1) From the classification of algebraic surface, we see now that for any surface of special type, the universal covering space is holomorphically convex (because for ruled, rational, $K-3$, abelian surfaces this is easy to verify and all Enriques surfaces are known to have elliptic fibrations).

(2) In the above Proposition, when C is rational, it can be shown that the only possible fundamental groups of S are finite cyclic group, dihedral group of $2n$ elements, tetrahedral group with 12 elements, octahedral group with 24 elements or icosahedral group on 60 elements.

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