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<http://www.numdam.org/item?id=CM_1985__54_1_95_0>

COVERING SPACES OF AN ELLIPTIC SURFACE

R.V. Gurjar and A.R. Shastri

Introduction

Not enough is known about covering spaces of a projective, non-singular variety/$\mathbb{C}$ of dimension bigger than 1. In this connection, the following question remains unanswered.

"Is the universal covering space of a projective, nonsingular variety/$\mathbb{C}$ holomorphically convex?."

See [9, Chapter IX] for a discussion of this question. Recall that a complex manifold X is holomorphically convex if given any sequence of points $x_1, \ldots, x_n, \ldots$ without a limit point, there exists a holomorphic function f on X such that the sequence $f(x_n)$ is unbounded. A compact, complex manifold is clearly holomorphically convex. In this paper, we will prove the following.

THEOREM: *Let S be an irreducible, non-singular, projective surface/$\mathbb{C}$ with an elliptic fibration $\pi: S \to \Delta$. If π has at least one singular fibre which is not of the type mI_0 (see §1 for the notation), then any unramified covering of S is holomorphically convex. If all the singular fibres of π are of mI_0 type, then the universal covering space of S is holomorphically convex.*

We will give an example (cf. Morimoto [7], p. 262) of an abelian surface (which is actually a product of elliptic curves) having a regular, unramified cover which is not holomorphically convex. As a corollary of the theorem, we get the following:

Let S be a projective, irreducible, non-singular elliptic surface/$\mathbb{C}$ such that the elliptic fibration $S \to \Delta$ has at least one singular fibre which is not of the type mI_0. Suppose $C \subset S$ is an irreducible curve with $C^2 > 0$. Then the image of the fundamental group of the non-singular model of C has finite index in the fundamental group of S.

In particular if C is rational, then π must have at least one singular fibre not of mI_0 type and hence $\pi_1(S)$ is finite. This result has been conjectured by M. Nori for arbitrary projective, non-singular, irreducible surface. See [8] for some results about this question.

One result in this paper is that the image I, of the fundamental group

of a good fibre of an elliptic fibration $S \to \Delta$ (having at least one singular fibre not of the type mI_0) in $\pi_1(S)$ is a cyclic group of odd order (I is trivial if $\Delta \approx \mathbb{P}^1$). This fact is crucial for the holomorphic convexity of coverings of S. S Iitaka has described the fundamental group of an Elliptic Surface in [3]. The extra information about $\pi_1(S)$ given in this paper supplements Iitaka's results.

We would like to thank M. Nori and R.R. Simha for many useful comments about the problems dealt in this paper.

§1. Notation and preliminaries

For a compact, complex surface S, we will use the following notation.

$$P_g(S) = \dim H^2(S, \mathcal{O}) = \dim H^0(S, \Omega^2)$$

$$q(S) = \dim H^1(S, \mathcal{O}).$$

We will use the definitions of elliptic surface, multiple fibre, multiplicity of a singular fibre as in K. Kodaira's fundamental papers [4]. Kodaira has described the possible singular fibres of an elliptic fibration $S \to \Delta$ where S and Δ need not be compact. Only possible multiple fibres are of the type mI_b for $b = 0, 1, \ldots$. Here mI_0 stands for an elliptic curve occuring with multiplicity m.

First, let S be an irreducible, projective, non-singular surface/$\mathbb{C}$ and $S \xrightarrow{\pi} \Delta$ be an elliptic fibration with Δ a compact Riemann surface of genus g. Since π_1 is a birational invariant and the conclusions about holomorphic convexity of coverings of S are preserved after blowing up points on S, we will assume throughout that no fibre of π contains an exceptional curve of the 1st kind.

We will recall some basic results about the neighbourhoods of singular fibres of π. For these, see [5,6]. Let $a \in \Delta$ be a point such that $\pi^*(a)$ is a singular fibre. Choose a small disc D around a in Δ and let $\delta = \partial D$ be the loop going around a once in the counter clock-wise direction. Choose a point $b \in \partial D$. Let $\pi^{-1}(D) = U, \pi^{-1}(b) = E, p \in E, U' = U - F, i: U' \to U$ be the inclusion map. Then $\pi_1(E)(\approx \mathbb{Z} \oplus \mathbb{Z})$ is a subgroup of $\pi_1(U')$.

LEMMA A [6]:

(1) *If F is not of the type mI_h ($h > 0$), then $\pi_1(F) = (1)$, and hence $\pi_1(U) = (1)$, since F is a strong deformation retract of U.*

(2) *Let F be of the type mI_h. Then $\exists$ loops β, γ in E at p and a loop α in U' at p such that $\pi_\#(\alpha) = \delta$ and β, γ generate $\pi_1(E)$. $\pi_1(U')$ is given by*

$$\pi_1(U') = \langle \alpha, \beta, \gamma / [\alpha, \beta] = 1 = [\beta, \gamma], [\alpha, \gamma] = \beta^h \rangle.$$

Also $i_\#(\alpha^m) \in i_\#(\pi_1(E))$.

Further if $h \geqslant 1$, then $i_\#(\beta) = 1$, $\pi_1(F) \approx \mathbb{Z}$ hence $\pi_1(U) \approx \mathbb{Z}$ is generated by $i_\#(\alpha)$ and $i_\#(\gamma)$. If $m = 1$, then $i_\#(\alpha) = 1$.

If F is of the type mI_0, then $\pi_1(E)$ injects into $\pi_1(U)$ which is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

LEMMA B [5]: *Let F be of the type mI_h ($h > 0$). Then there is an elliptic fibration $\tilde{S} \xrightarrow{\tilde{\pi}} \Delta$ such that $\tilde{F} = \tilde{\pi}^*(a)$ is a singular fibre of type $_1I_h$ (so, for $h = 0$, $\tilde{F}$ is a good fibre of $\tilde{S}$) and $\tilde{S} - \tilde{F}$ is complex-analytically isomorphic to $S - F$. Furthermore, the kernels of the homomorphisms $\pi_1(E) \to \pi_1(U)$ and $\pi_1(E) \to \pi_1(\tilde{U})$ are the same, where $\tilde{U} = \tilde{\pi}^{-1}(D)$.*

In Kodaira's terminology, S is obtained from $\tilde{S}$ by performing a logarithmic transformation in U.

LEMMA C [1]: *Let $S \xrightarrow{\pi} \Delta$ be an elliptic fibration with Δ and S compact, as above. Assume π has at least one singular fibre which is not of the type mI_0. Then any torsion, analytic line bundle on S comes from a divisor supported on the fibres of π. Further, if π has no multiple fibres, then any torsion line bundle on S is the pull-back of a torsion line bundle on Δ.*

PROOF: This is essentially proved in Dolgacev's paper [1].

LEMMA D: *Let $S \xrightarrow{\pi} \Delta$ be as in Lemma C and assume that π has no multiple fibres. Then $H_1(S, \mathbb{Z})$ is torsion-free.*

PROOF: By Lemma C, any analytic, torsion line bundle L on S is of the form $\pi^*(\mathcal{L})$, where $\mathcal{L}$ is a torsion line bundle on Δ. If $H_1(S, \mathbb{Z})$ has torsion, then $H^2(S, \mathbb{Z})$ also has torsion. From the long exact cohomology sequence $\cdots \to H^1(S, \mathcal{O}) \xrightarrow{\lambda} H^1(S, \mathcal{O}^*) \to H^2(S, Z) \to H^2(S, \mathcal{O}) \to \ldots$ any torsion-element z in $H^2(S, \mathbb{Z})$ is the 1st chern class of a line bundle L', $c_1(L') = z$. Suppose $nz = 0$. Then $\exists\ \omega \in H^1(S, \mathcal{O})$, with $\lambda(\omega) = nL'$. Let $L'' = \lambda(1/n\omega)$, then $nL' = nL''$ in Pic S. But then $n(L) = 0$ where $L = L' - L''$. Also $c_1(L) = c_1(L')$ since $c_1(L'') = 0$. But $L = \pi^*(\mathcal{L})$ where $\mathcal{L}$ is a torsion-line bundle on Δ. $c_1(L) = \pi^* c_1(\mathcal{L})$. But $H^2(\Delta, \mathbb{Z}) \approx \mathbb{Z}$ hence $c_1(\mathcal{L}) = 0$, so $c_1(L) = c_1(L') = 0$ i.e. $z = 0$.

§2. Description of $\pi_1(S)$

Let $a_1, \ldots, a_r$ be all the points in Δ for which $\pi^*(a_i)$ is a singular fibre with multiplicity $m_i \geqslant 1$. Let $\Delta' = \Delta - \{a_1, \ldots, a_r\}$ and $S = \pi^{-1}(\Delta')$, $S' \subset S$. For $i = 1, \ldots r$ choose small open discs D_i in Δ around $a_i (D_i \cap D_j = \emptyset$ for $i \neq j)$. *Choose p_i in $\pi^{-1}(D_i')$ as a base point for $U_i = \pi^{-1}(D_i')$ and $U_i = \pi^{-1}(D_i)$. By choosing arcs from p_0 to p_i in S' and conjugating by them we obtain isomorphisms $\pi_1(S', p_i) \approx \pi_1(S', p_0)$ under which we are*

going to identify $\pi_1(S', p_i)$ with $\pi_1(S', p_0)$. Then $\pi_1(E_i, p_i)$ gets identified with $\pi_1(E, p_0)$ where E_i is the fibre of π through p_i. If α_i, β_i, γ_i, are chosen in U_i' as in Lemma A, we shall continue to denote their images in $\pi_1(S', p_0)$ also by the same symbols. We also dispense away with writing down base points.

$S' \xrightarrow{\pi} \Delta'$ is a C^∞-fibration, so we have an exact sequence $1 \rightarrow \pi_1(E) \rightarrow$ $\pi_1(S') \rightarrow \pi_1(\Delta') \rightarrow 1$. Clearly

$$\pi_1(\Delta') = \langle x_j', y_j', \delta_1, \ldots, \delta_r, 1 \leqslant j \leqslant g / \prod_{j-1}^{r} [x_j', y_j'] \delta_1 \ldots \delta_r = 1 \rangle$$

is a free group and hence the above sequence splits. Choose lifts x_j, y_j, and α_i for x_j', y_j' and δ_i respectively in $\pi_1(S')$. We can then consider $\pi_1(\Delta')$ as a subgroup of $\pi_1(S')$ generated by $x_j, y_j, \alpha_1, \ldots, \alpha_r$.

For a base $\{\beta, \gamma\}$ of $\pi_1(E)$ and any $x \in \pi_1(\Delta')$, write

$$\begin{aligned} x\beta x^{-1} &= a_x\beta + b_x\gamma \\ x\gamma x^{-1} &= c_x\beta + d_x\gamma \end{aligned} \qquad a_x d_x - b_x c_x = \pm 1. \qquad (**)$$

For $1 \leqslant i \leqslant s$, let F_i be a fibre of type $m_i I_{h_i}$ with $h_i \geqslant 1$. For $s + 1 \leqslant i \leqslant t$, let F_i be of type $m_i I_0$ and for $t + 1 \leqslant i \leqslant r$ let F_i be a simply-connected singular fibre. By applying Van-Kampen's theorem finitely many times, we obtain $\pi_1(S)$ as the quotient of $\pi_1(S')$ by the following set of relations:

(i) $\beta_i = 1$, $\alpha_i^{m_i} = \gamma_i^{n_i}$, $[\alpha_i, \gamma_i] = 1$ for $1 \leqslant i \leqslant s$.

(ii) $[\alpha_i, \gamma_i] = 1$; $\alpha_i^{m_i} = \omega_i(\beta_i, \gamma_i)$ for $s + 1 \leqslant i \leqslant t$, where $\omega_i(\beta_i, \gamma_i)$ are some words in β_i and γ_i.

(iii) $\alpha_i = \beta_i = \gamma_i = 1$ for $t + 1 \leqslant i \leqslant r$.

We are now ready to prove:

THEOREM 1: *Let S be an irreducible, projective nonsingular surface$/\mathbb{C}$ with an elliptic fibration $S \xrightarrow{\pi} \Delta$ over a compact Riemann surface of genus $g \geqslant 0$. Let I denote the image of $\pi_1(E)$ in $\pi_1(S)$ where E is a nonsingular fibre. Then we have an exact sequence*

$$1 \rightarrow I \rightarrow \pi_1(S) \xrightarrow{\varphi} \Gamma \rightarrow 1 \qquad (*)$$

where

$$\Gamma = \langle x_i, y_i, 1 \leqslant i \leqslant g,$$

$$\alpha_j, 1 \leqslant j \leqslant r / \prod_{i=1}^{} [x_i, y_i] \prod_{j=1}^{r} \alpha_j = 1, \alpha_j^{m_j} = 1 \rangle$$

If π has at least one singular fibre not of the type mI_0, then I is a cyclic group of odd order; further if $g = 0$, then $I = (e)$.

PROOF: That we have the exact sequence $(*)$ follows immediately from the considerations above. If π has at least one simply connected singular fibre F_i, $t + 1 \leqslant i \leqslant r$, from relations (iii) it follows that $I = (e)$. Hence from now on we shall assume that π has no simply connected fibres, i.e. $r = t$.

Let B be the subgroup of $\pi_1(E)$ generated by $\beta_1, \ldots \beta_s$. By our assumption, $s \geqslant 1$. Now I is a quotient of $\pi_1(E)/B \approx \mathbb{Z}/(d)$ for some $d \geqslant 0$. Let $\{\beta, \gamma\}$ be a basis of $\pi_1(E)$ such that $\beta \in B$ and γ generates $\pi_1(E)/B$. From $(**)$ it follows that $\pi_1(S)$ is generated $\gamma, x_j, y_j, 1 \leqslant j \leqslant g$ and α_i, $1 \leqslant i \leqslant r$ with the following set of relations:

(i) $\displaystyle\prod_{j=1}^{g} [x_j, y_j] \prod_{i=1}^{r} \alpha_i = 1$

(ii) $\gamma^d = 1$, $\gamma^b x = 1$, $x\gamma x^{-1} = \gamma^d x$ $\forall x \in \pi_1(\Delta')$

(iii) $[\alpha_i, \gamma] = 1$, $1 \leqslant i \leqslant r$.

(iv) $\alpha_i^{m_i} = \gamma^{m_i}$ with $n_i = 0$ if $m_i = 1$, $1 \leqslant i \leqslant r$.

Let $\tilde{S} \xrightarrow{\pi} \Delta$ be the elliptic fibration as in Lemma B (replacing all the multiple fibres of π by simple singular fibres). If $\tilde{B}$ denotes the subgroup of $\pi_1(E)$ generated by $\tilde{\beta}_i (1 \leqslant i \leqslant s)$, then from Lemma B, it follows that $B = \tilde{B}$. Since $\tilde{\pi}$ has no multiple fibres, in the presentation $\pi_1(\tilde{S})$, $\alpha_i = 1$ for $1 \leqslant i \leqslant s$) Thus $\pi_1(\tilde{S})$ is the group

$$\langle \gamma, x_j, y_j, 1 \leqslant j \leqslant g / \prod_{j=1}^{g} [x_j, y_j] = 1, \gamma^d = \gamma^b x = 1,$$

$$x\gamma x^{-1} = \gamma^d x, \forall_x \in \pi_1(\Delta') \rangle$$

If H is the subgroup of integers generated by the integers $d, b_x, d_x - 1$ for $x \in \pi_1(\Delta)$, then it follows that

$$H_1(\tilde{S}, \mathbb{Z}) \approx \mathbb{Z}/_H \oplus H_1(\Delta, \mathbb{Z}).$$

Now $\tilde{\pi}$ has at least one singular fibre of type $_1I_h$, so the sheaf $\mathcal{G}$ on $\tilde{S}$ constructed by taking the 1st homology groups (co-efficients in $\mathbb{Z}$) of regular fibres of $\tilde{\pi}$ is nonconstant. By a result of Kodaira, this implies that $b_1(\tilde{S})$ is even; see [4, Theorem 11.8]. But by Lemma D, $H_1(\tilde{S}, \mathbb{Z})$ is torsion free. Thus $\mathbb{Z}/H = (0)$. If one of the integers b_x is odd, then clearly the image of $\pi_1(E)$ in $\pi_1(S)$ is a cyclic group of odd order. If b_x is even for all $x \in \pi_1(\Delta')$, then d_x is odd for all x since $a_x d_x - b_x c_x = \pm 1$. But then $d_x - 1$ is even for all x. In this case $\mathbb{Z}/_H$ cannot be trivial unless d is odd. Thus I is a cyclic group of odd order.

If $\Delta \approx \mathbb{P}^1$, then $b_x = 0 = d_x - 1$ for all $x \in \Gamma$, α_t commute with γ. Thus H is generated by d. But then $d = 1$ otherwise $H_1(\tilde{S}, \mathbb{Z})$ will have either odd rank or non-trivial torsion, which is not possible. This completes the proof of the theorem.

§3. Applications

THEOREM 2: *Let S be an irreducible non-singular, projective surface/$\mathbb{C}$ with an elliptic fibration $S \xrightarrow{\pi} \Delta$.*

  (*i*) *If π has at least one singular fibre not of the type mI_0, then any unramified covering of S is holomorphically convex.*
  (*ii*) *In general, the universal covering space $\tilde{S}$ of S is holomorphically convex.*

REMARK: We will give an example of an abelian surface $S \approx E_1 \times E_2$ with E_i elliptic curves such that S has an infinite sheeted unramified covering $\bar{S}$ with no non-constant holomorphic functions; in particular $\bar{S}$ is not holomorphically convex. We shall need the following:

LEMMA E: *Suppose $\Delta = \mathbb{P}^1$ and π has at most two singular fibres and these are of type mI_0. Then S is birationally a ruled surface and hence its universal covering $\tilde{S}$ is holomorphically convex.*

PROOF: Let $\pi^*(a_i)$ be a singular fibre of type $m_i I_0$, $i = 1, 2$. For the canonical bundle K_S of S, Kodaira has proved the formula

$$K_S \approx \pi^*\left({}_{\mathbb{P}^1}\left(-2 + \chi(S, \mathcal{O})\right)\right) \otimes \left[P_{a_1}\right]^{\otimes(m_1-1)} \otimes \left[P_{a_2}\right]^{\otimes(m_2-1)}$$

Here P_{a_i} is the divisor such that $\pi^*(a_i) = m_i P_i$ and $[P_{a_i}]$ is the corresponding complex-analytic line bundle.

By Noether's formula $\chi(S, \mathcal{O}) = (K_S^2 + c_2(S))/12$. But $K_S^2 = 0$ for our elliptic surface and $c_2(S)$ is equal to the sum of the topological Euler-characteristics of all the singular fibres of π. Thus $c_2(S)$ is also 0 and $\chi(S, \mathcal{O}) = 0$.

Since any two points in $\mathbb{P}^1$ are rationally equivalent, we see that

$$\pi^*\left({}_{\mathbb{P}^1}(1)\right) \approx \left[P_{a_1}\right]^{\otimes m_1} \approx \left[P_{a_2}\right]^{\otimes m_2}$$

We see easily that $K_S \approx [P_{a_1}]^{-1} \otimes [P_{a_2}]^{-1}$, thus forcing $|nK_S| = \emptyset$ for $n \geqslant 1$. This means that S is a birationally ruled surface.

Similar argument shows that when π has at most one singular fibre (and that too of type mI_0), S is a birationally ruled surface. To see that $\tilde{S}$, the universal cover of S is holomorphically convex we can assume that S is a relatively minimal model with a $\mathbb{P}^1$-bundle $S \xrightarrow{\Psi} \Delta$. Then $\Delta \neq \mathbb{P}^1$ since

$\chi(S, \mathcal{O}) = 0$. Pulling back the fibration Ψ to the universal cover $\tilde{\Delta}$ of Δ we see that $S \times_\Delta \tilde{\Delta}$ is complex analytically isomorphic to $\tilde{\Delta} \times \mathbb{P}^1$, which is holomorphically convex.

PROOF OF THEOREM 2: Let $\sigma: W \to \Delta$ be the ramified covering with Γ as the group of analytic automorphisms such that $W/\Gamma \approx \Delta$. (In particular W is simply connected). For any subgroup Γ_1 of Γ, the pull back fibration $W/\Gamma_1 \times_\Delta S$ over W/Γ_1, yields an elliptic fibration $X_{\Gamma_1} \to W/\Gamma_1$ after normalization, such that $X_{\Gamma_1} \to S$ is the unramified covering corresponding to the subgroup $\pi_1(X_{\Gamma_1}) = \varphi^{-1}(\Gamma_1)$ of $\pi_1(S)$.

To prove (i) let H be any subgroup of $\pi_1(S)$. Put $\Gamma_1 = \varphi(H)$. Then $IH = \varphi^{-1}(\Gamma_1)$. By theorem 1, I is finite and hence H is of finite index in IH. If $\tilde{S}_H$ is the covering of S with $\pi_1(\tilde{S}_H) = H$, then clearly $\tilde{S}_H \to X_{\Gamma_1}$ is a finite covering. Since $X_{\Gamma_1} \to W/\Gamma_1$ has compact fibres and W/Γ_1, being a Riemann surface, is holomorphically convex, it follows that $\tilde{S}_H$ is holomorphically convex.

To prove (ii), by (i) it suffices to consider the case when all singular fibres are of the type mI_0. By lemma E, we can further assume that either $g > 0$ or $g = 0$ and there are at least three singular fibres. But then one easily checks that $\mathrm{ord}(\alpha_i) = m_i$ in Γ. This implies that the ramification index of σ at $Q \in \sigma^{-1}(a_i)$ is precisely m_i. Hence the fibration $X_{(I)} \to W$ is non-singular. If W is noncompact then it is contractible. Hence by a theorem of Grauert (see [2]) $X_{(I)} \approx W \times E$, and hence $\tilde{S} \approx W \times \mathbb{C}$. If W is compact then $W \approx \mathbb{P}^1$ and again by lemma E, $X_{(I)}$ is a ruled surface and hence in any case $\tilde{S}$ is holomorphically convex. This completes the proof of theorem 2.

AN EXAMPLE: For any irrational number λ let

$$v_1 = (1, 0), \quad v_2 = (0, 1)$$

$$v_3 = \left(\frac{\log 2}{2\pi i}, \frac{\lambda \log 2}{2\pi i} \right), \quad v_4 = \left(\frac{-\log 2}{2\pi i}, \frac{\lambda \log 2}{2\pi i} \right)$$

$$v_3' = \left(\frac{\log 2}{\pi i}, 0 \right), \quad v_4' = \left(0, \frac{\lambda \log 2}{\pi i} \right)$$

be the vectors in $\mathbb{C} \times \mathbb{C}$. For any set A of elements in $\mathbb{C}$ (or $\mathbb{C} \times \mathbb{C}$) let $L[A]$ denote the additive subgroup of $\mathbb{C}$(or $\mathbb{C} \times \mathbb{C}$) generated by A. Then clearly

$$L[v_1, v_2, v_3', v_4'] = L[v_1, v_3'] \oplus L[v_2, v_4']$$

$$\simeq L\left[1, \frac{\log 2}{\pi i}\right] \times L\left[1, \frac{\lambda \log 2}{\pi i}\right]$$

If $E_1 = \mathbb{C}/L[1, \log 2/\pi i]$ and $E_2 = \mathbb{C}/L[1, \lambda \log 2/\pi i]$ are the elliptic curves then it follows that $S = E_1 \times E_2 \simeq \mathbb{C} \times \mathbb{C}/L[v_1, v_2, v_3', v_4']$. Since $L[v_1, v_2, v_3', v_4']$ is a subgroup of inxed 2 in $L[v_1, v_2, v_3, \cdot v_4]$ it follows that S is a double cover of $X = \mathbb{C} \times \mathbb{C}/L[v_1, v_2, v_3, v_4]$. Let $W = \mathbb{C} \times \mathbb{C}/L[v_1, v_2, v_3]$. Then $W \to X$ is an infinite cyclic cover. Pulling back this via the double covering $S \to X$ yields an infinite cyclic cover $\bar{S} \to S$.

We claim that $\bar{S}$ admits no nonconstant holomorphic function. Since $\bar{S}$ is a double cover of W it suffices to show that W does not admit any nonconstant holomorphic function. Since W is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*/\mathbb{Z}$ where $\mathbb{Z} = (g)$ acts on $\mathbb{C}^* \times \mathbb{C}^*$ via,

$$g(z_1, z_2) = (2z_1, 2\lambda z_2)$$

it follows that a holomorphic function f on W is given by a holomorphic function $\tilde{f}$ on $\mathbb{C}^* \times \mathbb{C}^*$ invariant under the $\mathbb{Z}$-action. Let

$$\tilde{f}(z_1, z_2) = \sum q_{ij} z_1^i z_2^j$$

be the Laurent series for $\tilde{f}$. It follows that $a_{ij} = a_{ij} 2^{i+\lambda j}$ for every (i, j). Thus if $a_{ij} \neq 0$ then $i + \lambda j = 0$; since λ is irrational this means $(i, j) = (0, 0)$. Thus $\tilde{f}$ and hence f is a constant.

As a corollary of Theorem 2 we prove the

PROPOSITION: *Let S be an irreducible, non-singular projective surface with an elliptic fibration $S \xrightarrow{\pi} \Delta$. Let $C \subset S$ be an irreducible, complete curve with $C^2 > 0$. Assume that $\chi(S, \mathcal{O}) > 0$. Then $[\pi_1(S): \mathrm{Im}\, \pi_1(\bar{C})] < \infty$, where $\bar{C} \to C$ is the non-singular model of C.*

CROLLARY: *If C is rational, then $\pi_1(S)$ is finite.*

REMARKS:
   (1) If C is rational, we will show later that $q(S) = 0$ and hence $\chi(S, \mathcal{O}) > 0$.
   (2) M.V. Nori has given an example of an elliptic fibration $S \to \Delta$ with $\chi(S, \mathcal{O}) = 0$ and an irreducible curve $C \subset S$ with $C^2 > 0$ such that $[\pi_1(S): \mathrm{Im}\, \pi_1(C)] = \infty$.
   (3) The arguments in the proof of the proposition show that if we delete the condition $C^2 > 0$, then we can still conclude that $[\mathrm{Im}\, \pi_1(C); \mathrm{Im}\, \pi_1(\bar{C})] < \infty$ in $\pi_1(S)$. Further, if C is a connected chain of rational curves then we can conclude that $\mathrm{Im}\, \pi_1(C)$ is finite in $\pi_1(S)$.

PROOF OF THE PROPOSITION: By the argument at the end of Proof of Lemma E $\chi(S, \mathcal{O}) > 0$ implies that π has at least one singular fibre not of

the type mI_0. Let $H = \operatorname{Im} \pi_1(\overline{C}) \subset \pi_1(S)$. Consider the covering $\tilde{S} \xrightarrow{\varphi} S$ such that $\varphi_{\#}\pi_1(\tilde{S}) = H$. By Theorem 2, $\tilde{S}$ is holomorphically convex. Since $C^2 > 0$, it can be shown that $\pi_1(C) \to \pi_1(S)$ is surjective. See [8] for a proof. Hence $\varphi^{-1}(C)$ is a connected curve on $\tilde{S}$. Also, by construction, $\overline{C} \to S$ lifts to $\overline{C} \to \tilde{S}$. Let $\varphi^{-1}(C) = \bigcup_{i=1}^{r} C_i$ where $C_1, \ldots,$ are the irreducible components of $\varphi^{-1}(C)$. Corresponding to each C_i, $\exists$ a unique lift of the map $\overline{C} \to S$ to $\overline{C} \to \tilde{S}$ which has C_i as the image of $\overline{C}$. Hence each C_i is compact. Choose points $x_i \in C_i$. If the set $\{x_1, x_2, \ldots\}$ is infinite, it has no limit point. In this case $\exists$ a holomorphic function f such that $f(x_n)$ is unbounded as $n \to \infty$. But since each C_i is compact and $\cup C_i$ is connected, f has to be constant on $\cup C_i$. This means r is finite and the covering $\tilde{S} \to S$ is of finite degree i.e. $[\pi_1(S): \operatorname{Im} \pi_1(\overline{C})] < \infty$.

For the proof of the corollary, we need the following.

LEMMA: *If S is a non-singular, irreducible, projective surface and $C \subset S$ an irreducible, complete, rational curve with $C^2 > 0$, then $q(S) = 0$.*

PROOF: There are several proofs of this result. We will give one due to M.P. Murthy.

Let $S \xrightarrow{\Psi} Alb\ S$ be the morphism from S to its Albanese variety. Clearly $\Psi(C)$ is a point. If the image of S in $AlB\ S$ is a surface V, then by the negative definiteness of the intersection form on the inverse image of a point of V, C^2 will have to be negative, which is not true. If $\varphi(S)$ is a curve, then the intersection form on any fibre of the map $S \to \Psi(S)$ can be seen to be negative semi-definite. This shows that $Alb\ S$ is a point i.e. $q(S) = 0$.

REMARK:

(1) From the classification of algebraic surface, we see now that for any surface of special type, the universal covering space is holomorphically convex (because for ruled, rational, $K - 3$, abelian surfaces this is easy to verify and all Enriques surfaces are known to have elliptic fibrations).

(2) In the above Proposition, when C is rational, it can be shown that the only possible fundamental groups of S are finite cyclic group, dihedral group of $2n$ elements, tetrahedral group with 12 elements, octahedral group with 24 elements or icosahedral group on 60 elements.

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