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J. W. SANDS

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ABELIAN FIELDS AND THE BRUMER-STARK CONJECTURE

J.W. Sands

Stark's conjectures [11] envision an elegant arithmetic interpretation of the values of Artin L-functions at $s = 1$. In [8], Stark considered abelian $L(s, \chi)$ with a first order zero at $s = 0$ and formulated a refinement of his general conjecture for that case. Considering instead L-functions having first order zeroes due to the imprimitivity of the characters χ leads to an analogous conjecture discovered by Stark [9], Tate [10] and Gross [2]. Tate dubbed this the Brummer-Stark conjecture, observing that it refines an earlier (unpublished) conjecture of Brumer [6] and generalizes Stickelberger's theorem. Until now, this classical theorem directly accounted for the most general family of cases where the conjecture had been proved, namely finite abelian field extensions $K/\mathbb{Q}$. In theorem (2.1) we enlarge this family to include all extensions K/k where K is abelian over the rationals and k is any base field. Our result handles exactly a situation of special interest singled out by Tate. The constructive proof is quite elementary, and entirely algebraic. A second theorem, (2.2), notes that the construction produces Jacobi-sum Hecke characters. Weil's theorem [12] plays a major role in both proofs. Another feature of our method is that we deduce the integrality results of Deligne and Ribet [1] for all field extensions under consideration from the knowledge of the Hurwitz zeta function.

1. The conjecture

To develop the Brumer-Stark conjecture, we use Tate's excellent article [10] as a guide.

Initially, let K/k be an arbitrary finite abelian extension of number fields with Galois group $\mathscr{G}$. The character group of $\mathscr{G}$ is denoted $\hat{\mathscr{G}}$ and we fix a finite set of primes S of k. S is assumed to have cardinality at least 2, and to include all of the infinite primes as well as the primes which ramify in K. For χ in $\hat{\mathscr{G}}$, $L_S(s, \chi) = L_S(s, \chi, K/k)$ is the Artin L-function with Euler factors for primes in S removed.

When $\mathfrak{a}$ is an ideal of k relatively prime to each of those in S, we write $(\mathfrak{a}, S) = 1$ and denote by $\sigma_{\mathfrak{a}}$ the corresponding element of $\mathscr{G}$. Then to each σ in $\mathscr{G}$ is associated a partial zeta function

$$\zeta_S(s,\sigma)=\sum_{\substack{(\mathfrak{a},S)=1\\ \sigma_{\mathfrak{a}}=\sigma}} N\mathfrak{a}^{-s},$$

so that

$$L_S(s,\chi)=\sum_{\sigma\in\mathscr{G}}\chi(\sigma)\zeta_S(s,\sigma).$$

Then the Stickelberger element $\theta=\theta_S(K/k)=\sum_{\sigma\in\mathscr{G}}\zeta_S(0,\sigma)\sigma^{-1}$ is defined and lies in $\mathbb{Q}[\mathscr{G}]$.

If $\alpha=\sum_{\sigma}c_\sigma\sigma$ is in the group ring $\mathbb{C}[\mathscr{G}]$, we set $\chi(\alpha)=\sum_{\sigma\in\mathscr{G}}\chi(\sigma)c_\sigma$. Then θ is characterized by the property that

$$\bar{\chi}(\theta)=L_S(0,\chi)\text{ for all }\chi\textit{ in }\hat{\mathscr{G}}. \tag{1.1}$$

Also, if w_K is the order of the group μ_K of roots of unity in K, then $w_K\theta$ lies in $\mathbb{Z}[\mathscr{G}]$ by the theorem of Deligne and Ribet. The last ingredient for Tate's statement of the conjecture is the group K° of elements of K having absolute value 1 at all infinite primes of K.

1.2. Conjecture $BS(K/k,S)$: *For each ideal $\mathfrak{a}$ of K, there exists an element $\varepsilon(\mathfrak{a})$ in K such that*

(*a*) $\varepsilon(\mathfrak{a})$ *is in* K^{o}

(*b*) $(\varepsilon(\mathfrak{a}))=\mathfrak{a}^{w_K\theta}$

(*c*) $K\left(\varepsilon(\mathfrak{a})^{1/w_K}\right)$ *is an abelian Galois extension of k.*

Condition b) is Brumer's conjecture; a) and c) constitute Stark's refinement. Note that a) and b) specify $\varepsilon(\mathfrak{a})$ up to an element of μ_K, and multiplying $\varepsilon(\mathfrak{a})$ by such an element also preserves the verity of c). The set of ideals $\mathfrak{a}$ for which $\varepsilon(\mathfrak{a})$ exists clearly form a group, so one may in fact restrict attention to, for instance, the set of prime ideals of K. We point out that unless K is totally complex and k is totally real, the conjecture holds trivially since $\theta=0$ by (1.1).

In [10], Tate offers some useful reformations of the conjecture. Of special interest to us is the following proposition involving the annihilator

$$\mathrm{Ann}_{\mathbb{Z}[\mathscr{G}]}(\mu_K)\quad\text{of}\quad\mu_K\quad\text{in}\quad\mathbb{Z}[\mathscr{G}].$$

1.3. Proposition: *Given an ideal $\mathfrak{a}$ of K there exists an $\varepsilon(\mathfrak{a})$ satisfying* (1.2) *a*), *b*), *and c*) *if and only if there is an assignment* $\alpha\to a_\alpha=a_\alpha(\mathfrak{a})$

from $\mathrm{Ann}_{\mathbf{Z}[\mathscr{G}]}(\mu_K)$ *to* K^o *such that* $(a_\alpha) = \mathfrak{a}^{\alpha\theta}$ *and* $a_\alpha^\gamma = a_\gamma^\alpha$ *for all* α *and* γ *in* $\mathrm{Ann}_{\mathbf{Z}[\mathscr{G}]}(\mu_K)$. *Then* $\varepsilon(\mathfrak{a}) = a_{w_K}(\mathfrak{a})$.

For a principal ideal $\mathfrak{a} = (\delta)$ one can set $a_\alpha = \delta^{\alpha\theta}$ and verify the conditions in (1.3). Thus to prove the conjecture it suffices to consider a set of ideals which generate the ideal class group of K. In particular, we will later choose the set of ideals relatively prime to a fixed modulus.

Our results are closely tied to the relations among cases of the conjecture. As a primary example of such relations, we mention (but do not use) the following.

1.4. PROPOSITION (Tate): *If* $K \supset K' \supset k$ *is a tower of fields then* $BS(K/k, S)$ *implies* $BS(K'/k, S)$.

To complement (1.4) one would like to know that $BS(K/k, S)$ implies $BS(K/K', S')$ when S' denotes the set of primes of K' dividing those in S. However, the only currently available application of such a result (or of (1.4) for that matter) would be to the specific situation handled in this paper. A weaker version of the general statement is proved in the author's thesis [7]. At present we find it instructive to work directly with the simple and illustrative special case.

2. Statements of theorems

Fixing a positive integer f, we let μ_f be the group of fth roots of unity and $K(f) = \mathbf{Q}(\mu_f)$. So $w_f = w_{K(f)}$ is f or $2f$, according as f is even or odd. We henceforth assume that K is contained in $K(f)$, k is a totally real subfield of K, and S is the set of primes of k which are either infinite or divisors of f. The following principal results are proved in Section 6.

2.1. THEOREM: $BS(K/k, S)$ *is true. For each ideal* $\mathfrak{a}$ *of* K *relatively prime to* $2f$, $\varepsilon(\mathfrak{a})$ *is given explicitly as a product of powers of Gauss sums.*

2.2. THEOREM: *The explicit assignment* $\mathfrak{a} \to \varepsilon(\mathfrak{a})$ *is a Hecke character with defining ideal* $(w_f)^2$.

3. Stickelberger elements

Let $G(f) = \mathrm{Gal}(K(f)/\mathbf{Q})$, $\bar{G}(f) = \mathrm{Gal}(K/\mathbf{Q})$, $G = \mathrm{Gal}(K(f)/k)$, and $\bar{G} = \mathrm{Gal}(K/k)$.

$$G(f)\left\{\begin{array}{l} K(f) \\ G\left\{\begin{array}{c} | \\ \left.\begin{array}{c} K \\ | \\ k \end{array}\right\}\bar{G} \end{array}\right. \\ \qquad | \\ \qquad \mathbf{Q} \end{array}\right. \qquad \bar{G}(f) = \left\{\begin{array}{c} K \\ | \\ \mathbf{Q} \end{array}\right.$$

Fixing $S(f)$ as the set consisting of the infinite prime of $\mathbb{Q}$ and the primes dividing f, we put $\theta_f = \theta(K(f)/\mathbb{Q}, S(f))$, $\theta = \theta(K(f)/k, S)$, and $\bar{\theta} = \theta(K/k, S)$. Via the characterizing property (1.1) of Stickelberger elements, the standard induction and inflation identities for Artin L-functions imply relations among θ_f, θ, and $\bar{\theta}$. These relations are captured by certain maps between the appropriate group rings.

First we extend the quotient map $\pi: G \to \bar{G}$ to a group ring homomorphism $\pi: \mathbb{C}[G] \to \mathbb{C}[\bar{G}]$. That $\bar{\theta} = \pi(\theta)$ follows immediately from the principles just mentioned. Consequently, we may denote the extended map by $\pi(\alpha) = \bar{\alpha}$ from now on.

Because $\mathbb{C}[G(f)]$ is a free module of finite rank $\mathrm{r} = (G(f):G)$ over $\mathbb{C}[G]$, a norm map is defined between the two. For $\alpha = \sum_{\sigma \in G(f)} c_\sigma \sigma$ in $\mathbb{C}[G(f)]$, $N\alpha$ in $\mathbb{C}[G]$ is the determinant of multiplication by α, regarded as a $\mathbb{C}[G]$-linear transformation on $\mathbb{C}[G(f)]$. Fixing a set of coset representatives $\{\gamma_i : i = 1, 2, \ldots, \mathrm{r}\}$ for $G(f)/G$ as a basis of $\mathbb{C}[G(f)]$ over $\mathbb{C}[G]$, one obtains matrix entries of the form $\sum_{\sigma \in G} c_{\sigma\rho}\sigma$, with ρ in $G(f)$. We define $\alpha(\rho) = \sum_{\sigma \in G} c_{\sigma\rho}\sigma\rho$, which depends only on the coset $\rho^* = \rho G$, and then $N\alpha = \det[\alpha(\gamma_i^{-1}\gamma_j)\gamma_j^{-1}\gamma_i]$. Multiplying the ith row of the matrix by γ_i^{-1} and the jth column by γ_j for every i and j preserves the determinant, so that $N\alpha = \det[\alpha(\gamma_i^{-1}\gamma_j)]$. In particular, $N\theta_f = \det A$ upon defining $A = \theta_f(\gamma_i^{-1}\gamma_j)$.

We would like an interpretation of θ involving characters χ of $G(f)$. For such a χ, and an arbitrary α as before, define $\chi[\alpha] = \sum_{\sigma \in G} c_\sigma \chi(\sigma)\sigma$. Also let χ_0 denote the principal character.

3.1. PROPOSITION

(a) $$\theta = \prod_{\substack{\chi \in \hat{G(f)} \\ \chi(G)=1}} \chi[\theta_f] = N\theta_f$$

(b) Define $$\beta = \prod_{\substack{\chi \neq \chi_0 \\ \chi(G)=1}} \chi[\theta_f].$$

Then $\theta = \beta\theta_f$ and β is the determinant of the matrix B obtained from A by substituting 1 *for every entry in the last row.*

PROOF:

(a) The characterizing property (1.1) of θ reduces the first identity to the induction property of Artin L-functions.

From representation theory for $G(f)/G$, we have a formal identity in the indeterminates $y(\rho^*)$ indexed by the elements of this group:

$$\det_{\gamma^*,\rho^*}\left[y(\gamma^{*-1}\rho^*)\right] = \prod_{\chi\in\widehat{G(f)/G}}\left[\sum_{\rho^*\in G(f)/G}\chi(\rho^*)y(\rho^*)\right]. \tag{3.2}$$

Upon substituting $\theta(\rho)$ for $y(\rho^*)$ one obtains the second identity.

(b) That $\theta = \beta\theta_f$ is clear from a) and the definition of β. To show that $\beta = \det B$, we start with identity (3.2). Note that when $\chi = \chi_0$, the corresponding factor on the right is just $\sum_{\rho^*} y(\rho^*)$. The determinant on the left is unchanged upon adding all rows of $[y(\gamma^{*-1}\rho^*)]$ to the last row, which then has every entry also equal to $\sum_{\rho^*} y(\rho^*)$. Removing this factor from the entire row and from the product on the right yields an identity which gives the result upon substituting $\theta(\rho)$ for $y(\rho^*)$ again.

4. Integrality properties

The values at $s = 0$ of partial zeta functions possess integrality properties which are aptly expressed in terms of Stickelberger elements and the annihilation of roots of unity by group ring elements. In general, this follows from the theorem of Deligne and Ribet [1], but our methods yield a much more elementary proof for the case of absolutely abelian fields K/k. The proof parallels the derivation of similar integrality properties for the element β of the last section. Both sets of properties are essential to the proof of the conjecture, and we find it natural to develop them simultaneously.

Our starting point is the formula for θ_f known by the classical results of Hurwitz [3]. When $(b, f) = 1$, σ_b in $G(f)$ is defined as usual by $\omega^{\sigma_b} = \omega^b$ for ω in μ_f. Then

$$\theta_f = \frac{1}{f}\sum_{\substack{b=1\\(b,f)=1}}^{f}\left(\frac{f}{2} - b\right)\sigma_b^{-1}.$$

4.1. Proposition: *If α is in $\mathrm{Ann}_{\mathbb{Z}[G(f)]}(\mu_{K(f)})$, then $\alpha\theta_f$ lies in $\mathbb{Z}[G(f)]$.*

Proof: This is easily checked for $\alpha = w_f$ and α of the form $\sigma_b - b$, with $(b, w_f) = 1$. Such elements generate $\mathrm{Ann}_{\mathbb{Z}[G(f)]}(\mu_{k(f)})$ as an additive group.

4.2. PROPOSITION: *If α is in* $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_{K(f)})$, *then $\alpha\theta$ and $\alpha\beta$ lie in* $\mathbb{Z}[G(f)]$. *In particular* $\beta \in \frac{1}{w_f}\mathbb{Z}[G(f)]$.

PROOF: Again let A be the matrix $\theta_f(\gamma_i^{-1}\gamma_j)$ and obtain B by replacing every entry in the last row of A by 1. So $\theta = \det A$ and $\beta = \det B$ as in (3.1). We may assume for convenience that $\gamma_1 = \sigma_1$, and for each i choose b_i relatively prime to w_f such that $\gamma_i = \sigma_{b_i}$.

The determinant of A (resp. B) is unchanged if for each i, $1 < i \leq r$ (resp. $1 < i \leq r-1$), we subtract $b_i\sigma_{b_i}^{-1}$ times the first row from the ith row to produce a new matrix we call A' (resp. B'). The new ith row then has entries in $\mathbb{Z}[G(f)]$ by (4.1). Indeed, this statement is equivalent to the fact that $(1 - b_i\sigma_{b_i}^{-1})\theta_f$ lies in $\mathbb{Z}[G(f)]$. To conclude that $\alpha\theta = \alpha \det A'$ and $\alpha\beta = \alpha \det B'$ lie in $\mathbb{Z}[G(f)]$, we multiply the first row $\theta_f(\gamma_j)$ of A′ and B′ by α and observe that we again obtain entries in $\mathbb{Z}[G(f)]$. This fact simply restates (4.1) in the case where α lies in $\mathbb{Z}[G]$.

Note that the first $r-1$ rows of A and B each contain a factor of $1 - \sigma_{-1}$. Consequently, one could show that all of our results remain valid with θ, β, $\bar{\theta}$ and $\bar{\beta}$ replaced by $(1/2^{r-2})\theta$, $(1/2^{r-2})\beta$, $(1/2^{r-2})\bar{\theta}$, and $(1/2^{r-2})\bar{\beta}$. We leave the details to the interested reader.

4.3. PROPOSITION: *If λ is in* $\mathrm{Ann}_{\mathbb{Z}[\bar{G}]}(\mu_K)$, *then $\lambda\bar{\theta}$ and $\lambda\bar{\beta}$ lie in* $\mathbb{Z}[\bar{G}(f)]$.

PROOF: It suffices to prove the proposition for $\lambda = w_K$ and $\lambda = \bar{\sigma}_b - b$, since these generate $\mathrm{Ann}_{\mathbb{Z}[\bar{G}]}(\mu_K)$, as an additive group. The case of $\lambda = \bar{\sigma}_b - b$ follows from (4.2). To handle the remaining case of $\lambda = w_K$, we introduce the corestriction map

$$\mathrm{cor}: \mathbb{Z}[\bar{G}(f)] \to \mathbb{Z}[G(f)].$$

This homomorphism of additive groups is defined by the property that

$$\mathrm{cor}(\bar{\delta}) = \delta \sum_{\bar{\sigma}_b = \bar{\sigma}_1} \sigma_b \qquad \text{for} \quad \delta \text{ in } G(f);$$

the equation then holds for all δ in $\mathbb{Z}[G(f)]$. Clearly $w_K\bar{\theta}$ and $w_K\bar{\beta}$ lie in $\mathbb{Z}[\bar{G}(f)]$ if and only if their corestrictions $w_K\theta\Sigma\sigma_b$ and $w_K\beta\Sigma\sigma_b$ lie in $\mathbb{Z}[G(f)]$. But the last condition follows from (4.2) once we verify that $w_K\Sigma\sigma_b$ is in $\mathrm{Ann}_{\mathbb{Z}[G(f)]}(\mu_{K(f)})$. For ω in $\mu_{K(f)}$ we have $\omega^{\Sigma\sigma_b} = N_K\omega$, which is in μ_K. Thus $\omega^{w_K\Sigma\sigma_b} = (N_K\omega)^{w_K} = 1$ and the proof is complete

5. Gauss sums

The solution of the conjecture for $K(f)/\mathbb{Q}$ is provided by Gauss sums. We review the facts which play a role in this and our more general result. For more details, see Lang [5].

Fixing a prime ideal $\mathfrak{P}$ of $K(f)$, we let the absolute norm be $N\mathfrak{P} = q = p^s$, where p is a rational prime. The power residue symbol $\chi_{\mathfrak{P}} = \chi_{\mathfrak{P},f}: (\mathfrak{O}_{K(f)}/\mathfrak{P})^* \to \mu_f$ is defined when p is relatively prime to f:

$$\chi_{\mathfrak{P}}(u) \equiv u^{(q-1)/f} \bmod \mathfrak{P}.$$

Letting Tr denote the trace to $\mathbb{Z}/p\mathbb{Z}$, and letting a be an element of $\mathbb{Z}$ or $\mathbb{Z}/f\mathbb{Z}$, the Gauss sum $J_f(a, \mathfrak{P}) = -\sum_{u(\bmod \mathfrak{P})} \chi_{\mathfrak{P}}^a(u)\, e^{2\pi i \mathrm{Tr}(u)/p}$ lies in $K(fp)$, and $J_f(a, \mathfrak{P})^f$ lies in $K(f)$. Extending σ_t in $G(f)$ to be the identity on $K(p)$ we obtain an element $\tilde{\sigma}_t$ in $\mathrm{Gal}(K(fp)/\mathbb{Q})$. Then

$$J_f(a, \mathfrak{P})^{\tilde{\sigma}_t} = J_f(a, \mathfrak{P}^{\sigma_t}) = J_f(at, \mathfrak{P}). \tag{5.1}$$

For an arbitrary ideal $\mathfrak{A}$ of $K(f)$, relatively prime to f, the generalized Gauss sum $J_f(a, \mathfrak{A})$ of Weil is defined by multiplicativity in $\mathfrak{A}$ and lies in $K(fN\mathfrak{A})$. Furthermore, when $\mathfrak{a}$ is an ideal of the subfield K, $(\mathfrak{a}, f) = 1$, we let $J_f(a, \mathfrak{a}) = J_f(a, \mathfrak{a}\mathcal{O}_{K(f)})$. Using the transformation property (5.1), one then finds that $J_f(a, \mathfrak{a})^{w_K}$ lies in K and $J_f(a, \mathfrak{a})$ lies in $K(\mu_{N\mathfrak{a}})$. For the absolute value, one has $|J_f(a, \mathfrak{A})|^2 = N\mathfrak{A}$, whenever $a \not\equiv 0 \pmod f$.

Using the Jacobi symbol, we let

$$g_f(\mathfrak{A}) = J_f(1, \mathfrak{A}) \Big/ \left[\left(\frac{-1}{N\mathfrak{A}}\right) N\mathfrak{A}\right]^{1/2}$$

in $K(fN\mathfrak{A})$ and $g_f(\mathfrak{a}) = g_f(\mathfrak{a}\mathcal{O}_{K(f)})$.

5.2. Proposition: *$BS(K(f)/\mathbb{Q}, S(f))$ is true with $\varepsilon(\mathfrak{A}) = \varepsilon_f(\mathfrak{A}) = g_f(\mathfrak{A})^{w_f}$ for $\mathfrak{A}$ relatively prime to f.*

Proof: For the given $\varepsilon_f(\mathfrak{A})$, (1.2) *a*) and *c*) follow from the above discussion, and (1.2) *b*) is precisely Stickelberger's theorem. By the remarks accompanying (1.3), this suffices to establish the conjecture.

6. Jacobi-sum Hecke characters

Now we exploit the relations between Stickelberger elements to pass from $BS(K(f)/\mathbb{Q}, S(f))$ to $BS(K/k, S)$. The Gauss sums in the solution of the first precipitate Jacobi-sum Hecke characters in the second, as the integrality results perfectly afford the application of Weil's theorem [12,], [13].

We use Kubert and Lichtenbaum's [4] statement of this last theorem. For $K \subset K(f)$ as always, we let $I_K(f)$ be the group of ideals of K relatively prime to f. We also use $[b]$ to denote the least non-negative representative (mod f) of the integer b.

6.1. THEOREM: *For $r_1, r_2, \ldots, r_{f-1}$ in $\mathbb{Z}$, define χ on $I_K(f)$ by*

$$\chi(\mathfrak{a}) = \prod_{a=1}^{f-1} J_f(a, \mathfrak{a})^{r_a}.$$

Then χ is a Hecke character if and only if the element γ in $\mathbb{Q}[\bar{G}(f)]$ defined by

$$\gamma = \frac{1}{f} \sum_{a=1}^{f-1} r_a \sum_{\substack{t=1 \\ (t,f)=1}}^{f} [-at]\bar{\sigma}_t^{-1}$$

lies in $\mathbb{Z}[\bar{G}(f)]$. When this occurs, χ has defining ideal (f^2) and infinity type γ, and takes values in K.

In order to consider the $g_f(\mathfrak{A})$ collectively, we let $K(f)^{\perp} = \bigcup_{(n,f)=1} K(n)$. Then every $g_f(A)$ lies in $K(f)\cdot K(f)^{\perp}$ and we put $\tilde{G}(f) = \mathrm{Gal}(K(f)\cdot K(f)^{\perp}/\mathbb{Q})$. Inevitably then $\bar{G}(f)\tilde{} = \mathrm{Gal}/(K \cdot K(f)^{\perp}/\mathbb{Q})$ and we obtain monomorphisms

$$G(f) \to \tilde{G}(f) \quad \text{and} \quad \bar{G}(f) \to \bar{G}(f)\tilde{}$$

by extending elements to act as the identify on $K(f)^{\perp}$. Of course there are corresponding homomorphisms of group rings, denoted by $\rho \to \tilde{\rho}$.

Since $J_f(1, \mathfrak{a})^{\bar{\sigma}_a} = J_f(a, \mathfrak{a})$ when $(a, f) = 1$, we obtain a reformation of (6.1) which relates it to θ_f. For this let

$$\eta = \sum_{\substack{b=1 \\ (b,f)=1}}^{f-1} \sigma_b \quad in \quad \mathbb{Z}[G(f)].$$

6.2. COROLLARY: *Suppose $\lambda = \sum_{(a,f)=1} r_a\sigma_a$ is in $\mathbb{Z}[G(f)]$. Then $\chi(\mathfrak{a}) = J_f(1, \mathfrak{a})^{\tilde{\lambda}}$ is a Hecke character for K if and only if the element $\gamma = \bar{\lambda}(\bar{\theta}_f + \frac{1}{2}\bar{\eta})$ is in $\mathbb{Z}[\bar{G}(f)]$. In this event, χ has defining ideal (f^2) and infinity type γ, and takes values in K.*

6.3. THEOREM: *Fix α in $\mathrm{Ann}_{\mathbb{Z}[\bar{G}]}(\mu_K)$ and for each $\mathfrak{a}$ in $I_K(w_f)$ let $a_\alpha(\mathfrak{a}) = g_f(\mathfrak{a})^{\tilde{\alpha}\tilde{\bar{\beta}}}$. Then $\mathfrak{a} \to a_\alpha(\mathfrak{a})$ is a well-defined Hecke character for K with values in K and defining ideal (w_f^2).*

PROOF: Firstly, $\alpha\bar{\beta}$ is in $\mathbb{Z}[\bar{G}(f)]$ by (4.3) and $J_f(1, \mathfrak{a})$ and $[(\frac{-1}{N\mathfrak{a}^n})N\mathfrak{a}^n]^{1/2}$ are in $K(\mu_{N\mathfrak{a}})$ so that

$$g_f(\mathfrak{a})^{\tilde{\alpha}\tilde{\bar{\beta}}} = J_f(1, \mathfrak{a})^{\tilde{\alpha}\tilde{\bar{\beta}}} \Big/ \left(\left[\left(\frac{-1}{N\mathfrak{a}^n}\right)N\mathfrak{a}^n\right]^{1/2}\right)^{\tilde{\alpha}\tilde{\bar{\beta}}}$$

is well defined ($n = [K(f) : K]$). We claim that both the numerator and denominator are Hecke characters with defining ideal (w_f^2) and values in K.

The case of the numerator is covered by (6.2), for which we must show that $\alpha\bar{\beta}(\bar{\theta}_f + \frac{1}{2}\bar{\eta})$ is in $\mathbb{Z}[\bar{G}(f)]$. But $\alpha\bar{\beta}\bar{\theta}_f = \alpha\bar{\theta}$ by (3.1) and this is in $\mathbb{Z}[G(f)]$ by (4.3). For the remaining term $\frac{1}{2}\alpha\bar{\beta}\bar{\eta}$, there are two cases. If $\beta = 1$, one easily checks that this term lies in $\mathbb{Z}[\bar{G}(f)]$ when α is a standard generator. Otherwise we note that θ_f, and hence $\beta = \prod_{\substack{\chi(G)=1 \\ \chi \neq \chi_0}} \chi[\theta_f]$, lies in $(1 - \sigma_{-1})\mathbb{Q}[G(f)]$. (Recall that σ_{-1} is in G.). Thus $\beta\eta = 0$.

In this latter case $\tilde{\alpha}\bar{\tilde{\beta}}$ is in $(1 - \tilde{\sigma}_{-1})\mathbb{Z}[\bar{G}(f)]$. As $[(\frac{-1}{N_{\mathfrak{a}}^n})N_{\mathfrak{a}}^n]^{1/2}$ lies in $\mathbb{Q}(\mu_{N\mathfrak{a}})$, which is fixed by $\bar{G}(f)\tilde{}$, the denominator is identically 1. When $\beta = 1$, one checks that for α a standard generator, the denominator is a power of the Hecke character $(\frac{-1}{\mathrm{N}\mathfrak{a}})N\mathfrak{a}$, with defining ideal (4).

The next proposition, combined with (6.3), (1.3), and the accompanying remarks, establishes the theorems of section 2, with $\varepsilon(\mathfrak{a}) = a_{w_k}(\mathfrak{a})$.

6.4 Proposition: *For each* $\mathfrak{a}$ *in* $I_K(w_f)$, *and* α *and* γ *in* $\mathrm{Ann}_{\mathbb{Z}[\bar{G}]}(\mu_K)$,

a) $a_\alpha(\mathfrak{a})$ *is in* K^o.

b) $(a_\alpha(\mathfrak{a})) = \mathfrak{a}^{\alpha\bar{\theta}}$

c) $a_\alpha(\mathfrak{a})^\gamma = a_\gamma(\mathfrak{a})^\alpha$

Proof: Since $a_\alpha(\mathfrak{a})$ is in K by (6.3), a) and c) follow from the definition of $a_\alpha(\mathfrak{a})$ and the fact that the power $\varepsilon_f(\mathfrak{a}\mathcal{O}_{K(f)}) = g_f(\mathfrak{a})^{w_f}$ of $g_f(\mathfrak{a})$ is in K^o. As for b), it suffices to prove the equality when raised to the w_f power. Thus we can apply (5.2) and (3.1).

$$(a_\alpha(\mathfrak{a}))^{w_f} = \left(g_f(\mathfrak{a})^{w_f}\right)^{\alpha\bar{\beta}} = \left(\varepsilon_f(\mathfrak{a})\right)^{\alpha\bar{\beta}}$$

$$= \mathfrak{a}^{w_f\bar{\theta}_f\bar{\beta}\alpha} = \mathfrak{a}^{w_f\bar{\theta}\alpha} = \left(\mathfrak{a}^{\alpha\bar{\theta}}\right)^{w_f}.$$

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Department of Mathematics
University of Maine
Orono, ME 04469
USA