H. PINKHAM

Appendix to : "smoothings of cusp singularities via triangle singularities"

Compositio Mathematica, tome 53, n° 3 (1984), p. 317-324
<http://www.numdam.org/item?id=CM_1984__53_3_317_0>

[1] Compositio Mathematica 53 (1984) 317–324.

APPENDIX TO: "SMOOTHINGS OF CUSP SINGULARITIES VIA TRIANGLE SINGULARITIES"

H. Pinkham

The goal of this appendix is to prove the following slightly improved version of Theorems 1 and 4 of the preceding paper (which will be referred to as [FP], all other references being those of [FP]).

THEOREM: *Let D be a cusp singularity with $r \leqslant 3$ components and $m = r + 9$. Assume the dual cusp $\check{D}$ lies anticanonically on a rational surface. Then for every good embedding φ: $R \simeq T_{pqr} \hookrightarrow \Lambda$ except possibly the primitive embedding of $T_{3,6,12}$ there is a family $W \to \Delta$ as in [FP], Theorem 4, where the singularity D' of W_0 is in fact D. Furthermore $(-2, -5, -11)$ has a good, non primitive embedding.*

By the discussion following Theorem 4 of [FP] the result is already proved in all cases except $D = (-4, -5, -9)$ and $(-2, -5, -11)$. Before dealing with these cases we make some general remarks, valid for all $T_{p,q,r}$.

DEFINITION: Let T be an over lattice of T_{pqr}. Then the *geometric section* of the map $\mathrm{T} \to \mathrm{T}/\mathrm{T}_{\mathrm{pqr}}$ is the section σ: $T/T_{qqr} \to T$ which to $t \in T/T_{pqr}$ associates the unique $\sigma(t) \in T$ all of whose coefficients in the natural basis of T_{pqr} are $\geqslant 0$ and < 1. The *support* of $\sigma(t)$ consists of the sublattice of T_{pqr} spanned by the vertices where the coefficients of $\sigma(t)$ are non zero.

We will construct branched covers using $\sigma(t)$, whence the terminology.

DEFINITION: If φ: $R \hookrightarrow L$ is an embedding of lattices, then the primitive sublattice of L generated by R is called the *saturation* of R for φ.

We make the following hypothesis, which holds for all the overlattices considered in this note: (H) for every non-trivial $t \in T_{pqr}/T_{p',q',r'}$ the support of the geometric section $\sigma(t)$ has connected components with negative definite intersection matrix of type A_n. It seems that (H) holds for all overlattices of T_{pqr}, where $p + q + r = 21$, which is the case considered in this note, but J. Wahl has given an example of an overlattice T of $T_{4,7,9}$ for which it fails. Wahl shows that T is the saturation of a good embedding of $T_{4,7,9}$, so that Lemma 1 below fails without (H).

LEMMA 1: *Let φ: $T_{pqr} \hookrightarrow \Lambda$ be a good embedding in the sense of Looijenga, and let T be the saturation of T_{pqr} for φ. Assume hypothesis (H). Then for any non-trivial $t \in T/T_{pqr}$, the geometric section $\sigma(t)$ of t has square $\sigma(t)^2 \leqslant -4$.*

PROOF: Since φ is good there is a $K-3$ surface X with Picard group T and a T_{pqr} configuration ([FP], Lemma 2). Apply Riemann-Roch to $\sigma(t)$ considered as an element of Pic X. The hypotheses imply that $H^0(X, \mathcal{O}(\sigma(t)))$ and $H^2(X, \mathcal{O}(\sigma(t)))$ are trivial, so that the desired inequality is equivalent to the assertion that dim $H^1(X, \mathcal{O}(\sigma(t))) \geqslant 0$.

LEMMA 2: *Let V by a rational surface containing the minimal resolution of either the cusp D or the associated D_{pqr} singularity, and assume the orthogonal complement T_{pqr} of the components of the exceptional divisor contains a lattice $T_{p',q',r'}$ of finite index such that*

(i) *the vertices of the $T_{p',q',r'}$ are represented by $\mathbb{P}^{1'}$ s, so V has a $T_{p',q',r'}$ configuration, in the language of [FP].*

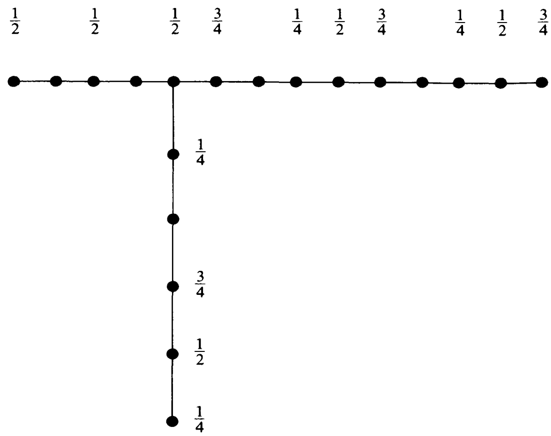
(ii) *hypothesis (H) holds.*

Then $\sigma(t)^2 = -2$ for every non trivial t, and the singularity is in fact the cusp if $T_{pqr}/T_{p',q',r'}$ is non-trivial.

PROOF: Contract the connected components of the support of $\sigma(t)$, considered as a fractional divisor on V, to rational double points. Call the resulting surface $\bar{V}$. As in Nikulin, "Finite automorphism groups of Kähler K-3 surfaces" (Trans. Moscow Math. Soc. 38(2), (1980)), §8.2, we can construct using $\sigma(t)$ a cyclic cover Z of $\bar{V}$ ramified only above these rational double points and with Z smooth above the double points. The dualizing sheaf on $\bar{V}$ is anti-effective, so that on Z is too. If the resolution on V is that of D_{pqr}, then π_1 of the exceptional locus is trivial so that D_{pqr} splits into several isomorphic singularities in the cover, if the cover is non trivial. By the classification of singular algebraic surfaces with trivial canonical divisor due to J.Y. Mérindol (C.R.A.S. Paris 293, 417-420 (1981), théorème 1.4) this cannot occur. So the resolution on V is that of D (where the exceptional locus has $\pi_1 \simeq \mathbb{Z}$). By Mérindol's classification again, we see that Z is rational, so $H^1(Z, \mathcal{O}_Z) = 0$. Use standard ramification theory and Riemann-Roch on V to compute $H^1(Z, \mathcal{O}_Z)$: it is zero if and only if $\sigma(t)^2 = -2$ for all non-trivial t.

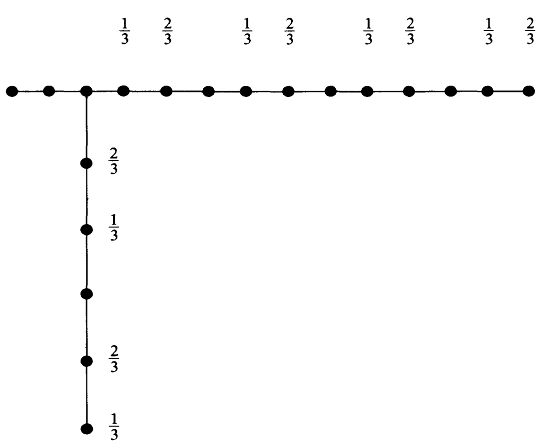
We now check the theorem in the two remaining cases.

Case I: $(-4, -5, -9)$
$T_{5,6,10}$ has one overlattice T_1 in which it has index 4. We represent it by listing the non-zero coefficients of the geometric section of a generator of $T_1/T_{5,6,10}$:

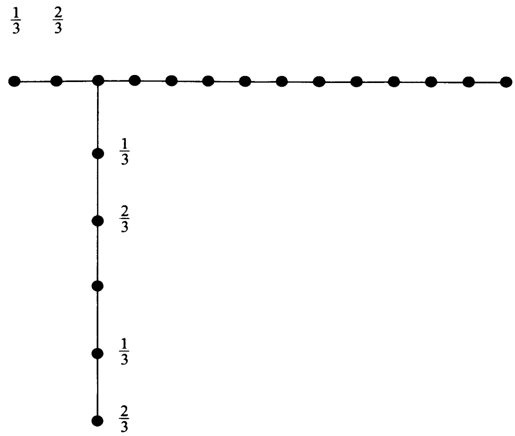


$T_{5,6,10}$ also has one overlattice T_2 in which it has index 2, corresponding to the subgroup of order 2 of $T_1/T_{5,6,10}$. These are the only overlattices of $T_{5,6,10}$ as can be checked using [P2], Lemmas 1 and 2. In both cases, for all non-trivial t, $\sigma(t)^2 = -4$ so that by Lemma 2 (and [FP] Lemma 9, (ii)) applied to $(p', q', r') = (5, 6, 10)$ the only cusp or Dolgachev singularity, corresponding as in [FP] Lemma 2 to T_{pqr}, with rank $T_{5,6,10} = 19$, and in the negative part of the versal deformation of $D_{5,6,10}$ is $(-4, -5, -9)$ or $D_{5,6,10}$ itself. The theorem now follows by examining the statement of [FP], theorem 4: all other possibilities for D'' have been ruled out.

Case II: $(-2, -5, -11)$
$T_{3,6,12}$ has two overlattices, and has index 3 in both. The first, which we call M_1 corresponds to the geometric section



The other, M_2, to



Note that for M_1 (resp. M_2) every non-trivial geometric section has square -4 (resp. -2), so that M_2 is not the saturation of a good embedding of $T_{3.16,12}$ (Lemma 1) and M_1 cannot correspond to a $T_{p,q,r}$ as in Lemma 2, where $(p', q', r') = (3, 6, 12)$. On the positive side we have:

LEMMA 3:
(i) *There exists a good embedding* $\varphi: T_{3,6,12} \hookrightarrow \Lambda$ *with saturation* M_1.
(ii) *There exists a smooth rational surface* V *containing a nodal rational curve* C *of self-intersection* -10 *and a* $T_{3,6,12}$ *configuration of rational curves orthogonal to* C. *The saturation of* $T_{3,6,12}$ *in* $H_2(V, \mathbb{Z})$ *is then necessarily* M_2.

PROOF: The existence of φ in (i) is standard, using [N]; the computation that φ is good is given in Lemma 4.

For (ii) we exhibit V explicitly: take two inflection points on a nodal cubic C in $\mathbb{P}^2$, and blow up 15 times at one, and 4 times at the other. The proper transforms of the two inflection lines and of the exceptional curves with self-intersection -2 form a $T_{3,6,12}$ diagram.

We now conclude the proof of the theorem. By Lemma 3, i) we construct using φ a family $W \to \Delta$ as in Theorem 4 of [FP]. We want to show that the special fiber W_0 has a $(-2, -5, -11)$ singularity. If not W_0 has a (-10) cusp singularity: indeed the only other possibility allowed by theorem 4 is a $D_{2,3,16}$ singularity, but this is ruled out by Lemma 2.

So assume W_0 has a (-10) cusp singularity. Call the singular point p. As in Lemma 11 of [FP] let B be a Milnor ball around p, and let $Y_t = W_t - B$. All the $(Y_t, \partial Y_t)$ are diffeomorphic, and ∂Y_t is a certain $S^1 \times S^1$ bundle over S^1 (described for example in [P3], but we will not need more detailed information).

Let $L = H_2(Y_t)$. Note that L has rank 20, and has a kernel of dim. 1. We have a natural inclusion of lattices for $t \neq 0$: $L = H_2(Y_t) \hookrightarrow H_2(W_t) = \Lambda$. Let L' be the saturation of L in Λ. Since $T_{3,6,12} \hookrightarrow L$ and the saturation of $T_{p,q,r}$ by the embedding into Λ is M_1, we have $M_1 \hookrightarrow L'$; and since M_1 is non degenerate $M_1 \hookrightarrow \overline{L}'$ where $\bar{\ }$ indicates the quotient by the kernel.

But we also have $L = H_2(Y_0)$, and by construction we have $M_2 \approx \overline{L} \subset \overline{L}'$. So finally we have

$$\begin{tikzcd} & M_1 \arrow[dr, hook] & \\ T_{3,6,12} \arrow[ur, hook] \arrow[dr, hook] & & \overline{L}' \\ & M_2 \arrow[ur, hook] & \end{tikzcd}$$

but this is absurd since $\overline{L}'$ has an integer valued quadratic form and M_1 and M_2 are distinct maximal overlattices of $T_{3,6,12}$.

To conclude we must show that the embedding of Lemma 3(i) is good. Using the definition of good given by Looijenga in [L2] it is possible to check, via a straightforward but involved computation, whether *any* given embedding of a T_{pqr} is good or not. (I thank Looijenga for explaining to me how to do this, and for showing me some unpublished manuscripts on this subject). There is one situation, covering the case we are interested in, which can be treated using a generalization of Looijenga's proof that primitive embeddings are good:

LEMMA 4: *Let* φ: $T_{pqr} \to \Lambda$ *and let* T *be the saturation. If for every non-trivial* $t \in T/T_{pqr}$ *the geometric section* $\sigma(t)$ *satisfies*:

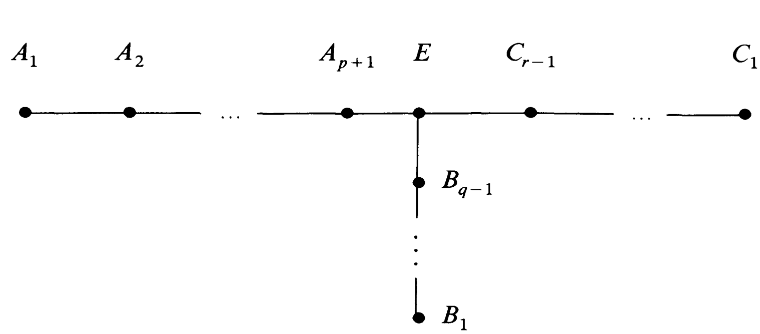
(i) $\sigma(t)^2 = -4$;

(ii) *the support of* $\sigma(t)$ *is contained in* 2 *branches of* T_{pqr};

then the embedding is good.

Hypothesis (ii) is probably unnecessary. It is rather restrictive, since for instance it forces $|T/T_{pqr}| \leqslant 3$.

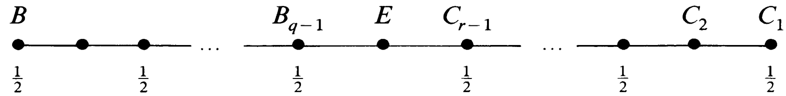
PROOF: We label the vertices of T_{pqr} as in [L2]:



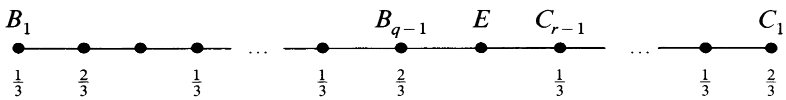
Let n be the fundamental isotropic element ([L2], §3): e.g., if $3 \leqslant p \leqslant q \leqslant r$ then $n = A_{p-2} + 2A_{p-1} + 3E + 2B_{q-1} + B_{q-2} + 2C_{r-1} + C_{r-2}$.

According to [L2], we must show that for any $t \in T$ such that $t.n = 0$ and $t^2 = -2$, then either t is orthogonal to all components in the support of n, or is supported on the support of n.

Write $t = s - u$, where $s \in T_{pqr}$ and u is the geometric section of an element of T/T_{pqr}. By Looijenga's result in the primitive case we may assume $u \neq 0$. By hypothesis u is supported on 2 branches of T_{pqr}, say for concreteness on $(E, B_j\ 1 \leqslant j \leqslant q-1,\ C_k\ 1 \leqslant k \leqslant r-1)$. It is easy to see that $|T/T_{pqr}| \leqslant 3$ and u is of the form:



if the order is 2. Thus q and r are even. The condition $u^2 = -4$ implies $q + r = 16$. This case was already treated in [P2]. Or if the order is 3:



(or interchange $1/3 \leftrightarrow 2/3$). So $q \equiv r \equiv 0 \bmod 3$, and $q + r = 18$.

Note that t is obviously not supported on n.

We will need the following two facts on the $-A_k$ root system. Let $\alpha_1, \ldots, \alpha_k$ be the standard basis, so that $\alpha_i^2 = -2$, $\alpha_i \cdot \alpha_j = 1$ if $|i - j| = 1$ and 0 if $|i - j| > 1$.

SUBLEMMA 1: (*Looijenga*) *If* $z = \sum_{j=1}^{k} z_j \alpha_j$ *is in the integral span of the* α_j, *then* $z \cdot z + z_1^2 \leqslant 0$ *with equality iff* $z = 0$ *and* $z \cdot z + z_1^2 = -1$ *iff* $\pm z = \alpha_1 + \ldots + \alpha_l$ *for some* $l \leqslant k$.

SUBLEMMA 2: *If* $k = mp - 1$ (m *and* p *positive integers*) *and*

$$u = \frac{1}{p}\big(\alpha_1 + 2\alpha_2 + \ldots + (p-1)\alpha_{p-1}$$

$$+\alpha_{p+1} + 2\alpha_{p+2} + \ldots + (p-1)\alpha_{2p-1}$$

$$\ldots$$

$$+\alpha_{(m-1)p+1} + \ldots + (p-1)\alpha_{mp-1}\big)$$

then for any w *in the integral span of the* α_j,

$$0 \leqslant 2u \cdot w - w^2$$

PROOF: We only prove Sublemma 2. Use the standard representation of $-A_k$ in $\mathbb{R}^{k+1}$ with basis $e_1,\dots,e_{k+1}$ and bilinear form $e_i \cdot e_j = -\delta_{ij}$. Then $\alpha_j = e_j - e_{j+1}$, $w = \Sigma_{j=1}^{k+1} t_j e_j$ with $\Sigma_{j=1}^{k+1} t_j = 0$, and $t_j \in \mathbb{Z}$.

$$u = \frac{1}{p}\big(e_1 + e_2 + \dots + e_{p-1} - (p-1)ep$$

$$+ e_{p+1} + \dots + e_{2p-1} - (p-1)e_{2p}$$

$$+ \dots - (p-1)e_{mp}\big).$$

So that

$$2u \cdot w - w^2 = 2(t_p + t_{2p} + \dots t_{mp}) + \sum_{j=1}^{k+1} t_j^2. \qquad (*)$$

Now make the change of variable to complete the square. The equation becomes $2u \cdot w - w^2 = \Sigma_{j=1}^{k+1} s_j^2 - m$ with constraint $\Sigma_{j=1}^{k+1} s_j = m$. Clearly this is always $\geqslant 0$, as required.

We go back to the original situation: $t = s - u$. Note that replacing t by $t +$ an for any integer a affects neither the hypothesis nor the conclusion of what we are trying to show, so we may assume the coefficient of s in A_{p-1} (the vertex in T_{pqr}, not the root system....) is $\geqslant 0$ and strictly less than the corresponding coefficient of n.

Write $s = z + w$, where z is supported on $(A_i, 1 \leqslant i \leqslant p-1)$ and w on $(E, B_j, 1 \leqslant j \leqslant q-1, C_k, 1 \leqslant k \leqslant r-1)$. Let z_1 be the coefficient of A_{p-1} in z and w_e that of E in w. Now

$$-2 = t^2 = (z + w - u)^2 = z^2 + 2z_1 w_e + w^2 - 2u \cdot w + u^2 \qquad (**)$$

If $z_1 = 0$, since $z^2 \leqslant 0$, $u^2 \leqslant -4$ by hypothesis and $w^2 - 2u \cdot w \leqslant 0$ by sublemma 2, there is no solution to this equation. The rest of the proof consists in checking case by case the remaining values of z_1: $0 < z_1 <$ coefficient of A_{p-1} in n.

For concreteness we only do the case $(p, q, r) = (3, 6, 12)$ and T the overlattice M_1 defined above. Then there is only one case to check: $z_1 = 1$. Note that $z \cdot n = u \cdot n = 0$ so the hypothesis on t implies $w \cdot n = 0$. Using sublemma 1, equation $(**)$ becomes:

$$4 \leqslant 2w_e + w^2 - 2u \cdot w \qquad (***)$$

Identify the chain $B_1,\dots,B_{q-1}$, E, $C_{r-1},\dots,C_1$ with the A_{17} root system in the obvious way and use the standard representation as in the proof of Sublemma 2. Then

$$w_e = t_1 + t_2 + \dots + t_6$$

and the condition $w \cdot n = 0$ is

$$2(t_1 + t_2 + t_3) + t_4 + t_5 + \dots + t_9 = 0. \qquad (****)$$

Thus we are trying to solve in integers (use equation $(*)$ in $(***)$):

$$\sum_{j=1}^{18} t_j^2 + 2(t_3 + t_6 + \dots + t_{18}) - 2(t_1 + t_2 + \dots + t_6) + 4 \leqslant 0$$

with constraints $\Sigma_{j=1}^{18} t_j = 0$ and $(****)$.
Make the obvious change of variable to complete the square. The equation becomes:

$$\sum_{j=1}^{18} s_j^2 - 4 \leqslant 0$$

with constraints

$$\sum_{j=1}^{18} s_j = 0$$

and

$$2(s_1 + s_2 + s_3) + s_4 + \dots s_9 = -5.$$

It is obvious there are no solutions in integers. Therefore there are no candidates for t, and the Lemma is proved.

Acknowledgements

This appendix was written at the Centre de Mathématiques of the Ecole Polytechnique. I thank its members and staff for their hospitality and financial support. I also thank Eduard Looijenga for help concerning "good" embeddings.

Reference

[FP] R. Friedman and H. Pinkham: Smoothings of cusp singularities via triangle singularities. *Comp. Math.* 53 (1984) 303–316.

(Oblatum 26-IV-1983)

Columbia University
New York, NY 10027
USA