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SMOOTHINGS OF CUSP SINGULARITIES VIA TRIANGLE SINGULARITIES

Robert Friedman [1] and Henry Pinkham [2]

Introduction

The purpose of this note is to verify a conjecture of Looijenga ([L1] III.2.11) concerning the existence of smoothings of cusp singularities in case the number r of components in the minimal resolution is less than or equal to 3. The cases $r \leq 2$ and $r = 3$ with multiplicity $m \leq 11$ were considered in [FM], with very different methods. We exhibit here smoothings for all smoothable cusps with $r \leq 3$ and $m = r + 9$. By a result of Wahl any smoothable cusp satisfies $m \leq r + 9$ [W2], and such cusps must at least satisfy an additional condition (the dual cusp must sit on a rational surface). The case $m = r + 9$ turns out to be the most delicate case as the cases $m < r + 9$ can frequently be deduced from the cases $m = r + 9$, using a method of Wahl [W1].

The essential point is to locate the cusps we wish to smooth on smoothing components of the negative part of the versal deformation of a Dolgachev (or triangle) singularity $D_{p,q,r}$, where (p, q, r) is determined by the cusp. We do this by constructing an appropriate degenerating family of $K3$ surfaces, using the period map. The tricky part is to arrange a model for the central fiber which is a singular rational surface with the right cusp singularity. We do this by looking at the versal deformation of the $D_{p,q,r}$ singularity and by using a combination of monodromy arguments and discriminant computations which will rule out various possibilities for the central fiber, until at the end we are left with the cusp we want. The relevant discriminant computations are summarized in a table at the end of this note. For more information on $T_{p,q,r}$ lattices and Dolgachev singularities, the reader is encouraged to look at [L1] and [L2], and, for information on cusp singularities, at [P3] and [FM] and the references cited there.

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The introduction of the Dolgachev singularities is somewhat artificial, and certainly peripheral to our main purpose. What we really need is a geometrically meaningful compactification of the moduli space of "M-marked $K3$ surfaces" where M is the orthogonal complement of the lattice generated by the components of the exceptional divisor of the cusp in the rational surface (this terminology is that of [P1]). By the results of [P1], we have such a compactification when $r \leq 3$. Analogous constructions for $r > 3$ would suffice to generalize our results to the case where the resolution of D sits as an anticanonical divisor on a rational surface (and $m = r + 9$), at least up to $r = 6$ since for $r > 6$ the lattice M is no longer an "extended affine root system" ([L1]) so that Looijenga's theory of "good embeddings" into K-3's no longer applies.

As the reader will doubtless notice we use in an essential way the work of Nikulin [N]. We would like to thank, among others, Lee McEwan, Rick Miranda, Dave Morrison, and Nick Shepherd-Barron for explaining this work to us and for many helpful conversations (in part during the course of a seminar organized at the Institute for Advanced Study, Princeton).

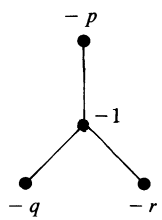
We would also like to thank J. Wahl for an interesting letter on the first version of this note, especially for providing an example (see Remark 5) showing that a statement we claimed was incorrect. Finally we thank J. Morgan for helping us find that statement.

§1. Statement of the theorem

The first assertion of the following theorem is Looijenga's conjecture for $r \leq 3$. For the definition and properties of cusps see [P3], [L1], [La2], and [FM]. Suffice it to say here that a cusp is a normal singularity whose minimal resolution consists in a cycle of rational curves. We usually identify the cusp by listing the self intersections of the rational curves in cyclic order.

THEOREM 1: *Let D be a cusp singularity with $r \leq 3$ components and multiplicity $m = r + 9$. If the minimal resolution of the dual cusp $\check{D}$ sits on a rational surface as an anticanonical divisor D is smoothable. More precisely D lies on a smoothing component in the negative part of the versal deformation of the appropriate $D_{p,q,r}$ singularity (the one with $\mathbb{C}^*$-action).*

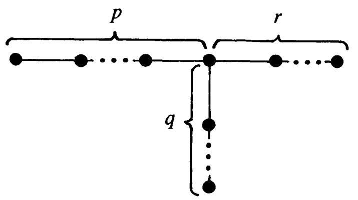
The $D_{p,q,r}$ are the so called Dolgachev or triangle singularities. They have a resolution with dual graph:



where each vertex represents a smooth rational curve and its weight is its self intersection. For more details see [L2].

The property of $\check{D}$ in the statement of Theorem 1 is rather difficult to use. Instead, we shall use a lattice-theoretic property of D, which we now explain. Under the assumption that $r \leq 3$, the (reduced) exceptional divisor E of minimal resolution of D itself sits as an anticanonical divisor on the rational surface V. Let $L \subseteq H_2(V;\, \mathbb{Z})$ be the sublattice generated by the components of E and $R = L^{\perp}$ in $H_2(V;\, \mathbb{Z})$., It follows from [L1] theorem 1.1 that $L \subseteq H_2(V;\, \mathbb{Z})$ is primitively embedded. Moreover R is even and nondegenerate.

As we shall see in a moment, R is a $T_{p,q,r}$ lattice ([L1], [L2]). This is a lattice with Dynkin diagram:



where each vertex has self-intersection -2, and $\bullet\!\!-\!\!-\!\!-\!\!\bullet$ means the corresponding vertices have intersection 1.

DEFINITION: A surface S has a $T_{p,q,r}$ configuration if there exist smooth rational curves in S, not passing through the singular locus of S, such that the lattice generated by these curves in $H_2(S;\, \mathbb{Z})$ is $T_{p,q,r}$ and such that the homology classes of the curves are the vertices of the $T_{p,q,r}$.

LEMMA 2:

(*i*) *R is a $T_{p,q,r}$ lattice for appropriate p, q, r.*

(*ii*) *There exist a rational surface with a cusp of type D such that V has a $T_{p,q,r}$ configuration.*

(*iii*) *R determines the unordered triple (p, q, r).*

(*iv*) *If the reduced exceptional divisor E' of the minimal resolution of the $D_{p,q,r}$ singularity sits as an anticanonical divisor on a rational surface V', then the orthogonal complement in $H_2(V';\, \mathbb{Z})$ to the components of E' is the $T_{p,q,r}$ lattice.*

PROOF:

(i) This is proved in [L1], I.2, and in any case follows easily from the construction of the $T_{p,q,r}$ configuration in (ii) plus a discriminant computation.

(ii) If $r = 1$, start with a nodal cubic and make $m + 9$ infinitely near blowups at an inflection point on the cubic. This lowers the self-intersection to $-m$, and we leave it to the reader to locate a $T_{p,q,r}$ configuration disjoint from E, where $(p, q, r) = (2, 3, m + 6)$.

If $r = 2$, start with the transverse intersection of a line and conic, and blow up appropriately many times at infinitely near points to the intersection of a tangent line to the conic at the conic and line. Again, the reader will easily locate the $T_{p,q,r}$ configuration. Here, if the self-intersections of the two components of E are $-x$ and $-y$, $(p, q, r) = (2, x+2, y+2)$.

If $r = 3$, start with three lines in general position and make all blowups at infinitely near points to the intersection of a general line with the given three. If the self-intersections of the components of E are $-a$, $-b$, $-c$, we have $(p, q, r) = (a+1, b+1, c+1)$.

(iii) In the case $m = r + 9$, this follows by a direct inspection of the table at the end of this paper, since no value of the discriminant appears twice in the list.

(iv) is well known, and indeed is the source of the terminology. The proof is the same as that of (i).

REMARK 3: Using (ii) of the lemma it can be shown that D appears on the negative part of the versal deformation of $D_{p,q,r}$, independent of the condition $m = r + 9$ (use [P4], 6.7). The problem is to decide if D actually lies on a smoothing component. We can show directly that if the cusp smooths at all, it will smooth on the negative part of the versal deformation of $D_{p,q,r}$; we omit the argument, since we do not need the result and since it is a special case of a new result of Looijenga which says that the negative part of the versal deformation of a $D_{p,q,r}$ is versal everywhere except at the origin.

This remark will motivate our construction.

We now fix notation which will be used throughout this paper.

$$\Lambda = \text{the } K3 \text{ lattice } (-E_8) \oplus (-E_8) \oplus H \oplus H \oplus H.$$

$$\Lambda' = (-E_8) \oplus (-E_8) \oplus H \oplus H.$$

Here, $-E_8$ is the unique negative definite even unimodular lattice of rank 8 and H is the hyperbolic plane [S]. We can now state a theorem which together with a result of [FM] will imply Theorem 1. We will need the notion of a *good* embedding φ of a $T_{p,q,r}$ lattice into Λ, notion due to Looijenga ([L2]); it is equivalent to saying there exists a K-3 surface S with a $T_{p,q,r}$ configuration such that the embedding $T_{p,q,r} \to \Lambda = H_2(S; \mathbb{Z})$ is equivalent to φ. Furthermore Looijenga shows all primitive embeddings are good, which is all we need to know here about good embeddings.

THEOREM 4: *Let D be a cusp singularity with $r \leq 3$ components and multiplicity $m = r + 9$. Assume that $R \cong T_{p,q,r}$ admits an embedding into Λ' (not necessarily primitive). Then for every good embedding φ: $R \hookrightarrow \Lambda$ there is a flat and proper family $\mathcal{W} \to \Delta$, $\Delta = \operatorname{disk}|t| < 1$ such that, if W_t denotes the fiber above t:*

(1) *for $t \neq 0$, W_t is a K-3 surface with a $T_{p,q,r}$ configuration embedded by φ in $\Lambda = H_2(W_t; \mathbb{Z})$*

(2) *W_0 is a rational surface with a $T_{p,q,r}$ configuration and exactly one singularity D' which is either a cusp or a $D_{p',q',r'}$ singularity with $(p', q', r') \neq (p, q, r)$. In both cases if r' denotes the number of components in the minimal resolution and m' the multiplicity, than $r' \leqslant 3$ and $m' = r' + 9$. Thus, if $\tilde{W}_0$ is the minimal resolution of W_0 then the components of the resolution of D'' generate the orthogonal complement of the $T_{p,q,r}$ lattice in $H_2(\tilde{W}_0; \mathbb{Z})$.*

(3) *If D' is a $D_{p',q',r'}$ and R' is the primitive lattice spanned by $T_{p,q,r}$ in $H_2(\tilde{W}_0; \mathbb{Z})$ (which is $T_{p',q',r'}$ by the last statement in (2)), then there exists a natural embedding ψ: $R' \to$ the primitive lattice spanned by* Im *$\varphi \subseteq \Lambda$. In particular, if φ is a primitive embedding, then this case cannot occur by (iii) of Lemma 2.*

REMARK: To a given φ: $R \to \Lambda$, the number of "different" $\mathscr{W} \to \Delta$ is, in the notation of Lemma 8 below, the number of points of $\bar{C} - C$.

Let us see how to deduce Theorem 1 from Theorem 4. First, by [P3] §3, if the resolution of $\check{D}$ sits as an anticanonical divisor on a rational surface, then R admits an embedding into Λ', so that the hypothesis of Theorem 1 implies that of Theorem 4. (Actually, the only fact used about R in Lemma 6 is that there is *some* embedding of R into Λ for which the orthogonal complement represents 0. This is really a statement about $\mathbb{Q}$-lattices, so it suffices to show that there is an embedding over $\mathbb{Q}$ of R into Λ'. This can easily be verified by the standard results of [S]; the corollary on *p.* 78 reduces this question to a computation of Hasse invariants, without the more refined theory of the discriminant form given in [N].)

Next note that for each D the corresponding R has a primitive embedding into Λ (Lemma 6), which by the result of Looijenga already mentioned ([L2], §3, remarks preceeding Prop. 3) is good, so that we do have a $\mathscr{W} \to \Delta$. If we knew the singularity D' of W_0 were D, we would have Theorem 1. So assume it is not. Theorem 4, (2) and (3) imply that D' is a cusp singularity given in the table at the end of this paper, and that $|\text{disc. } D| = n^2 |\text{disc. } D'|$, where n is the index of one lattice in the other. By inspection of the table below the only pairs of possibilities for which this can occur are (we list the p, q, r involved):

D	(p, q, r)	$\lvert$disc. $(D)\rvert$	D'	$\lvert$disc. $(D')\rvert$
$(-2, -5, -11)$	$(3, 6, 12)$	$3^2 \cdot 10$	(-10)	10
$(-4, -5, -9)$	$(5, 6, 10)$	$4^2 \cdot 10$	(-10)	10
$(-4, -5, -9)$	$(5, 6, 10)$	$2^2 \cdot 40$	$(-4, -11)$	40
$(-4, -11)$	$(2, 6, 13)$	$2^2 \cdot 10$	(-10)	10
$(-3, -3, -12)$	$(4, 4, 13)$	$2^2 \cdot 22$	$(-2, -13)$	22

Therefore for all cusps except those with p, q, r listed in the left-hand column, Theorem 1 is proved. By [FM], Prop. (4.8) and (4.9), the dual cusps of $(-4, -11)$ and $(-3, -3, -12)$ do not sit on rational surfaces and therefore do not satisfy the hypothesis of Theorem 1 (although they do satisfy that of Theorem 4) so we need not concern ourselves with them.

Two cases remain: $(-2, -5, -11)$ and $(-4, -5, -9)$. For $(-4, -5, -9)$ an easy additional argument (given in the appendix) shows that the singularity of W_0 is indeed the $(-4, -5, -9)$ cusp. The case of $(-2, -5, -11)$ is more complicated: see remark 5 below for further discussion. Fortunately in both these cases D is shown to be smoothable in [FM], so that the proof of Theorem 1 is complete, assuming Theorem 4 and the result of [FM].

REMARK 5:

(a) When $D = (-4, -11)$ or $(-3, -3, -12)$ and φ is primitive, the singularity on W_0 must be the cusp (-10) and $(-2, -13)$ respectively, as D does not smooth ([L2], III.2.9 and [FM], prop. (4.8) and (4.9)). This pathology is explained by the "exotic" elliptic deformations of Karras-Brieskorn-Wahl ([W1], 5.6 (b) and (d)). These examples pointed out to us by J. Wahl show that the lattice theoretic hypothesis of Theorem 4 is not equivalent to the more subtle condition in Theorem 1.

(b) The pair $(-2, -5, -11)-(-10)$ again corresponds to an exotic elliptic deformation ([W1], 5.6 (e)), but this time both cusps turn out to be smoothable. This fact makes the cusp $(-2, -5, -11)$ hard to treat by our method. As the proof that $(-2, -5, -11)$ and $(-4, -5, -9)$ smooth given in [FM] uses totally different techniques (those of [F]), it seemed worthwhile (to the second author) to give a proof, in the appendix, closely related to the techniques of this paper. What is shown is that there is a good, nonprimitive embedding of $T_{3,6,12}$ into Λ such that for the corresponding $\mathscr{W} \to \Delta$, W_0 has a $(-2, -5, -11)$ singularity. It is not known what happens for families associated to the primitive embedding, except that some of them at least must go to (-10).

(c) Using the techniques of [S], one checks easily that the only cases where the orthogonal of R in Λ represents 0 are either when $\check{D}$ sits as an anticanonical divisor on a rational surface or when $D = (-4, -11)$ or $(-3, -3, -12)$.

§2. Some arithmetic and Hodge theory

LEMMA 6: *With hypotheses on R as in Theorem* 4, *there exists a primitive embedding of R into Λ. Moreover, for every embedding of R in Λ, the orthogonal complement $R^{\perp}$ represents* 0.

PROOF: The existence of a primitive embedding follows from a result of Nikulin [N] (1.12.3) as R^*/R has at most two generators [L2], [P2] and is of rank 19 (and signature (1,18), by Hodge index).

Now let $R \subseteq \Lambda$ be any embedding. For the remainder of this proof, we view R, Λ as $\mathbb{Q}$-lattices, as this suffices to check whether or not $R^{\perp}$ represents 0.

By hypothesis, there is some embedding $R \subseteq \Lambda = \Lambda' \oplus H$, whose complement has the form (over $\mathbb{Q}$)

$$S = S' \oplus H,$$

S' one-dimensional. Hence S represents 0. But, over $\mathbb{Q}$,

$$\Lambda = R \oplus R^{\perp} = R \oplus S,$$

so, by [S], (IV.1.5), p. 59, corollary to Witt's theorem, $R^{\perp}$ and S are isomorphic over $\mathbb{Q}$. Since S represents 0, $R^{\perp}$ does as well, Q.E.D.

REMARK 7: From [N], theorem 1.14.4 one sees that, if there exists a *primitive* embedding of R into Λ', then the primitive embedding of R in Λ is unique. This is the case for all cusps with $r \leq 3$ and $m = r + 9$ whose duals sit on rational surfaces, except for $(-2, -2, -14)$, $(-2, -5, -11)$, and $(-6, -6, -6)$.

In these remaining cases, using a result of Kneser (cf. [N], 1.13.1), one can likewise show that the primitive embedding of R in Λ is unique (use Lemma 1 of [P2]).

On the other hand as pointed out to us by Wahl the two primitive embedding of the dual cusp to $(-2, -6, -10)$ into a rational surface constructed in [FM], §6 are lattice theoretically distinct.

LEMMA 8: *Let R be a lattice of signature* (1,18) *and* φ: $R \to \Lambda$ *an embedding such that $R^{\perp}$, the orthogonal complement of* Im φ, *represents* 0. *Let C be the coarse moduli space of all $K3$ surfaces W such that, under the map* φ: $R \to \Lambda = H_2(W; \mathbb{Z})$, *$R$ is of type* (1,1) (*so that C is a connected algebraic curve*). *We call a $K3$ surface W in C an R-marked $K3$ surface.*

(*i*) *C is not complete.*

(*ii*) *If $\bar{C}$ is the completion of C, then, locally around any point P of $\bar{C} - C$, there is a non constant map f: $\Delta \to \bar{C}$ with $f(0) = P$ and a family π: $\mathscr{W} \to \Delta$ such that the family $\mathscr{W}^* \to \Delta^*$ is a smooth family of R-marked $K3$ surfaces and f: $\Delta^* \to C$ coincides with the natural map.*

(*iii*) *For any such family π: $\mathscr{W} \to \Delta$ as in ii*), *if the monodromy, acting on $H_2(W_t)$, is unipotent and N is its logarithm, then $N^2 \neq 0$, i.e. $\mathscr{W} \to \Delta$ is birationally a Type III degeneration of $K3$ surfaces.*

PROOF: Consider the corresponding period space for $K3$ surfaces as in the statement:

$$\mathfrak{D} = \Big\{ F^2 \text{ a line in } R^{\perp}_{\mathbf{C}} = R^{\perp} \otimes_{\mathbf{Z}} \mathbf{C} : F^2$$

$$\text{is isotropic and, if}$$

$$\omega \in F^2,\ \omega \neq 0, \text{ then } \omega \cdot \bar{\omega} > 0 \Big\}.$$

Moreover, $\mathfrak{D} = SO(2,1)/K$ for an appropriate maximal compact K, and $\mathfrak{D}$ is a symmetric space isomorphic to the upper half plane. By local Torelli, C maps finite-to-one to $\Gamma \setminus \mathfrak{D}$, where $\Gamma = \{ g \in SO(\Lambda)_{\mathbf{Z}} : g|R = \mathrm{Id}\}$, an arithmetic group, and the main point of the lemma is that $\Gamma \setminus \mathfrak{D}$ is noncompact. While this follows immediately from a well known compactness criterion for quotients of symmetric spaces by arithmetic groups, we prefer to give an explicit construction which will be used later. We will locate an integral unipotent $T \in SO(\Lambda)_{\mathbf{Z}}$, which fixes R. If $N = \log T$, the corresponding monodromy weight filtration lives in $R^{\perp}$ and is necessarily of Type III as $R^{\perp}$, of signature (2,1), does not contain any two dimensional isotropic subspaces. (This shows that all cusps of Γ, to use a highly confusing phrase, correspond to Type III degenerations; in our construction, it will be clear that $N^2 \neq 0$.)

Since $R^{\perp}$ represents 0, there exists a nonzero isotropic vector $\gamma \in R^{\perp}$. Define

$$
\begin{array}{llll}
W_0 = \mathbf{Q} \cdot \gamma \subseteq R^{\perp}_{\mathbf{Q}} = R^{\shortparallel} \otimes_{\mathbf{Z}} \mathbf{Q} & & & \\
\cap| & & & \\
W_2 = W_0^{\perp} \quad \text{in} \quad R^{\perp}_{\mathbf{Q}}, & = \mathbf{Q}\gamma + \mathbf{Q}\delta, & \text{say (nonorthogonal sum)} & \\
\cap| & & & \\
W_4 = R^{\perp}_{\mathbf{Q}}, & = W_2 + \mathbf{Q} \cdot \gamma'. & &
\end{array}
$$

We may assume, after replacing γ' by $\gamma' + c\delta$ for an appropriate c, that $\gamma' \cdot \delta = 0$. Note that $\delta \cdot \delta \neq 0$ by looking at the signature and that $\gamma' \cdot \gamma \neq 0$ since $\gamma' \notin W_2$.

Define the rational nilpotent matrix N by

$$N(\gamma') = \delta$$

$$N(\delta) = -((\delta \cdot \delta)/(\gamma \cdot \gamma')) \cdot \gamma$$

$$N(\gamma) = 0,$$

extending by linearity. Note that $N^2 \neq 0$. A simple calculation gives

$$(*) \qquad N(\alpha)\cdot\beta + \alpha\cdot N(\beta) = 0, \quad \text{all} \quad \alpha, \beta \in R^{\perp}_{\mathbf{Q}}.$$

Via the splitting $\Lambda = R \oplus R^{\perp}$ over $\mathbf{Q}$, we may view N as a rational nilpotent matrix acting on Λ, with $N(R) = 0$ and such that (*) holds. After multiplying by a suitable integer, we may assume $T = \exp N$ is integral, hence $T \in \Gamma$.

To locate an explicit map $\Delta^* \to \Gamma \backslash \mathfrak{D}$ with the required monodromy, and corresponding to a cusp of Γ, it suffices to use

$$(T^n)\backslash \mathfrak{D} \to \Gamma \backslash \mathfrak{D},$$

noting that $(T^n)\backslash \mathfrak{D} \cong \Delta^*$. We have completed the proof of (i) of Lemma 8, and (ii) is now immediate.

As for (iii), note that it follows immediately for any point P lying over a cusp of $\Gamma \backslash \mathfrak{D}$ constructed in the course of the proof of (i), which is all we need for this paper. That all cusps of $\Gamma \backslash \mathfrak{D}$ satisfy $N^2 \neq 0$ is, as already remarked, an easy consequence of the fact that $R^{\perp}$ has signature (2,1). Finally, that a point of $\bar{C} - C$ must actually map onto a cusp of $\Gamma \backslash \mathfrak{D}$ follows from the surjectivity of the period map for $K3$ surfaces.

REMARK. It follows from global Torelli that C is actually isomorphic to $\Gamma \backslash \mathfrak{D}$. We do not, however, need this result.

§3. The deformation space of $D_{p,q,r}$

We begin by fixing some notation. Let S be the base of the negative part of the (mini)versal deformation of the $D_{p,q,r}$ singularity with $\mathbf{C}^*$ action, with $0 \in S$ the distinguished point. Thus, S is an affine scheme with good $\mathbf{C}^*$ action. By the general theory, there is a corresponding family $\mathscr{V} \to S$ with compact fibers, and $\mathbf{C}^*$ acts equivariantly on $\mathscr{V}$ as well. We will need the following facts about S:

LEMMA 9:

(*i*) *There are no $D_{p,q,r}$ singularities in the fibers of $\mathscr{V} \to S$ away from* 0 (*with or without $\mathbf{C}^*$-action*).

(*ii*) *All fibers of $\mathscr{V} \to S$ have a $T_{p,q,r}$ configuration. They are irreducible and are* (*birationally*) *either K-3 surfaces or rational.*

(*iii*) *The quotient scheme $(S - \{0\})/\mathbf{C}^*$ exists and is complete. It contains an open set which is a coarse moduli space for $K3$ surfaces with a $T_{p,q,r}$ configuration.*

PROOF:

(i) By a result of Laufer [La1], there are precisely two isomorphism classes of $D_{p,q,r}$ singularities: those with $\mathbf{C}^*$ action, and those without.

Since S is part of a miniversal deformation, we may ignore those with $\mathbb{C}^*$ action: they occur only at 0. By [La3] a family of $D_{p,q,r}$ singularities without $\mathbb{C}^*$ action degenerating to one with $\mathbb{C}^*$ action can be resolved simultaneously. But then the deformation is equisingular in the sense of Wahl, and by [P4], 4.6 cannot be in S. This proves (i).

(ii) and (iii). The existence and compactness of $(S - \{0\})/\mathbb{C}^*$ are standard, and the remaining assertions follow from [P1] and [P4] (cf. also [L2]).

Let m and r be the multiplicity and number of components in the minimal resolution of the $D_{p,q,r}$ singularity; hence they are equal to the corresponding m and r for the cusp. (The 2 uses of r in this context are regrettable but unavoidable).

For V_t the fiber of $\mathscr{V} \to S$ over t, let $\tilde{V}_t$ denote the minimal resolution of the corresponding singular surface. Note that V_t has at most one nonrational singular point, which is then minimally elliptic (this is because V_t is in the deformation space of a minimally elliptic singularity, which implies its singularities are Gorenstein, and the sum of their genera is ≤ 1), and if such a point exists, then as $\tilde{V}_t$ has a non trivial effective anticanonical divisor and a $T_{p,q,r}$ configuration of rational curves, $\tilde{V}_t$ is a rational surface. Assume V_t has a minimally elliptic singularity and let E' be its fundamental cycle, i.e. simply an anticanonical divisor supported on the exceptional set of the minimal resolution. Let r' be the number of components of E' and $m' = -E' \cdot E'$. R', the orthogonal complement in $H_2(\tilde{V}_t; \mathbb{Z}) = \operatorname{Pic} \tilde{V}_t$ to the components of E', contains a $T_{p,q,r}$lattice by Lemma 9.

As $\tilde{V}_t$ is rational, rank Pic $\tilde{V}_t = 10 + m'$. On the other hand since $\tilde{V}_t$ contains the $T_{p,q,r}$ lattice which has rank 19, and in its orthogonal complement a lattice of rank r', we have rank Pic $\tilde{V}_t \geq 19 + r'$. So

$$10 + m' \geqslant 19 + r' = 10 + m - r + r',$$

so that

$$m' - r' \geqslant m - r = 9.$$

Note that by [La2] m' is the multiplicity of V_t at the elliptic singularity.

LEMMA 10:

(*i*) *For any good embedding* φ: $R \to \Lambda$ *there exists a map* ρ: $\Delta \to (S - \{0\})/\mathbb{C}^*$ *and hence after finite base change a lifting* $\tilde{\rho}$: $\Delta \to S - \{0\}$ *such that if* $\mathscr{W} \to \Delta$ *is the family pulled back from* $\mathscr{V} \to S$ *by* $\tilde{\rho}$, *then* $\mathscr{W} \to \Delta$ *is a type III degeneration as constructed in Lemma* 8.

(*ii*) *If* $W_0 = V_t$ *is the surface corresponding to* $t = \tilde{\rho}(0)$, *then* W_0 *has a minimally elliptic singularity* D' *which is either a cusp with at most* 3 *components or a triangle singularity with* $(p', q', r') \neq (p, q, r)$.

By construction D' *is a smoothable singularity.*

PROOF:

(i) Via Lemmas 8 and 6, we may construct a family $\mathscr{W}^* \to \Delta^*$, using the embedding φ. Since φ is good in the sense of Looijenga, the fibers W_t, $t \neq 0$, have a $T_{p,q,r}$ configuration, ([L2], Prop. 3) and by Lemma 9 we obtain a map ρ^*: $\Delta^* \to (S - \{0\})/\mathbb{C}^*$ which by compactness of the right hand side extends to Δ. Thus we obtain (i).

(ii) Since $m' - r' \geq m - r$, then $r - r' \geq m - m' \geq 0$, since m is the multiplicity of $D_{p,q,r}$ and m' is the multiplicity of a singularity in its deformation space. So $r - r' \geq 0$, therefore $r' \leq r \leq 3$.

Examining the list of minimally elliptic singularities [La2], (3.4) and (3.5), we see that W_0 must be simple elliptic, a cusp, or a Dolgachev $D_{p',q',r'}$ (with $(p', q', r') \neq (p, q, r)$, by (i) of Lemma 8). (The case of only rational double points or W_0 smooth are impossible since the monodromy is not of finite order.) If W_0 had a simple elliptic singularity, the corresponding degeneration of $K3$ surfaces would be Type II ($N^2 = 0$), as one sees easily by considering the semi-stable model. Another proof would consist in noting that simple elliptic singularities with $m' - r' \geq 9$ are not smoothable ([P0], 7.5: note that $r' = 1$ so $m' \geq 10$) and W_0 comes with a smoothing. We have proved (ii).

§4. The end of the proof of Theorem 4

To finish the proof it remains to show:

(1) $m' - r' = m - r$;

(2) if the singularity of W_0 is a Dolgachev singularity $D_{p'q'r'}$, then $T_{p',q',r'}$ is a sublattice of the primitive lattice spanned by the image of $T_{p,q,r}$ in Λ under φ.

We already have $m' - r' \geq m - r = 9$. Since cusps with $m' - r' > 9$ are not smoothable by [W2], 5.6, we may now assume that D'' is a Dolgachev singularity. Our main goal here again is to prove $m' - r' = 9$, since that is all we really need to prove Theorem 1 (it is the condition that permits us to examine discriminants), but for the proof of (1) we need (2). The argument that ruled out simple elliptics and cusps will not work for triangle singularities since there are smoothable triangle singularities with $m' - r' > 9$ ([P2]).

We first do (2).

LEMMA 11: *Let $\mathfrak{Z} \to \Delta$ be a deformation of the compact analytic surface Z_0, where as usual Z_t is the fiber above t. Assume that Z_t, $t \neq 0$, is smooth, and that aside from rational double points, Z_0 has a unique normal singular point p. Let $\tilde{Z}_0$ be the minimal resolution of Z_0, and $R' \subseteq H_2(\tilde{Z}_0, \mathbb{Z})$ the orthogonal complement of the lattice generated by the components of the exceptional divisor E' at p. Finally assume that $H^1(E'; \mathbb{Z}) = 0$. Then there is a natural inclusion $R' \hookrightarrow H_2(Z_t; \mathbb{Z})$, $t \neq 0$, and the image of R' is invariant under some power of the monodromy.*

PROOF: We may assume Z_0 has the unique singular point p since by assumption the other singularities, which are rational double points, may be simultaneously resolved after finite base change.

Using the exact sequence of the pair $(\tilde{Z}_0, \tilde{Z}_0 - E)$ as in [L1], I.5.1 and writing (coefficients in $\mathbb{Z}$)

$$\begin{array}{ccccccc} H_3(\tilde{Z}_0, \tilde{Z}_0 - E') & \to & H_2(\tilde{Z}_0 - E') & \xrightarrow{\alpha} & H_2(\tilde{Z}_0) & \xrightarrow{\beta} & H_2(\tilde{Z}_0, \tilde{Z}_0 - E) \\ \downarrow \wr & & & & & & \downarrow \wr \\ H^1(E') & & & & & & H^2(E') \end{array}$$

we see that α is an injection and that R' is the image of $H_2(\tilde{Z}_0 - E')$ by α.

Let B be a Milnor ball ([W2]) around p for the smoothing $\mathfrak{Z} \to \Delta$. Let $Y_t = Z_t - B$. For appropriate B and if necessary after shrinking Δ we may assume that for all t including 0, $(Y_t, \partial Y_t)$ is a compact manifold with boundary. By the Ehresmann theorem, all the $(Y_t, \partial Y_t)$ are diffeomorphic. Thus we have the composite map:

$$\psi\colon R' \approx H_2(\tilde{Z}_0 - E') \approx H_2(Y_0) \approx H_2(Y_t) \to H_2(Z_t).$$

Since ψ is clearly an isometry and R' is nondegenerate, ψ is injective. Since there is a representative for the geometric monodromy on Z_t which is the identity on Y_t, R' is invariant under monodromy. Going back to the original family (before the finite base change made at the beginning of the proof), we get the last statement.

REMARK:

(a) The hypotheses of Lemma 11 apply to $\mathcal{W} \to \Delta$ when W_0 has a triangle singularity, but *not* when it has a cusp.

(b) If $H^1(E'; \mathbb{Z}) \neq 0$, there is still some lift of R' to $H_2(\tilde{Z}_0 - E')$ and an inclusion $R' \to H_2(Z_t; \mathbb{Z})$ which is invariant under some power of the monodromy. In the case of a cusp, the kernel of $H_2(\tilde{Z}_0 - E') \to H_2(\tilde{Z}_0)$ is of the form $\mathbb{Z} \cdot \gamma$, where γ is isotropic. In particular, the lift of R' to $H_2(\tilde{Z}_0 - E')$ need not contain the given $T_{p,q,r}$ lattice R in this case.

Assertion (2) is now easy: R' contains $T_{p,q,r}$, and, under ψ, $T_{p,q,r}$ is sent to "the" $T_{p,q,r}$ on W_t, so that if $T_{p,q,r}$ and R' are of the same rank we are done. So suppose not. Then the rank of R' is at least 20. Referring to the proof of Lemma 8, this forces $R' = \operatorname{Ker} N$ (over $\mathbb{Q}$). But R' is nondegenerate, whereas ker N contains γ which is orthogonal to ker N, a contradiction.

$$\begin{array}{rcl} \text{Thus rank } R' & = & \text{rank } T_{p,q,r} \\ \| & & \| \\ 10 + m' - r' & & 10 + m - r \end{array}$$

$\Rightarrow m' - r' = m - r$ which proves (1), and finishes the proof of Theorem 4.

A table of discriminants

	cusp	(p, q, r)	discriminant $= \pm(pq + pr + qr - pqr)$
$r = 1$	$-m$	$(2, 3, m+6)$	$-m$
	-10	$(2, 3, 16)$	-10
$r = 2$	$(-x, -y)$	$(2, x+2, y+2)$	$xy - 4$
	$(-2, -13)$	$(2, 4, 15)$	22
	$(-3, -12)$	$(2, 5, 14)$	32
	$*(-4, -11)$	$(2, 6, 13)$	40
	$(-5, -10)$	$(2, 7, 12)$	46
	$(-6, -9)$	$(2, 8, 11)$	50
	$*(-7, -8)$	$(2, 9, 10)$	52
$r = 3$	$(-a, -b, -c)$	$(a+1, b+1, c+1)$	$2 + a + b + c - abc (= 20 - abc)$
	$(-2, -2, -14)$	$(3, 3, 15)$	-36
	$(-2, -3, -13)$	$(3, 4, 14)$	-58
	$*(-2, -4, -12)$	$(3, 5, 13)$	-76
	$(-2, -5, -11)$	$(3, 6, 12)$	-90
	$(-2, -6, -10)$	$(3. 7, 11)$	-100
	$(-2, -7, -9)$	$(3, 8, 10)$	-106
	$*(-2, -8, -8)$	$(3, 9, 9)$	-108
	$*(-3, -3, -12)$	$(4, 4, 13)$	-88
	$*(-3, -4, -11)$	$(4, 5, 12)$	-112
	$(-3, -5, -10)$	$(4, 6, 11)$	-130
	$(-3, -6, -9)$	$(4, 7, 10)$	-142
	$*(-3, -7, -8)$	$(4, 8, 9)$	-148
	$*(-4, -4, -10)$	$(5, 5, 11)$	-140
	$(-4, -5, -9)$	$(5, 6, 10)$	-160
	$*(-4, -6, -8)$	$(5, 7, 9)$	-172
	$*(-4, -7, -7)$	$(5, 8, 8)$	-176
	$*(-5, -5, -8)$	$(6, 6, 9)$	-180
	$(-5, -6, -7)$	$(6, 7, 8)$	-190
	$(-6, -6, -6)$	$(7, 7, 7)$	-196

* Indicates that the dual cusp does not sit on a rational surface as an anticanonical divisor. ([FM] prop. 4.8).
Note the relations: $p + q + r = 21 = 12 + (m - r)$ and $m - r = 9$.

References

[F] R. FRIEDMAN: Global smoothings of varieties with normal crossings, to appear in *Annals of Math.*

[FM] R. FRIEDMAN and R. MIRANDA: Smoothing cusp singularities of small length, to appear in *Math. Annalen.*

[La1] H. LAUFER: Taut two dimensional singularities. *Math. Ann.* 205 (1973) 131–164.

[La2] H. LAUFER: On minimally elliptic singularities. *Am. J. Math.* 99 (1977) 1257–1295.

[La3] H. LAUFER: Simultaneous resolution of some families of isolated surface singularities, to appear in the *Arcata* singularities volume.

[L1] E. LOOIJENGA: Rational surfaces with an effective anticanonical divisor. *Annals of Math* 114 (1981) 267–322.

[L2] E. LOOIJENGA: The smoothing components of a triangle singularity, to appear in the *Arcata* singularities volume.

[N] V.V. NIKULIN: Integral symmetric bilinear forms and some of their applications. *Math. USSR Izvestiya* 14 (1980) 103–167.

[P0] H. PINKHAM: Deformations of algebraic varieties with G_m action. *Asterisque* 20 (1974) Soc. Math. France.

[P1] H. PINKHAM: Groupe de monodromie des singularités unimodulaires exceptionnelles, *C.R. Acad. Sc. Paris* 284 (A) (20 Juin 1977) 1515–1518.

[P2] H. PINKHAM: Smoothings of the $D_{p,q,r}$ singularities, $p + q + r = 22$. Appendix to a paper of E. LOOIJENGA, to appear in the *Arcata* singularities volume.

[P3] H. PINKHAM: Automorphisms of cusps and Inoue-Hirzebruch surfaces. *Comp. Math.* 52 (1984) 299–313.

[P4] H. PINKHAM: Deformations of normal surface singularities with $\mathbb{C}^*$ action. *Math. Ann.* 232 (1978) 65–84.

[S] J.-P. SERRE: *Cours d'arithmétique*. Paris, P.U.F. (1970).

[W1] J. WAHL: Elliptic deformations of minimally elliptic singularities. *Math. Ann.* 253 (1980) 241–262.

[W2] J. WAHL: Smoothings of normal surface singularities. *Topology* 20 (1981) 219–246.

[W3] J. WAHL: Derivations of negative weight and non-smoothability of certain singularities. *Math. Ann.* 258 (1982) 383–398.

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