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GUDRUN BRATTSTRÖM
STEPHEN LICHTENBAUM

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JACOBI-SUM HECKE CHARACTERS OF IMAGINARY QUADRATIC FIELDS

Gudrun Brattström * and Stephen Lichtenbaum **

In this paper we formulate a hypothesis concerning the values of the L-series of Jacobi-sum Hecke characters of an abelian number field k, and verify that this hypothesis is true if k is imaginary quadratic of odd class number. Roughly, this hypothesis asserts that if ψ is a Jacobi-sum Hecke character of k, such that the infinity-type of ψ is in the "good range", then $L(\psi, 0)$ is equal up to a rational number to the inverse of a product of values of the Γ-function at rational numbers which is associated to ψ, multiplied by the square-root of the discriminant of the maximal real subfield. More precisely, we have the following:

Using the notation of paragraph 1, let $\psi = \prod_{j=1}^{t}(J(\theta_j, k_j) \circ \mathbb{N}_{k/k_j})$ for subfields $k_1, k_2, \dots, k_t$ contained in k. The functional equation of the L-series of ψ may be written $\Gamma(\psi)L(0, \psi) = W_\psi \Gamma(\psi^{-1}\mathbb{N}^{-1})$-$L(0, \psi^{-1}\mathbb{N}^{-1})$. Here W_ψ is a non-zero constant and $\Gamma(\psi)$ and $\Gamma(\psi^{-1}\mathbb{N}^{-1})$ are, up to non-zero numbers, products of some values of the Γ-function. If both $\Gamma(\psi)$ and $\Gamma(\psi^{-1}\mathbb{N}^{-1})$ are finite and non-zero, then we say that the infinity-type $I(\psi)$ of ψ is in the good range, or, following Deligne ([Del]), that ψ is "critique". As Katz points out in ([Ka], p. 203), if k is imaginary (hence CM since we are assuming k to be abelian) this is equivalent to saying that there exists a CM-type Σ of k such that $I(\psi)$ is in what we may call $C(\Sigma)$, i.e. that $I(\psi) = \Sigma_{\sigma \in \Sigma} - n\sigma + d_\sigma(\bar{\sigma} - \sigma)$, and either (a) $n \geqslant 1$ and all $d_\sigma \geqslant 0$ or (b) $n < 1$ and all $d_\sigma \geqslant 1 - n$.

It is easy to verify that Σ must be unique. Let d^+ be the discriminant of the maximal totally real subfield of k. We then state the Γ-hypothesis as follows:

$$L(\psi, 0)\Gamma_\Sigma(\psi, \theta)(d^+)^{1/2} \quad \text{lies in } \mathbb{Q}. \tag{0.1}$$

REMARKS:

(1) This does not preclude $L(\psi, 0)$ being zero, in which case the Γ-hypothesis is automatically true.

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(2) It is by no means obvious that $\Gamma_\Sigma(\psi, \theta)$ is independent of θ in $\mathbb{R}^\times/\mathbb{Q}^\times$, as is implied by (0.1) if $L(\psi, 0) \neq 0$. For k imaginary quadratic, this follows from the generalized Deligne's theorem proved by Kubert and one of the authors in [K-L] and quoted here as Theorem 1.9, but it is still unknown in general.

(3) If k is real, we may make the analogous statement with Σ replaced by the set of all embeddings of k into $\mathbb{C}$, and this has been proven by Brattström in [B].

(4) Using results of Shimura, it is proved in [L] that (0.1) is true for any k if $\mathbb{Q}$ is replaced by $\overline{\mathbb{Q}}$.

(5) G. Anderson has recently shown ([A]) that the Γ-hypothesis is a consequence of Deligne's conjecture ([Del], p. 323).

(6) All of the above should be extended to the still more general Jacobi-sum characters defined in Kubert [Ku].

Some of the intermediate results obtained in this paper are themselves of independent interest. In paragraph 3 we show that if k is imaginary quadratic with class number one and is not equal to $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, or $\mathbb{Q}(\sqrt{-3})$, then the Jacobi-sum Hecke characters are exactly the Hecke characters of k of type A_0 which are Galois-equivariant. In paragraph 4, we prove Damerell's theorem for all imaginary quadratic fields of odd class number up to an element of the imaginary quadratic field in question (rather than up to an algebraic number as in Damerell's original paper [Da] or in Weil [W3]). A related version of Damerell's theorem is proved in [G-S].

We should like to express here our debt to André Weil, who first suggested that there should be a relationship between the values of L-series of Jacobi-sum Hecke characters and corresponding products of values of the Γ-function. We should also like to thank Dan Kubert for his contributions to this paper. Finally we thank the referee for suggesting valuable simplifications of the arguments in paragraph 4.

§1. Jacobi-sum Hecke characters

This paragraph is devoted to reviewing the basic definitions and results about Jacobi-sum Hecke characters, to be found in [W1], [W2], and [K-L].

First, let F be a finite field, χ a non-trivial additive character of F, and ψ a non-trivial multiplicative character of F. We define the (modified) Gauss sum $G(\chi, \psi)$ associated to χ and ψ to be $-\sum_{a\in F^\times}\chi(a)\psi(a)$.

Next, let N be an integer greater than 1, and let $K = \mathbb{Q}(\zeta_N)$ be the field obtained by adjoining a primitive N-th root of unity ζ_N to $\mathbb{Q}$. For $b \in (\mathbb{Z}/N\mathbb{Z})^\times$, we define $\sigma_b \in G(K/\mathbb{Q})$ by $\sigma_b(\zeta_N) = \zeta_N^b$, thus identifying $(\mathbb{Z}/N\mathbb{Z})^\times$ with $G(K/\mathbb{Q})$. For each rational prime p, let ψ_p be the additive character $\psi_p(a) = e^{2\pi i a/p}$, $a \in \mathbb{Z}/p\mathbb{Z}$. For any finite field F of characteristic p, let ψ_F be the additive character obtained by composing ψ_p with the trace from F to $\mathbb{Z}/p\mathbb{Z}$. Fix once and for all an embedding of $K=\mathbb{Q}(\zeta_N)$ into $\mathbb{C}$. Let $\mathfrak{p}$ be a prime ideal of K prime to N, and let $\mathbb{N}_{\mathfrak{p}}=q=p^f$. Let $a\in\mathbb{Z}/N\mathbb{Z}$ be different from 0.

Define $\chi^a_{\mathfrak{p}}(x)$, for x in $\kappa(\mathfrak{p})$, to be $t(x^{a(q-1)/N})$, where $t(\lambda)$ is the unique N-th root of unity in K reducing to the N-th root of unity λ in $\kappa(\mathfrak{p})$. Then define a function $J_N(a)$ from the set of prime ideals of K prime to N to the complex numbers by $J_N(a)(\mathfrak{p})=G(\chi^a_{\mathfrak{p}},\psi_{\kappa(\mathfrak{p})})$. Extend $J_N(a)$ multiplicatively to a homomorphism from the group of fractional ideals of K prime to N into $\mathbb{C}^\times$. Let θ be an element of the free abelian group $\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}-\{0\}]$, and define $J_N(\theta)$ be extending $J_N(a)$ multiplicatively.

Now let k be an abelian extension of $\mathbb{Q}$, and assume that $k\subseteq\mathbb{Q}(\zeta_{N_i})$ for $i=1,\ldots s$, and that all the N_i are distinct. Let θ be an element of the free abelian group on the disjoint union of $\mathbb{Z}/N_i\mathbb{Z}-\{0\}$, $i=1,\ldots s$, and write $\theta=\Sigma^s_{i=1}\theta_i$, $\theta_i=\Sigma n_i(a)[a]_{N_i}$. If $\mathfrak{a}$ is an ideal of k prime to every N_i, define $J_{N_i}(\theta_i,k)(\mathfrak{a})$ to be $J_{N_i}(\theta_i)(\mathfrak{a}\mathfrak{o}_{N_i})$, where $\mathfrak{o}_{N_i}$ is the ring of integers in $\mathbb{Q}(\zeta_{N_i})$. Then define $J(\theta,k)(\mathfrak{a})$ to be $\Pi^s_{i=1}J_{N_i}(\theta_i,k)(\mathfrak{a})$. When is $J(\theta,k)$ a Hecke character? The answer is given by the following result ([K-L]) which is a straight-forward generalization of the similar result of Weil ([W1], [W2]):

If α is in $\mathbb{Z}/N\mathbb{Z}$, let a be any integer representing α, and write $\langle\alpha/N\rangle=\langle a/N\rangle=$ the fractional part of a/N. Let $\theta=\Sigma^s_{i=1}\theta_i$ and $\theta_i=\Sigma n_i(a)[a]_{N_i}$. Let $K_i=\mathbb{Q}(\zeta_{N_i})$. Define $I(\theta,k)$ to be $\Sigma^s_{i=1}\rho_i(\Sigma_a\Sigma_b n_i(a)\langle -ab/N_i\rangle\sigma_b^{-1})$, where b runs through $(\mathbb{Z}/N_i\mathbb{Z})^\times$ and ρ_i is the natural map from $\mathbb{Q}[G(K_i/\mathbb{Q})]$ to $\mathbb{Q}[G(k/\mathbb{Q})]$.

THEOREM 1.1:

(a) *The following are equivalent:*

(i) *$J(\theta,k)$ is a Hecke character.*

(ii) *$I(\theta,k)$ lies in $\mathbb{Z}[G(k/\mathbb{Q})]$.*

(b) *If these conditions are satisfied, then*

(iii) *$I(\theta,k)$ is the infinity-type of $J(\theta,k)$.*

(iv) *$J(\theta,k)$ has values in k.*

(v) *If σ is any automorphis in $G(k/\mathbb{Q})$, then $J(\theta,k)(\mathfrak{a}^\sigma)=(J(\theta,k)(\mathfrak{a}))^\sigma$, i.e. $J(\theta,k)$ is Galois-equivariant.*

(vi) *If $\theta=\Sigma\Sigma n_i(a)[a]_{N_i}$ and b is any integer prime to all the N_i, then $(J(\theta,k)(\mathfrak{a}))^{\sigma_b}=J(b^*\theta,k)(\mathfrak{a})$, where $b^*\theta=\Sigma\Sigma n_i(a)[ba]_{N_i}$.*

DEFINITION 1.2: A Hecke character of k which may be written in the form $J(\theta,k)$ for θ as above is called a *strict Jacobi-sum character* of k.

DEFINITION 1.3: If $s=1$, so $\theta=\theta_1=\Sigma n(a)[a]_N$, we say that $J(\theta,k)$ is *pure* of *level N*.

DEFINITION 1.4: If k is any number field, then ψ is a *Jacobi-sum character* of k if there exist abelian extensions k_j of $\mathbb{Q}$, $j = 1, \dots t$, such that $k_j \subseteq k$, and corresponding θ_j, such that $\psi = \prod_{j=1}^{t}(J(\theta_j, k_j) \circ \mathbb{N}_{k/k_j})$, where each $J(\theta_j, k_j)$ is a strict Jacobi-sum character of k_j.

DEFINITION 1.5: Let $\theta = \Sigma_i \Sigma_a n_i(a)[a]_{N_i}$, and let σ be in $G(k/\mathbb{Q})$. Choose b_i prime to N_i such that $\sigma = \sigma_{b_i}$. Define $\Gamma_\sigma(\theta, k)$ in $\mathbb{R}^\times/\mathbb{Q}^\times$ to be the class of $\prod_i \prod_a \prod_c \Gamma(cb_i a/N_i)^{n_i(a)}$, where c runs through $G(\mathbb{Q}(\zeta_{N_i})/k)$. (It is immediate that this is independent of the choice of b_i.) In particular, we let $\Gamma(\theta, k)$ be $\Gamma_\sigma(\theta, k)$ when σ is the identity element of $G(k/\mathbb{Q})$.

DEFINITION 1.6: Let $\psi = \prod_{j=1}^{t} J(\theta_j, k_j) \circ \mathbb{N}_{k/k_j}$. Define $\Gamma_\sigma(\psi, \theta)$ to be $\prod_{j=1}^{t} \Gamma_\sigma(\theta_j, k_j)$ in $\mathbb{R}^\times/\mathbb{Q}^\times$. (We write $\Gamma_\sigma(\psi, \theta)$ to emphasize the dependence of Γ_σ upon the choice of Jacobi-sum representation θ for ψ.)

Let k be a CM-field, so a totally imaginary quadratic extension of a totally real number field. Let Σ be a CM-type of k; so Σ contains exactly one element from each pair of conjugate complex embeddings of k. Assume that $k \subset \mathbb{C}$, and identify embeddings with elements of the Galois group $G(k/\mathbb{Q})$.

DEFINITION 1.7:
(a) $\Gamma_\Sigma(\theta, k) = \prod_{\sigma \in \Sigma} \Gamma_{\sigma^{-1}}(\theta, k)$.
(b) $\Gamma_\Sigma(\psi, \theta) = \prod_{\sigma \in \Sigma} \Gamma_{\sigma^{-1}}(\psi, \theta)$.

LEMMA 1.8: *Let $\psi = J(\theta)$ be a Jacobi-sum character of a field $k \subset \mathbb{C}$. Let $\sigma = \sigma_c$ be an element of $G(k/\mathbb{Q})$. Define ψ^σ by $\psi^\sigma(\mathfrak{p}) = (\psi(\mathfrak{p}))^\sigma$ for $\mathfrak{p}$ a prime ideal of k. Then $\Gamma(\psi^\sigma, c^*\theta) = \Gamma_\sigma(\psi, \theta)$.*

PROOF: We may assume that $\psi = J(\theta)$, $\theta = \Sigma_i \Sigma_a n_i(a)[a]_{N_i}$. Then $\psi^\sigma = J(c^*\theta)$, by Theorem 1.1, (vi). So

$$\Gamma(\psi^\sigma) = \prod_i \prod_a \prod_b \Gamma\left(\frac{bca}{N_i}\right)^{n_i(a)},$$

where b runs through $G(\mathbb{Q}(\zeta_{N_i})/k)$. On the other hand, by definition $\Gamma_\sigma(\psi) = \prod_i \prod_a \prod_d \Gamma(da/N_i)^{n_i(a)}$, where d in $G(\mathbb{Q}(\zeta_{N_i})/\mathbb{Q})$ restricts to $\sigma = \sigma_c$ in $G(k/\mathbb{Q})$. But clearly σ_{bc} restricts to σ_c if and only if σ_b leaves k fixed.

THEOREM 1.9 (*Generalized Deligne's theorem*): *Let $\psi = J(\theta, k)$ be a strict Jacobi-sum character of k, and assume that ψ is of the form $\chi\mathbb{N}^r$, where $r \in \mathbb{Z}$ and χ is a Dirichlet character of k. let*

$$\Gamma^*(\theta, k) = \Gamma(\theta, k)(2\pi i)^{-r}.$$

Then $\Gamma^*(\theta, k)$ *transforms via* χ, *in the sense that*:

(i) $\Gamma^*(\theta, k)$ *generates the abelian extension* K *of* k *corresponding to* χ;

(ii) *if* σ *is in* $G(K/k)$, *then* $\sigma(\Gamma^*(\theta, k)) = \chi(\sigma)\Gamma^*(\theta, k)$.

PROOF: This is the main result of [K-L]. For pure characters it is due to Deligne ([De1], [De2]).

§2. Jacobi-sum characters of $\mathbb{Q}$

In this section we completely identify all Jacobi-sum characters of $\mathbb{Q}$ and show that the Γ-hypothesis holds for $\mathbb{Q}$. Along the way, we demonstrate various results which will prove useful in subsequent sections. The complete identification of the jacobi-sum characters is a special case of a result of one of us (Brattström), who solved the analogous problem for arbitrary real abelian fields.

Let $\theta = \sum_{i=1}^{s} \theta_i$, $\theta_i = \sum n_i(a)[a]_{N_i}$. Since σ_{-1} is in $G(K_i/\mathbb{Q})$ and is different from 1 if $N_i > 2$, we see immediately from Theorem 1.1 that:

PROPOSITION 2.1: *The only condition for* $J(\theta, \mathbb{Q})$ *to be a Hecke character is that if* $N_i = 2$, $n_i(a)$ *must be even.*

It then follows that if $J(\theta, \mathbb{Q})$ *is a Hecke character, so are all the* $J(\theta_j, \mathbb{Q})$, *and it suffices to assume* $s = 1$ *to determine all Jacobi-sum Hecke characters of* $\mathbb{Q}$. *So let* $\theta = \sum n(a)[a]_N$, *with* $n(1)$ *even if* $N = 2$, *and write* $J_N(\theta, \mathbb{Q}) = J(\theta, \mathbb{Q})$. *Since* $J(\theta, \mathbb{Q})$ *is a Hecke character of* $\mathbb{Q}$ *with values in* $\mathbb{Q}$, *it must be of the form* $\chi_d \mathbb{N}^r$, *where* r *is an integer,* $\mathbb{N}$ *is the norm character of* $\mathbb{Q}$ *and* χ_d *is the character corresponding to the extension* $\mathbb{Q}(\sqrt{d})$ *of* $\mathbb{Q}$ *of degree* 1 *or* 2. *We write* $J(\theta, \mathbb{Q}) = \chi_{d(\theta)} \mathbb{N}^{r(\theta)}$. *Let* $\epsilon(\theta) = \sum n(a)$. *Let* ϕ *denote the Euler* ϕ*-function. Then we have, precisely,*

THEOREM 2.2: *The infinity-type of* $J_N(\theta, \mathbb{Q}) = r = \phi(N)\epsilon(\theta)/2$. *If* $\theta = [a]_N$, *then* $d = 1$, *unless*:

(i) $N = l^k$, *l prime* $\equiv 1 (\mathrm{mod}\ 4)$, *when* $d = l$.

(ii) $N = l^k$, *l prime* $\equiv 3 (\mathrm{mod}\ 4)$, *when* $d = -l$.

(iii) $N = 2^k$, $k \geqslant 3$, *a odd, when* $d = 2$.

(iv) $N = 2l^k$, *l prime* $\equiv 1 (\mathrm{mod}\ 4)$, *$a$ even, when* $d = l$.

(v) $N = 2l^k$, *l prime* $\equiv 3 (\mathrm{mod}\ 4)$, *when* $d = -l$ *if a is even and* $d = -1$ *if a is odd*

(vi) $N = 4$, *when* $d = -1$ *if a is even and* $d = -2$ *if a is odd.*

If $N = 2$ *and* $\theta = 2[1]_2$, *then* $d = -1$.

The proof is given in [B].

We now wish to describe completely when a character $\chi_d \mathbb{N}^r$ can be of the form $J(\theta, \mathbb{Q})$, i.e. is a Jacobi-sum Hecke character of $\mathbb{Q}$.

COROLLARY 2.3: *Let d be square-free and let $r \in \mathbb{Z}$. $\chi_d \mathbb{N}^r$ is a Jacobi-sum Hecke character if and only if either r is even d is positive, or r is odd and d is negative.*

PROOF: Let S be the subgroup of Jacobi-sum characters. From Theorem 2.2, we know $\chi_{-3}\mathbb{N} \in S$, hence $\mathbb{N}^2 \in S$.
If $l \equiv 1 (\text{mod } 4)$, $\chi_l \mathbb{N}^{l-1/2} \in S$, hence $\chi_l \in S$.
If $l \equiv 3 (\text{mod } 4)$, $\chi_{-l} \mathbb{N}^{l-1/2} \in S$, hence $\chi_{-l}\mathbb{N} \in S$.
From (vi) $\chi_{-1}\mathbb{N}$ and $\chi_{-2}\mathbb{N} \in S$.
From (iii) $\chi_2 \in S$, and it is easily checked that these generate all characters of the desired type and that no other occur.

We now define $\Gamma_N(a)$ to be the class of $\prod_{b_{(b,N)=1}} \Gamma(ab/N)$ in $\mathbb{R}^\times/\mathbb{Q}^\times$, for $a \in \mathbb{Z}$, $a \not\equiv 0$ (mod N), where the product is taken over b strictly between 0 and N and prime to N. We wish to compute this in all cases up to a rational number. (Note that $\Gamma_N(a) = \Gamma(\theta, \mathbb{Q})$ for $\theta = [a]_N$.)

THEOREM 2.4:

(i) *If $a \equiv c (\text{mod } N)$, $\Gamma_N(a) = \Gamma_N(c)$.*

(ii) *If N is divisible by two odd primes, or by 4 and an odd prime, then $\Gamma_N(a) = \pi^{\phi(N)/2}$.*
If $N = 2^k$, $k \geqslant 2$ and a is even, $\Gamma_N(a) = \pi^{\phi(N)/2}$.
If $N = 2^k$, $k \geqslant 2$ and a is odd, $\Gamma_N(a) = \pi^{\phi(N)/2} \cdot 2^{1/2}$.
If $N = l^k$, and l is odd, $\Gamma_N(a) = \pi^{\phi(N)/2} \cdot l^{1/2}$.
If $N = 2l^k$ and a is odd, $\Gamma_N(a) = \pi^{\phi(N)/2}$.
If $N = 2l^k$ and a is even, $\Gamma_N(a) = \pi^{\phi(N)/2} \cdot l^{1/2}$.

(All equalities are of course in $\mathbb{R}^\times/\mathbb{Q}^\times$.)

PROOF: (i) follows immediately, since $\Gamma(z+1) = z\Gamma(z)$. For (ii), let us first assume $(a, N) = 1$ and $N > 2$, and let $K = \mathbb{Q}(\zeta_N)$. Since $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$,

$$\prod_{\substack{b=1 \\ (b,N)=1}}^{N} \Gamma\left(\frac{b}{N}\right) = \pi^{\phi(N)/2} \prod_{\substack{b=1 \\ (b,N)=1}}^{N} \left(\sin \frac{\pi b}{N}\right)^{-1/2}.$$

Let $R_N = \prod_{b=1_{(b,N)=1}}^{N} \sin(\pi b/N)$. Then R_N is obviously real and positive. Observe that if $\zeta_N = e^{2\pi i/N}$,

$$\sin \frac{\pi b}{N} = \frac{\zeta_{2N}^b - \zeta_{2N}^{-b}}{2i} \in \mathbb{Q}(i, \zeta_{2N}).$$

But $\zeta_{2N}^b - \zeta_{2N}^{-b} = \zeta_{2N}^{-b}(\zeta_{2N}^{2b} - 1)$. As b runs from 1 to N, with $(b, N) = 1$, $(\zeta_{2N}^{2b} - 1) = (\zeta_N^b - 1)$ runs through a complete set of conjugates for $(\zeta_N - 1)$. So $R_N = \zeta_{2N}^{-\Sigma b} \cdot \mathbb{N}_{K/\mathbb{Q}}(\zeta_N - 1)/(2i)^{\phi(N)}$. Now $\mathbb{N}_{K/\mathbb{Q}}(\zeta_N - 1) = l$ if N is a power of l, l prime, and is equal to 1 otherwise. Since R_N is real and positive, $R_N = |R_N| = l/2^{\phi(N)}$ if N is a power of l, $1/2^{\phi(N)}$ if not. So if $(a, N) = 1$, $\Gamma_N(a) = \prod_{b=1_{(b,N)=1}}^{N} \Gamma(ab/N) = \prod_{b=1_{(b,N)=1}}^{N} \Gamma(b/N)$, which equals $\pi^{\phi(N)/2} \cdot l^{-1/2}$ if N is a power of l, $\pi^{\phi(N)/2}$ if not. Next we consider the general case. Let $(a, N) = d$, and write $a = kd$, $N = md$, with $(k, m) = 1$. Since $(\mathbb{Z}/N\mathbb{Z})^\times$ maps onto $(\mathbb{Z}/m\mathbb{Z})^\times$, we have:

$$\Gamma_N(a) = \prod_{(b,N)=1} \Gamma\left(\frac{kbd}{md}\right) = \prod_{(b,N)=1} \Gamma\left(\frac{kb}{m}\right)$$

$$= \prod_{(b,m)=1} \Gamma\left(\frac{kb}{m}\right)^{\phi(N)/\phi(m)} = \Gamma_m(k)^{\phi(N)/\phi(m)}.$$

Since we have just computed $\Gamma_m(k)$, an easy case-by-case analysis concludes the proof of the theorem.

PROPOSITION 2.5: *Let χ_d be the character corresponding to $\mathbb{Q}(\sqrt{d})$, and let $r \in \mathbb{Z}$. Then $L(\chi_d, -r) = \pi^{-r}\sqrt{|d|}$ in $\mathbb{R}^\times/\mathbb{Q}^\times$ if r is negative and even and d is positive, or if r is negative and odd and d is negative.*

PROOF: This follows immediately from standard results on the L-functions of Dirichlet character. (See [I], especially pp. 9 and 12.)

Now, if ψ is any Hecke character of $\mathbb{Q}$, let $L(\psi) = L(\psi, 0)$. Observe that $L(\chi_d \mathbb{N}^r) = L(\chi_d, -r)$. If N is even, let $\epsilon_1(\theta) = \Sigma n(a)$, a odd, and $\epsilon_2(\theta) = \Sigma n(a)$, a even, so $\epsilon(\theta) = \epsilon_1(\theta) + \epsilon_2(\theta)$. For convenience, let $J(\theta) = J_N(\theta, \mathbb{Q})$, $L(\theta) = L(J(\theta))$, and $\Gamma(\theta) = \Gamma(\theta, \mathbb{Q})$. It follows immediately from our previous results that we have:

PROPOSITION 2.6: *Let $m = \phi(N)/2$.*
(1) *If N is divisible by two odd primes or by $4l$, l an odd prime, then $J(\theta) = N^{m\epsilon(\theta)}$ and $\Gamma(\theta) = \pi^{m\epsilon(\theta)}$.*

(2) *If $N = l^k$, l* prime $\equiv 1 \pmod 4$, *then $J(\theta) = \chi_l^{\epsilon(\theta)} \mathbb{N}^{m\epsilon(\theta)}$ and $\Gamma(\theta) = \pi^{m\epsilon(\theta)}(l^{1/2})^{\epsilon(\theta)}$.*

(3) *If $N = l^k$, l* prime $\equiv 3 \pmod 4$, *then $J(\theta) = \chi_{-l}^{\epsilon(\theta)} \mathbb{N}^{m\epsilon(\theta)}$ and $\Gamma(\theta) = \pi^{m\epsilon(\theta)}(l^{1/2})^{\epsilon(\theta)}$.*

(4) *If $N = 2l^k$, l* prime $\equiv 1 \pmod 4$, *then $J(\theta) = \chi_l^{\epsilon_2(\theta)} \mathbb{N}^{m\epsilon(\theta)}$ and $\Gamma(\theta) = \pi^{m\epsilon(\theta)}(l^{1/2})^{\epsilon_2(\theta)}$.*

(5) *If $N = 2l^k$, l* prime $\equiv 3 \pmod 4$, *then $J(\theta) = \chi_{-l}^{\epsilon_2(\theta)} \chi_{-1}^{\epsilon_1(\theta)} \mathbb{N}^{m\epsilon(\theta)}$ and $\Gamma(\theta) = \pi^{m\epsilon(\theta)}(l^{1/2})^{\epsilon_2(\theta)}$.*

(6) *If $N = 2^k$, $k \geqslant 3$, then $J(\theta) = \chi_2^{\epsilon_1(\theta)} \mathbb{N}^{m\epsilon(\theta)}$ and $\Gamma(\theta) = \pi^{m\epsilon(\theta)}(2^{1/2})^{\epsilon_1(\theta)}$.*

(7) *If $N = 4$, $J(\theta) = \chi_{-2}^{\epsilon_1(\theta)} \chi_{-1}^{\epsilon_2(\theta)} \mathbb{N}^{m\epsilon(\theta)}$ and $\Gamma(\theta) = \pi^{m\epsilon(\theta)}(2^{1/2})^{\epsilon_1(\theta)}$.*

Now let $\theta = \Sigma_{i=1}^{s} \theta_i$, and let $\epsilon(\theta) = \Sigma_{i=1}^{s} \epsilon(\theta_i)$, and $\Gamma(\theta) = \prod_{i=1}^{s} \Gamma(\theta_i)$.

THEOREM 2.7: *Assume $r(\theta) < 0$. Then $L(J(\theta)) = \Gamma(\theta)^{-1}$ in $\mathbb{R}^\times/\mathbb{Q}^\times$.*

PROOF: If ψ is a Hecke character of $\mathbb{Q}$ of the form $\chi_d \mathbb{N}^r$, with either r even and d positive or r odd and d negative, define $E(\psi)$ to be $\pi^r\sqrt{|d|}$. Clearly $E(\psi)$ is multiplicative in ψ. By corollary 2.3, $E(\psi)$ is defined for every Jacobi-sum character. We claim that $E(J(\theta)) = \Gamma(\theta)$. Both sides are obviously multiplicative in θ, so we may assume that θ is pure. (See Definition 1.3.) But then the claim follows immediately from Proposition 2.6. $J(\theta) = \chi_{d(\theta)}\mathbb{N}^{r(\theta)}$, so if $r(\theta) < 0$, then Proposition 2.5 gives $L(J(\theta)) = E(J(\theta))^{-1}$ and hence $L(J(\theta)) = \Gamma(\theta)^{-1}$ in $\mathbb{R}^\times/\mathbb{Q}^\times$.

REMARK 2.8: This is indeed the full Γ-hypothesis for $\mathbb{Q}$ since if $r \geqslant 0$ then $I(\psi)$ does not lie in the good range. By Corollary 2.3 either χ_d is real and r even or χ_d is complex and r odd. In the former case the functional equation involves the factors $\Gamma((1-s)/2)$ and $\Gamma(s/2)$ (along with some exponential factors which never have poles or zeros). Since r is even $\Gamma(s/2)$ has a pole at $s = -r$ if $r \geqslant 0$ and is otherwise finite and non-zero at this point, whereas $\Gamma((1+r)/2)$ is always finite and non-zero. Hence $I(\psi)$ is in the good range if and only if $r < 0$. Similar considerations show that this is true when r is odd also.

§3. The independence and existence theorems

Let k be an arbitrary imaginary quadratic field, with ring of integers $\mathfrak{o} = \mathfrak{o}_k$. In this section we will prove that our main theorem is independent of the choice of the Jacobi-sum representation for a given Hecke character. If k has class number one and is not $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ we will give a complete characterization of all Jacobi-sum Hecke characters, and we will obtain a partial characterization in the cases $k = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$.

We also will explicitly compute the periods of our basic elliptic curves with complex multiplication by $\mathfrak{o}_k$ if $k = \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, or $\mathbb{Q}(\sqrt{-3})$.

Let $G = G(k/\mathbb{Q})$, and let e and ρ be the identity and non-identity elements of G, respectively. Denote the norm character of k by $\mathbb{N}$.

LEMMA 3.1:

(a) *Let* $k = \mathbb{Q}(\sqrt{-p})$, *$p$ prime*, $p \equiv 3 (mod\ 4)$, $p \neq 3$. *Let* $\psi = J([1]_p, k)$, *and let h be the class number of k. Then the infinity-type of* $\psi = \frac{1}{4}(p - 1 + 2h)e + \frac{1}{4}(p - 1 - 2h)\rho$.

(b) *If* $k = \mathbb{Q}(\sqrt{-1})$, *let* $\psi = J([1]_4 + [2]_4 - [3]_4, k)$. *Then the infinity-type of* ψ *is* e.

(c) *If* $k = \mathbb{Q}(\sqrt{-2})$, *let* $\psi = J([1]_8 - [5]_8, k)\mathbb{N}$. *Then the infinity-type of* ψ *is* $2e$.

(d) *If* $k = \mathbb{Q}(\sqrt{-3})$, *let* $\psi = J([2]_6 + [3]_6 - [5]_6, k)$. *Then the infinity-type of* ψ *is* e.

PROOF: (a) By Theorem 1.1, the infinity-type of ψ is equal to $Ae + B\rho$, where $A = \Sigma_a\langle -a/p\rangle$ and $B = \Sigma_b\langle -b/p\rangle$, where a runs through quadratic residues mod p and b runs through quadratic non-residues. By the analytic class number formula for imaginary quadratic fields (see for instance [B-S], p. 344), $A - B = h$. On the other hand $A + B = (p-1)/2$, so the lemma follows immediately. The proofs of (b), (c), and (d) are just simple computations from Theorem 1.1.

LEMMA 3.2: *Let $k = \mathbb{Q}(\sqrt{-2})$, and let ψ be any Jacobi-sum character of k. Let the infinity-type of ψ be $Ae + B\rho$. Then $A \equiv B \pmod 2$.*

PROOF: Any character induced from $\mathbb{Q}$ via the norm has infinity-type a multiple of the norm. It follows readily from the results of [K-L] that any strict Jacobi-sum character of k has an infinity-type which differs by a multiple of the norm from the infinity-type of a pure character of level 8. An explicit computation of all Jacobi-sum characters of level 8 completes the proof.

REMARK 3.3: This exceptional lack of a Jacobi-sum character of infinity-type e is due to a not sufficiently general definition of our Jacobi-sum characters. For the correct definition, see [Ku].

Let k be an imaginary quadratic field with odd class number; this is wellknown to imply that k is of the form $\mathbb{Q}(\sqrt{-p})$, where p is either 1, 2 or a prime congruent to 3 modulo 4. For $p > 3$, let E be the $\mathbb{Q}$-curve $A(p)$ defined in [G], p. 35.

If $p = 1$ let E be the curve $y^2 = x^3 - 4x$,

if $p = 2$ let E be $y^2 = x^3 + x^2 - 3x + 1$ and

if $p = 3$ let E be $y^2 = x^3 + 1$.

More precisely, we choose an embedding $\iota : \bar{k} \to \mathbb{C}$ (which will remain fixed for the rest of the paper) such that the modular invariant of E is mapped to $j(\mathfrak{o}) \in \mathbb{C}^\times$. Then E has complex multiplication by $\mathfrak{o}$ and in all cases except $p = 3$ (where $\Delta = -2^4 \cdot 3^3$) E has bad reduction only at primes dividing the discriminant of k. Let H be the Hilbert class field and h the class number of k.

Let θ be an isomorphism $k \xrightarrow{\sim} \mathrm{End}(E) \otimes \mathbb{Q}$ such that $\omega \circ \theta(\alpha) = \alpha\omega$ for all differential forms ω of the first kind and all $\alpha \in k$. (See [La2] p. 119.) Given θ and the embedding $\iota : H \to \mathbb{C}$, we can associate to E a (complex-valued) Hecke character χ_E of H. Composing it with the inclusion of the ideals of k into those of H produces a Hecke character ψ_E of k. ψ_E is of type A_0, has infinity-type he (e being the trivial element of $G(k/\mathbb{Q})$) and takes values in k. Our assumption that E is a $\mathbb{Q}$-curve implies that $\chi_{E^\sigma} = \chi_E$ for any $\sigma \in G$ ([G], Lemmas 9.3.1 and 11.1.1), so if we define ψ_{E^σ} in analogy with ψ_E then we have $\psi_{E^\sigma} = \psi_E$. (The latter statement is in fact true even if we do not assume E to be a $\mathbb{Q}$-curve; this follows readily from Lemma 9.3.1 in [G] and the fact that χ_E takes values in k.) By [G], Lemma 11.1.1, χ_E is Galois-equivariant, and hence so is ψ_E.

LEMMA 3.4:

(a) *Let* $k \neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, *or* $\mathbb{Q}(\sqrt{-3})$, *so* $k = \mathbb{Q}(\sqrt{-p})$, p prime, $p \equiv 3 (\text{mod } 4)$, $p \neq 3$. *Let* $\psi_J = J([1]_p, k)$. *Then* $\psi_E = \psi_J \mathbb{N}^{-d}$, *where* $d = \frac{1}{4}(p - 1 - 2h)$.

(b) *Let* $k = \mathbb{Q}(\sqrt{-1})$. *Let* $\psi_J = J([1]_4 + [2]_4 - [3]_4, k)$. *Then* $\psi_E = \chi_{2,k}\psi_J$ *where* $\chi_{2,k}$ *is the quadratic character of* k *corresponding to* $k(\sqrt{2}) = \mathbb{Q}(\mu_8)$.

(c) *Let* $k = \mathbb{Q}(\sqrt{-2})$, *Let* $\psi_J = J([1]_8 - [5]_8)$. *Then* $\psi_E^2 = \psi_J \mathbb{N}$.

(d) *Let* $k = \mathbb{Q}(\sqrt{-3})$. *Let* $\psi_J = J([2]_6 + [3]_6 - [5]_6)$. *Then* $\psi_E = \psi_J$.

PROOF: (a) By Lemma 3.1, ψ_E and $\psi_J \mathbb{N}^{-d}$ have the same infinity-type, so there exists a Dirichlet character χ_0 such that $\psi_E = \chi_0 \psi_J \mathbb{N}^{-d}$. By [W2] the conductor of ψ_J is a power of $\mathfrak{p} = (\sqrt{-p})$. (On p. 14 of [W2], Weil states that the conductor of a Jacobi-sum Hecke character $J_N(\theta, k)$ is divisible only by primes dividing $2N$, but what he actually proves is the same statement with $2N$ replaced by N.) By Theorem 11.2.4, p. 33 in [G], χ_E is ramified only at primes lying over p, so the conductor of χ_E is also a power of $\mathfrak{p}$. Hence so is the conductor of χ_0. Moreover, since ψ_E, ψ_J and $\mathbb{N}$ all values in k so does χ_0, so χ_0 is either quadratic or trivial. If we let k_0 be the extension of k corresponding to χ_0 then the conductor of χ_0 equals the discriminant $D_{k_0/k}$. Since the $\mathfrak{p}$-ramification in k_0/k is tame, $D_{k_0/k}$ is either $\mathfrak{p}$ or $\mathfrak{o}$ (see [La1], p. 62), whence $k_0 \subset H_\mathfrak{p}$, the ray class field mod $\mathfrak{p}$ of k. However, $G(H_\mathfrak{p}/H) \cong (\mathfrak{o}/\mathfrak{p})^\times/\{\pm 1\}$, *so* $[H_\mathfrak{p} : k] = [H_\mathfrak{p} : H][H : k] = \frac{1}{2}(p-1)h$ *is odd and we conclude that* $k_0 = k$ *and* $\chi_0 \equiv 1$. *Hence* $\psi_E = \psi_J \mathbb{N}^{-d}$.

(*b*) *The curve* $y^2 = x^3 - 4x$ *is birational over* $\mathbb{Q}$ *to the curve* $E' : y^2 = x^4 + 1$. *It is shown in* [*W*1] *that the Hecke character* $\psi_{E'}$ *is equal to* $\chi_{2,k} J([1]_4 + [2]_4 - [3]_4)$. (*This is the special case of Weil's paragraph* 2 *where* $e = 2$, $f = 4$, $\gamma = \delta = 1$. *Then in Weil's notation*, $m_0 = 4$, *and the only Hecke character to occur is* $J_{1,2}$, *which is easily seen to be* $\chi_{2,k}(J([1]_4 + [2]_4 - [3]_4)$ in our notation.)

(c) By Lemma 3.1, ψ_E^2 and $J([1]_8 - [5]_8)\mathbb{N}$ have the same infinity-type. Since they are both Galois-equivariant and unramified outside of 2, they must either be equal or differ by the quadratic character of $\mathbb{Q}(\sqrt{-2})$ corresponding to $\mathbb{Q}(\mu_8)$. Since this quadratic character is equal to -1 on the prime ideal $\mathfrak{p} = (1 + \sqrt{-2})$ of $\mathbb{Z}[\sqrt{-2}]$ lying above 3, if suffices to check whether $\psi_E^2(\mathfrak{p})/9$ and $(J([1]_8 - [5]_8)\mathbb{N})/9$ are congruent mod $\mathfrak{p}$.

Since $\psi_E(\mathfrak{p}) = \pm(1+\sqrt{-2})$, $\psi_E^2(\mathfrak{p}) = (1+\sqrt{-2})^2$, and $(1+\sqrt{-2})^2/9 \equiv 1 \pmod{\mathfrak{p}}$. We then use Stickelberger's theorem to show that

$$(J([1]_8 - [5]_8)\mathbb{N})/9 \equiv 1 (\mathrm{mod}\ \mathfrak{p}),$$

and we are done.

(d) Weil shows in [W1] that $\psi_E = J([2]_6 - [3]_6 - [5]_6)$. (This is the special case of paragraph 2 where $e = 2$, $f = 3$, $\gamma = \delta = 1$. Then in Weil's notation, $m_0 = 6$, and the only Hecke character to occur is $J_{2,3}$, which is easily seen to be $\chi_{-1,k} J([2]_6 - [3]_6 - [5]_6)$ in our notation, where $\chi_{-1,k}$ is the quadratic character of $k = \mathbb{Q}(\sqrt{-3})$ corresponding to $k(\sqrt{-1})$.)

THEOREM 3.5: *Let ψ be a Hecke character of k, and assume that k has class number one.*

(a) *Let $k \neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, or $\mathbb{Q}(\sqrt{-3})$. Then the following are equivalent*:

(1) *ψ is a Jacobi-sum character, i.e. $\psi = J(\theta_1, k)(J(\theta_2, \mathbb{Q}) \circ \mathbb{N}_{k/\mathbb{Q}})$ for some θ_1 and θ_2.*

(2) *ψ is of type A_0, takes values in k, and is Galois-equivariant.*

(3) *ψ may be written in the form $\chi_{d,k}\psi_E^{a-b}\mathbb{N}^b$, where $a, b \in \mathbb{Z}$, $d \in \mathbb{Q}$, $d > 0$ and $\chi_{d,k}$ is the quadratic character of k corresponding to $k(\sqrt{d})$.*

(b) *Let k be $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, or $\mathbb{Q}(\sqrt{-3})$. Then the following are equivalent*:

(1) *ψ is a Jacobi-sum character.*

(2) *ψ may be written in the form $\chi\psi_E^{a-b}\mathbb{N}^b$, where χ is a Jacobi-sum Dirichlet character, and $a - b$ is an arbitrary integer if k is $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, and an even integer if k is $\mathbb{Q}(\sqrt{-2})$.*

PROOF: That 1) $\Rightarrow$ 2) has been shown in [W2] and in [K-L]. We next show 2) $\Rightarrow$ 3). If ψ is of type A_0, it has infinity-type $ae + b\rho$ with $a, b \in \mathbb{Z}$, and so can be written in the form $\psi = \chi\psi_E^{a-b}\mathbb{N}^b$, with χ a Dirichlet character. Since ψ, ψ_E and $\mathbb{N}$ have values in k and are Galois-equivariant, the same must be true of χ. Let F be the extension of k corresponding to χ via class field theory. Then since χ is Galois-equivariant F is left fixed as a set by any element of $G(\overline{\mathbb{Q}}/\mathbb{Q})$, hence is Galois over $\mathbb{Q}$, hence abelian, being of degree four over $\mathbb{Q}$. Moreover, $G(F/\mathbb{Q})$ is non-cyclic since F contains the two distinct quadratic subfields k and F^+, the maximal real subfield of F. From this it follows that $F = k(\sqrt{d})$, with $d \in \mathbb{Q}$, $d > 0$, so $\chi = \chi_{d,k}$.

We now show 3) $\Rightarrow$ 1). We have seen in Section 2 (Corollary 2.3) that $\chi_{-p}\mathbb{N}_{\mathbb{Q}}$ is a Jacobi-sum character of $\mathbb{Q}$. Composing this with $\mathbb{N}_{k/\mathbb{Q}}$, we see that $\mathbb{N} = \mathbb{N}_k$ is a Jacobi-sum character of k. It then follows from Lemma 3.4 that ψ_E is a Jacobi-sum character of k. Again, by Corollary 2.3, if $d > 0$, then χ_d is a Jacobi-sum character of $\mathbb{Q}$, hence $\chi_{d,k}$ is a

Jacobi-sum character of k. It follows that ψ is a Jacobi-sum character of k.

(b) Let $k = \mathbb{Q}(\sqrt{-3})$. Then ψ_E is a Jacobi-sum character by Lemma 3.4d. Let $k = \mathbb{Q}(\sqrt{-1})$. Then $\chi_{2,k} = \chi_2 \circ \mathbb{N}_{k/\mathbb{Q}}$ is a Jacobi-sum character by Corollary 2.3, so ψ_E is a Jacobi-sum character by Lemma 3.4b. Let $k = \mathbb{Q}(\sqrt{-2})$. Then ψ_E^2 is a Jacobi-sum character by Lemma 3.4c. (Note that $\mathbb{N}$ is a Jacobi-sum character as above.) This shows that $2) \Rightarrow 1)$.

To show $1) \Rightarrow 2)$, we observe that by Lemma 3.1 if $k = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, and by Lemmas 3.1 and 3.2 if $k = \mathbb{Q}(\sqrt{-2})$, any Jacobi-sum character has the same infinity-type as $\psi_E^{a-b}\mathbb{N}^b$ for suitable integral a and b where $a-b$ is even if $k = \mathbb{Q}(\sqrt{-2})$. Since ψ_E and $\mathbb{N}$ are Jacobi-sum characters, so is $\chi = \psi(\psi_E^{b-a}\mathbb{N}^{-b})$, and χ is clearly Dirichlet.

For an imaginary quadratic field we prove the generalized Deligne's theorem for all Jacobi-sum characters of the form $\chi\mathbb{N}^r$, not just the strict ones:

THEOREM 3.6: *Let k be an arbitrary imaginary quadratic field. Let ψ be a Hecke character of k, and suppose $\psi = J(\theta_1, k)(J(\theta_2, \mathbb{Q}) \circ \mathbb{N}_{k/\mathbb{Q}}) = \chi\mathbb{N}^r$ where $r \in \mathbb{Z}$ and χ is a Dirichlet character. Then $\Gamma^*(\psi, \theta) = \Gamma(\theta_1, k)\Gamma(\theta_2, \mathbb{Q})(2\pi i)^{-r}$ transforms via χ in the sense of Theorem* 1.9.

PROOF: As in paragraph 2 we have $J(\theta_2, \mathbb{Q}) = \chi_2\mathbb{N}_{\mathbb{Q}}^{r_2}$, where $r_2 \in \mathbb{Z}$ and χ_2 is a Dirichlet character of $\mathbb{Q}$. (So for this proof we are deviating – temporarily – from our usual notation, in which χ_2 would have meant the character corresponding to $\mathbb{Q}(\sqrt{2})$.) Thus $J(\theta_2, \mathbb{Q}) \circ \mathbb{N}_{k/\mathbb{Q}} = (\chi_2 \circ \mathbb{N}_{k/\mathbb{Q}})\mathbb{N}^{r_2}$. On the other hand we are assuming $\psi = \chi\mathbb{N}^r$, so it follows that $J(\theta_1, k) = \chi_1\mathbb{N}^{r_1}$, with r_1 an integer and χ_1 a Dirichlet character of k. $J(\theta_1, k)$ is strict, so by Theorem 1.9 $\Gamma^*(\theta_1, k) = \Gamma(\theta_1, k)(2\pi i)^{-r_1}$ transforms via χ_1, Also by Theorem 1.9 $\Gamma^*(\theta_2, \mathbb{Q})$ transforms via χ_2 over $\mathbb{Q}$, hence transforms via $\chi_2 \circ \mathbb{N}_{k/\mathbb{Q}}$ over k. (See for instance [C-F], Prop. 3.2 p. 166.) From this it readily follows that $\Gamma^*(\psi, \theta) = \Gamma^*(\theta_1, k)\Gamma^*(\theta_2, \mathbb{Q})$ transforms via $\chi = \chi_1(\chi_2 \circ \mathbb{N}_{k/\mathbb{Q}})$.

COROLLARY 3.7: *If k is imaginary quadratic then $\Gamma(\psi, \theta)$ and $\Gamma_\rho(\psi, \theta)$ only depend on ψ and not on θ.*

PROOF: It suffices to show that $\Gamma(\psi, \theta)$ and $\Gamma_\rho(\psi, \theta)$ are rational if ψ is the trivial Dirichlet character. By Theorem 3.6 $\Gamma(\psi, \theta)$ then lies in k, and since $\Gamma(\psi, \theta)$ is real, it lies in $\mathbb{Q}$. By Lemma 1.8 $\Gamma_\rho(\psi, \theta) = \Gamma(\psi^\rho, c^*\theta)$ for some integer c. But $\psi^\rho = \psi \equiv 1$ so the corollary again follows from Theorem 3.6.

DEFINITION 3.8: If $\psi = J(\theta_1, k)(J(\theta_2, \mathbb{Q}) \circ \mathbb{N}_{k/\mathbb{Q}}$ we may define $\Gamma(\psi)$ to be $\Gamma(\theta_1)\Gamma(\theta_2)$ and $\Gamma_\rho(\psi)$ to be $\Gamma_\rho(\theta_1)\Gamma(\theta_2)$, and these definitions are independent of the choice of θ_1 and θ_2 by Corollary 3.7.

COROLLARY 3.9: *If $\psi = \psi_1\psi_2$, then $\Gamma(\psi) = \Gamma(\psi_1)\Gamma(\psi_2)$ and $\Gamma_\rho(\psi) = \Gamma_\rho(\psi_1)\Gamma_\rho(\psi_2)$.*

PROOF: Immediate from the fact that Definition 3.8 is independent of the choices made.

In the remainder of this section, we will compute the real periods for curves with complex multiplication by $\mathfrak{o}_k$, where $k = \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, or $\mathbb{Q}(\sqrt{-3})$. This is necessary because the theorem of Gross on which we rely heavily in paragraph 5 leaves out these exceptional cases.

First, let k be an arbitrary imaginary quadratic field, and let $L = \mathfrak{o}_k$ have the basis $\{1, \tau\}$ as a lattice. Let $\eta(\tau) = \mathrm{e}^{\pi i \tau/12}\prod_{n=1}^{\infty}(1 - \mathrm{e}^{2\pi n i \tau})$, and $\tilde{\omega} = 2\pi|\eta(\tau)|^2$. It follows then that

$$\tilde{\omega}^{12} = |\Delta(\mathfrak{o}_k)| = \pm\Delta(\mathfrak{o}_k),$$

where $\Delta(L)$ is the discriminant of the lattice L.

LEMMA 3.10: *Let $e_6(L) = \sum'_{x \in L} 1/x^6$, where the sum is taken over all non-zero x in L. Then if $k \neq \mathbb{Q}(\sqrt{-1})$, $e_6(L)$ is real and positive. If $k = \mathbb{Q}(\sqrt{-1})$, $e_6(L) = 0$. Let $e_4(L) = \sum'_{x \in L} 1/x^4$. Then if $k = \mathbb{Q}(\sqrt{-1})$, $e_4(L) > 0$.*

PROOF: If we let $L = [1, \tau]$, then $e_6(L)$ has the q-expansion $C(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n)$, where $\sigma_5(n)$ is the sum of the fifth powers of the divisors of n, $q = \mathrm{e}^{2\pi i \tau}$, and C is the positive constant $1/252 \cdot (2\pi)^6/(5!)$. (See for example [W3], p. 20.) If $\tau = \alpha i$ with α real, then $q^n = \mathrm{e}^{-2\pi n \alpha}$ and $e_6([1, \tau])$ is a strictly increasing function of α. If $\alpha = 1$, then $e_6(L) = 0$ since changing x to ix shows that $e_6(L) = -e_6(L)$. So $e_6([1, \alpha i]) > 0$ for $\alpha > 1$.

If $\tau = \frac{1}{2} + \alpha i$, α real, then $q^n = (-1)^n \mathrm{e}^{-2\pi n \alpha}$. Then

$$C^{-1}e_6 = 1 - 504\sum_n \sigma_5(n)(-1)^n \mathrm{e}^{-2\pi n \alpha} > C^{-1}e_6([1, \alpha i]),$$

and so is positive if $\alpha > 1$. We have now covered all imaginary quadratic fields except $k = \mathbb{Q}(\sqrt{-3})$. But if $\alpha = \frac{1}{2}\sqrt{3}$, easy estimates yield that

$$\left|\sum_n \sigma_5(n)(-1)^n \mathrm{e}^{-2\pi i n \alpha} - 0.003733212\right| < 10^{-8}$$

so e_6 is still positive. Finally, if $k = \mathbb{Q}(\sqrt{-1})$, $e_4(L)$ is a positive constant multiplied by $1 + \sum_{n \geqslant 1} 240\sigma_3(n)q^n$, so positive.

Now, let L be the lattice corresponding to E and the differential $\mathrm{d}x/2y$. Let ω_E be the real period of E. Since L has complex multiplication by $\mathfrak{o}_k$ and $h(k) = 1$, $L = \Omega\mathfrak{o}_k$ for some complex number Ω, defined up to a root of unity in k. Since E and $\mathrm{d}x/2y$ are defined over $\mathbb{Q}$, $L = \bar{L}$.

PROPOSITION 3.11:

(*a*) *If* $k = \mathbb{Q}(\sqrt{-3})$, Ω *may be taken to be purely imaginary, and then* $\omega_E = \Omega\sqrt{-3} = \tilde{\omega} \cdot 3^{1/4} \cdot 2^{-1/3}$.

(*b*) *If* $k = \mathbb{Q}(\sqrt{-2})$, Ω *is purely imaginary and* $\omega_E = \Omega\sqrt{-2} = \tilde{\omega} \cdot 2^{-3/4}$.

(*c*) *If* $k = \mathbb{Q}(\sqrt{-1})$, Ω *may be taken to be real and* $\omega_E = \Omega = 2^{-1}\tilde{\omega}$, *where all equalities are up to sign.*

PROOF: (a) As before, we must have $\Omega = \zeta\bar{\Omega}$, with ζ a root of unity in k. If $\Omega = \pm\rho\bar{\Omega}$ with $\rho^3 = 1$, replacing Ω by $\rho\Omega$ we may assume that $\Omega = \pm\bar{\Omega}$, i.e. that Ω is real or purely imaginary. By Lemma 3.10, $e_6(\mathfrak{o}_k) > 0$, so, since $e_6(L) = \Omega^{-6}e_6(\mathfrak{o}_k)$, Ω is imaginary if and only if $e_6(L) < 0$. but $e_6(L) = g_3(L)/140$ and $g_3(L)$ is easily computed from the formularly in [T], p. 180 to be -4. Since Ω is imaginary, $\omega_E = \Omega\sqrt{-3}$, and since $-3^3 \cdot 2^4 = \Delta_E = -\Omega^{-12}\tilde{\omega}^{12}$, it follows that $\omega_E = \pm\tilde{\omega}3^{1/4} \cdot 2^{-1/3}$.

(b) Again Ω is real or purely imaginary. Since $e_6(L) = g_3(L)/140 = -2^8 \cdot 7/(140 \cdot 216) < 0$, Ω is imaginary, and $\omega_E = \Omega\sqrt{-2}$. Since $2^9 = \Delta_E = \Omega^{-12}\tilde{\omega}^{12}$, and $\omega_E^{12} = \tilde{\omega}^{12} \cdot 2^{-3}$, we have $\omega_E = \tilde{\omega} \cdot 2^{-3/4}$.

(c) We have $e_4(\mathfrak{o}_k) > 0$, by Lemma 3.10. Also $\Omega = \zeta\bar{\Omega}$ with $\zeta^8 = 1$. So Ω^8 is real, and Ω, which is only defined up to a fourth root of unity, may be taken to be either real or a real multiple of (1 + i). With $E = (y^2 = x^3 - 4x)$, $g_2(E) = 16$, hence $e_4(L) > 0$. But $e_4(L) = \Omega^{-4}e_4(\mathfrak{o}_k)$, so Ω may be taken to be real. Hence $\omega_E = \Omega$, and since $-2^{12} = \Delta_E = -\Omega^{-2}\tilde{\omega}^{12}$, $\omega_E^{12} = 2^{-12}\tilde{\omega}^{12}$ and $\omega_E = \pm 2^{-1}\tilde{\omega}$.

§4. A refined version of Damerell's theorem

Damerell's original theorem (see [Da] or [W3]) computes the value of the L-function of a Hecke character of an imaginary quadratic field up to an algebraic number. However, to prove the Γ-hypothesis we need a result up to an element of the imaginary quadratic field itself; hence the present section. For the proof of our version of Damerell's theorem we shall rely heavily on [G-S].

Let k be an imaginary quadratic field with odd class number and let E be a $\mathbb{Q}$-curve in the sense of [G] with complex multiplication by $\mathfrak{o}_k$; for this section it is not necessary to take E to be the special $\mathbb{Q}$-curve used in paragraph 3. However, we are still assuming that our chosen embedding ι maps the modular invariant of E to $j(\mathfrak{o}_k) \in \mathbb{C}$. Then there exists $\Omega \in \mathbb{C}^*$ such that $E \cong \mathbb{C}/\Omega\mathfrak{o}_k$, the isomorphism being given by the Weierstrass $\wp$-function. The lattice $L = \Omega\mathfrak{o}_k$ determines a Weierstrass model for E:

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in H,$$

along with a differential $\omega = dx/y$ on E. For each $\sigma \in G = G(H/k)$ let

E^σ be the curve

$$y^2 = 4x^3 - g_2^\sigma x - g_3^\sigma$$

and let ω^σ be the differential dx/y on E^σ. Then the pair $(E^\sigma, \omega^\sigma)$ determines a complex lattice L_σ. Also, for any non-zero ideal $\mathfrak{a}$ of k we shall write $E^{\mathfrak{a}}$ and $L_{\mathfrak{a}}$ for $E^{\sigma_{\mathfrak{a}}}$ and $L_{\sigma_{\mathfrak{a}}}$ respectively, where $\sigma_{\mathfrak{a}}$ is the Artin symbol of $\mathfrak{a}$ with respect to the extension H/k. By the general theory of complex multiplication $L_{\mathfrak{a}}$ is homothetic to $\mathfrak{a}^{-1}$. Hence there exists $\Omega_{\mathfrak{a}} \in \mathbb{C}^\times$, determined up to a unit in k, such that $L_{\mathfrak{a}} = \Omega_{\mathfrak{a}} \mathfrak{a}^{-1}$. In fact, as is shown in [G-S], paragraph 4, $\Omega_{\mathfrak{a}}$ can be determined without ambiguity once we have chosen Ω. Letting e be the identity element of G we then have $L = L_e = L_{\mathfrak{o}k}$ and $\Omega = \Omega_{\mathfrak{o}_k}$.

Since E is a $\mathbb{Q}$-curve, [G], paragraph 15 and [G-S], Theorem 4.1 allow us to deduce the existence of a Hecke character ϕ of k such that $\chi_E = \phi \circ \mathbb{N}_{H/k}$.

Now let ψ be a Galois-equivariant Hecke character of type A_0 of k with infinity-type $a\rho$, where ρ is the non-trivial element of $G(k/\mathbb{Q})$. Also assume that ψ takes values in k and that a is a positive multiple of h, the class number of k. Then ψ can be written

$$\psi = \chi \bar{\phi}^a$$

where ϕ is as above and χ is an at most quadratic character of k if $k \neq \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$; if $k = \mathbb{Q}(\sqrt{-1})$ then χ has at most order 4 and if $k = \mathbb{Q}(\sqrt{-3})$ then χ has at most order 6. Note that ϕ^h equals ψ_E which is Galois-equivariant since χ_E is. Thus, since h divides a, so is $\bar{\phi}^a$. ψ is Galois-equivariant by assumption and we conclude that χ is Galois-equivariant as well.

Let d_k be the discriminant of k and let $a_0 = a/h$.

THEOREM 4.1 ("*The Refined Damerell's Theorem*"): *Let b be an integer such that $1 \leqslant b \leqslant a$, let $c = a - b$ and let ψ be as above. Then*

$$L(\psi, b) = \left(\sqrt{|d_k|}\,/\pi\right)^c \left(\prod_{\mathfrak{a}} \Omega_{\mathfrak{a}}\right)^{a_0} U,$$

where $\mathfrak{a}$ ranges over a set of ideal class representatives of k and U transforms via $\bar{\chi}$, i.e., if $\eta \in G(\mathbb{C}/k)$ then $\eta U = \bar{\chi}(\eta) U$.

PROOF: First assume $0 < a/2 < b \leqslant a$. Let $\mathfrak{g} \neq \mathfrak{o}_k$ be an integral ideal of k which is divisible by the conductors of ϕ and χ. Let $\mathfrak{A} = \{\mathfrak{a}\}$ be a set of representatives for the ideal classes of k and assume that all the representatives are prime to $\mathfrak{g}$.

Let Λ be the function defined on p. 198 of [G-S]. Then for any ideal $\mathfrak{c}$

of k $\Lambda(\mathfrak{c}) = \Omega_{\mathfrak{c}}\Omega^{-1} \in H^*$. As in paragraph 5 of the same paper, for $\tau \in G(H(E_{\mathfrak{g}})/k)$ (an abelian group by [G] paragraph 15 and [G-S] Theorem 4.1), let $L(\psi, \tau, s)$ be the partial L-function, given by

$$L(\psi, \tau, s) = \sum_{\sigma_{\mathfrak{b}} = \tau} \psi(\mathfrak{b})\mathbb{N}\mathfrak{b}^{-s}$$

for $\operatorname{Re} s > 1 + a/2$, the sum being taken over all integral ideals $\mathfrak{b}$ of k which are prime to $\mathfrak{g}$ and whose Artin symbbls $\sigma_{\mathfrak{b}}$ with respect to the extension $H(E_{\mathfrak{g}})/k$ are equal to τ.

$$\sum_{\tau \in G(H(E_{\mathfrak{g}})/k)} L(\psi, \tau, s) = L_{\mathfrak{g}}(\psi, s) = \prod_{\mathfrak{p} \nmid \mathfrak{g}} (1 - \psi(\mathfrak{p})\mathbb{N}\mathfrak{p}^{-s})^{-1},$$

$$\operatorname{Re} s > 1 + \frac{a}{2}.$$

$L_{\mathfrak{g}}(\psi, s)$ differs from $L(\psi, s)$ by a finite product of Euler factors. Since $b > a/2$, $(\psi(\mathfrak{p})\mathbb{N}\mathfrak{p}^{-b})\mathfrak{o}_k = \bar{\mathfrak{p}}^{a-b}\mathfrak{p}^{-b}$ cannot equal $\mathfrak{o}_k$, so this product is a non-zero rational number at $s = b$. Hence from the view-point of rationality statements it makes no difference whether we consider $L(\psi, s)$ or $L_{\mathfrak{g}}(\psi, s)$, as long as $b > a/2$.

Let $\rho \in \Omega k^* \subset \mathbb{C}^*$ be such that $\rho\Omega^{-1}\mathfrak{o} = \mathfrak{g}^{-1}\mathfrak{h}$, where $\mathfrak{h}$ is an integral ideal of k, prime to $\mathfrak{g}$. Then by Corollary 5.7 of [G-S] we have

$$\left(\frac{2\pi i}{\mathbb{N}\mathfrak{h}}\right)^c \cdot \frac{\phi^a(\mathfrak{c}\mathfrak{h})}{(\Lambda(\mathfrak{c})\rho)^a} \cdot L(\bar{\phi}^a, \sigma_{\mathfrak{c}\mathfrak{h}}, b) = \sum_{\mathfrak{b} \in \mathfrak{B}} \mathscr{E}_{b,a}(\mathfrak{c}, \mathfrak{b}), \tag{4.1}$$

where $\mathfrak{B}$ is such that $\sigma_{\mathfrak{b}}$ runs over $G(H(E_{\mathfrak{g}})/H)$ without repetition when $\mathfrak{b}$ runs over $\mathfrak{B}$, and $\mathscr{E}_{b,a}(\mathfrak{c}, \mathfrak{b})$ is, up to some factors, Weil's double Eisenstein series – see [G-S] and [W3], Chapter VI. Since $L(\psi, \sigma_{\mathfrak{c}\mathfrak{h}}, b) = \chi(\mathfrak{c}\mathfrak{h})L(\bar{\phi}^a, \sigma_{\mathfrak{c}\mathfrak{h}}, b)$ we may rewrite the left-hand side as

$$\left(\frac{2\pi i}{\mathbb{N}\mathfrak{h}}\right)^c \cdot \frac{\phi^a(\mathfrak{c}\mathfrak{h})\bar{\chi}(\mathfrak{c}\mathfrak{h})}{(\Lambda(\mathfrak{c})\rho)^a} \cdot L(\psi, \sigma_{\mathfrak{c}\mathfrak{h}}, b).$$

Imitating Corollary 4.11 of [G-S] we may show that $\Phi^a(\mathfrak{c})\Lambda(\mathfrak{c})^{-a}$ only depends on the ideal class of $\mathfrak{c}$ and that $\sigma \mapsto \Phi(\sigma) = \phi^a(\mathfrak{c})\Lambda(\mathfrak{c})^{-a}$, where $\mathfrak{c}$ is any ideal whose Artin symbol with respect to the extension H/k is σ, is a cocycle on G. Since $h|a$, ϕ^a takes values in k^*, so we have $\Phi(\sigma) \in H^*$. By Hilbert's Theorem 90 there exists $y \in H^*$ such that $y^{\sigma-1} = \Phi(\sigma)$ for all $\sigma \in G$. Also, since $\bar{\chi}$ takes values in k it may be viewed as a cocycle on $G(H(E_{\mathfrak{g}})/k)$ with values in $H(E_{\mathfrak{g}})^*$, so again by Hilbert's Theorem 90 there exists $U \in H(E_{\mathfrak{g}})^*$ such that $U^{\tau-1} = \bar{\chi}(\tau)$ for all $\tau \in G(H(E_{\mathfrak{g}})/k)$, i.e. U transforms via $\bar{\chi}$. Hence we see that

$$\frac{\phi^a(\mathfrak{c})\bar{\chi}(\mathfrak{c})}{\Lambda(\mathfrak{c})^a} = \frac{(yU)^{\sigma_{\mathfrak{c}}}}{yU},$$

where $\sigma_{\mathfrak{c}}$ is the Artin symbol of $\mathfrak{c}$ with respect to the extension $H(E_{\mathfrak{g}})/k$. Observing that $\bar{\psi}(\mathfrak{h})\Omega^a\rho^{-a}$ is in k^* and hence can be absorbed into U, we get from (4.1)

$$\left(\frac{2\pi \mathrm{i}}{\mathbb{N}h}\right)^c \cdot \frac{L(\psi, b)}{y\Omega^a U} = \sum_{\mathfrak{c},\mathfrak{b}} \frac{\mathscr{E}_{b,a}(\mathfrak{c}, \mathfrak{b})}{(yU)^{\sigma_{\mathfrak{c}}}},$$

where $\mathfrak{c}$ ranges over a set of ideals prime to $\mathfrak{g}$ representing $G(H(E_{\mathfrak{g}})/k)$. By the Galois properties of the Eisenstein series $\mathscr{E}_{b,a}(\mathfrak{c}, \mathfrak{b})$ ([G-S] Theorem 6.1) the right hand side is in k, so after multiplying U by an element of k (0 if $L(\psi, b) = 0$) we obtain

$$L(\psi, b) = (2\pi \mathrm{i})^{-c} y\Omega^a U.$$

For α, $\beta \in \mathbb{C}$ write $\alpha \sim \beta$ to mean that there exists $r \in k^*$ such that $\alpha = r\beta$. Then $(2\pi\mathrm{i})^{-1} \sim \sqrt{|d_k|}/\pi$ and we are reduced to proving the

LEMMA 4.2: $y \sim \prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a})^{a_0}$.

PROOF: Let $\sigma \in G$ and let $\mathfrak{b}$ be an ideal of k which is prime to $\mathfrak{g}$ and such that $\sigma = \sigma_{\mathfrak{b}}$. We have, by Proposition 4.10 (iv) in [G-S],

$$\left(\frac{y}{\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a})^{a_0}}\right)^{\sigma_{\mathfrak{b}}}$$

$$= \frac{y^{\sigma_{\mathfrak{b}}}}{y} \cdot \frac{\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a}\mathfrak{b})^{a_0}}{\left(\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a})^{a_0}\right)^{\sigma_{\mathfrak{b}}}} \cdot \frac{y}{\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a}\mathfrak{b})^{a_0}}$$

$$= \frac{\phi^a(\mathfrak{b})}{\Lambda(\mathfrak{b})^a} \cdot \frac{\prod_{\mathfrak{a}\in\mathfrak{A}} (\Lambda(\mathfrak{a})^{\sigma_{\mathfrak{b}}}\Lambda(\mathfrak{b}))^{a_0}}{\left(\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a})^{a_0}\right)^{\sigma_{\mathfrak{b}}}} \cdot \frac{y}{\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a}\mathfrak{b})^{a_0}}$$

$$= \frac{\phi^a(\mathfrak{b})y}{\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a}\mathfrak{b})^{a_0}}.$$

On the other hand, writing $\mathfrak{a}\mathfrak{b} = (\alpha_{\mathfrak{a}})\mathfrak{a}'$ where $\alpha_{\mathfrak{a}} \in k$ and $\mathfrak{a}' \in \mathfrak{A}$, we have, again using Proposition 4.10 (iv) in [G-S],

$$\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a}\mathfrak{b}) = \prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda\big((\alpha_{\mathfrak{a}})\mathfrak{a}'\big) = \prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda((\alpha_{\mathfrak{a}}))^{\sigma_{\mathfrak{a}'}}\Lambda(\mathfrak{a}')$$

$$= \prod_{\mathfrak{a}\in\mathfrak{A}} \phi((\alpha_{\mathfrak{a}}))^{\sigma_{\mathfrak{a}}} \prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a}),$$

since $\Lambda((\alpha_{\mathfrak{a}})) = \phi((\alpha_{\mathfrak{a}}))$ (by the definition of Λ) and $\mathfrak{a}'$ runs over $\mathfrak{A}$ exactly once when $\mathfrak{a}$ does. By [G-S], Lemma 4.9 $\phi((\alpha_{\mathfrak{a}})) \in k^*$ so

$$\prod_{\mathfrak{a}\in\mathfrak{A}} \phi((\alpha_{\mathfrak{a}}))^{\sigma_{\mathfrak{a}}} = \prod_{\mathfrak{a}\in\mathfrak{A}} \phi((\alpha_{\mathfrak{a}})) = \prod_{\mathfrak{a}\in\mathfrak{A}} \phi(\mathfrak{a})\phi(\mathfrak{b})\phi(\mathfrak{a}')^{-1} = \phi(\mathfrak{b})^{h}.$$

Hence

$$\left(\frac{y}{\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a})^{a_0}}\right)^{\sigma_{\mathfrak{b}}} = \frac{\phi^{a}(\mathfrak{b})y}{\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a}\mathfrak{b})^{a_0}} = \frac{y}{\prod_{\mathfrak{a}\in\mathfrak{A}} \Lambda(\mathfrak{a})^{a_0}}.$$

This concludes the proof of Damerell's theorem when $a/2 < b \leqslant a$. For the remaining range, i.e. $1 \leqslant b \leqslant a/2$, we use the functional equation of the L-function.

For a moment let k be any imaginary quadratic field and drop the assumption that ψ is Galois-equivariant. Define ψ^* by $\psi^*(\mathfrak{a}) = \overline{\psi(\bar{\mathfrak{a}})}$. Then we have:

THEOREM 4.3: *Let ψ be a Hecke character of k of infinity-type $a\rho$. Let*

$$\Lambda(\psi, s) = \Gamma(s)\Big(\sqrt{|d_k|\mathbb{N}\mathfrak{f}}\,/2\pi\Big)^{s} L(\psi, s),$$

where $\mathfrak{f}$ is now the exact conductor of ψ. Then

$$\Lambda(\psi, s) = W_{\psi}\Lambda(\psi^*, a+1-s),$$

where W_{ψ} is a non-zero constant. If ψ is Galois-equivariant, then $W_{\psi} = \pm 1$.

PROOF: The functional equation is immediate from [H] pp. 272–73 and 282 if one recalls that Hecke prefers to consider only characters of absolute value 1; to place ourselves in that situation, all we have to do is replace ψ by $\psi\mathbb{N}^{-a/2}$ and s by $s - a/2$. Then note that $L(\psi^*, s) = L(\bar{\psi}, s)$, and we have the functional equation as stated. Applying the functional equation to ψ^* and comparing, we see immediately that $W_{\psi}W_{\psi^*} = 1$. Hence if ψ is Galois-equivariant, so $\psi = \psi^*$, we have $W_{\psi} = \pm 1$.

Now let k be an imaginary quadratic field with odd class number, and assume ψ to be Galois-equivariant again. Setting $s = b$ so that $a + 1 - s = c + 1$, we obtain

$$L(\psi, b) = \frac{W_\psi \Gamma(c+1) L(\psi, c+1)\left(\sqrt{|d_k| \mathbb{N}\mathfrak{f}}/2\pi\right)^{c+1}}{\Gamma(b)\left(\sqrt{|d_k| \mathbb{N}\mathfrak{f}}/2\pi\right)^{b}}.$$

$W_\psi = \pm 1$ and since both b and $c + 1$ are positive integers so are $\Gamma(b)$ and $\Gamma(c+1)$. Hence

$$L(\psi, b) \sim L(\psi, c+1)\left(\sqrt{|d_k| \mathbb{N}\mathfrak{f}}/\pi\right)^{c-b+1},$$

Now $c + 1 \geqslant a - (c+1) = b - 1$, so we can apply the previously proved case to $L(\psi, c+1)$ and obtain

$$\begin{aligned} L(\psi, b) &\sim \left(\sqrt{|d_k|}/\pi\right)^{b-1}\left(\prod_{\mathfrak{a}} \Omega_{\mathfrak{a}}\right)^{a_0} U\left(\sqrt{|d_k| \mathbb{N}\mathfrak{f}}/\pi\right)^{c-b+1} \\ &= \left(\sqrt{|d_k|}/\pi\right)^{c}\left(\prod_{\mathfrak{a}} \Omega_{\mathfrak{a}}\right)^{a_0} U \sqrt{\mathbb{N}\mathfrak{f}}^{\,c-b+1} \end{aligned}$$

where U transforms via $\bar{\chi}$.

Hence what remains to be shown is that $\sqrt{\mathbb{N}\mathfrak{f}}^{\,c-b+1}$ is rational. If $c - b + 1$ is even the rationality of $\sqrt{\mathbb{N}\mathfrak{f}}^{\,c-b+1}$ is clear, so let us assume that $c - b + 1$ is odd. Then $c - b$ and hence $a = b + c$ is even, and since h is odd a_0 is also even.

Low let us assume, in addition, that $k \neq \mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. Then $\bar{\psi}_E^{a_0}$ has trivial conductor, for it is none other than the character which sends an ideal $\mathfrak{a}$ to $\bar{\alpha}^{a_0}$ where $(\alpha) = \mathfrak{a}^h$; this is well-defined since the only units in k are ± 1. Hence $\mathfrak{f} = \mathfrak{f}_\chi$, the conductor of χ. However, $\sqrt{\mathbb{N}\mathfrak{f}_\chi}$ is rational by the following lemma:

Lemma 4.4: *If $\mathfrak{f}$ is the conductor of a quadratic Galois-equivariant Dirichlet character of an imaginary quadratic field k then $\mathbb{N}\mathfrak{f}$ is a square.*

Proof: Let F be the extension of k corresponding to the given character χ by class field theory. We may assume χ to be non-trivial so that F is a proper quadratic extension of k. Since χ is Galois-equivariant we see, as in the proof of Theorem 3.5 a), that F is Galois over $\mathbb{Q}$ with non-cyclic Galois group. The lemma now follows from [B], Lemma 4.2.

Now let $k = \mathbb{Q}(\sqrt{-1})$. Let ψ be a Galois-equivariant character such that $\psi((\alpha)) = \epsilon(\alpha)\bar{\alpha}^a$, where ϵ is a primitive character of $(\mathfrak{o}_k/\mathfrak{f})^\times$. We wish to show that if a is even, $\mathfrak{f}$ is the extension of an ideal in $\mathbb{Z}$, i.e. of the form $f\mathfrak{o}$ for some $f \in \mathbb{Z}$. We first observe that necessarily $\epsilon(-1)=1$, since a is even. We may decompose $\mathfrak{f}$ into its p-primary components $\mathfrak{f}_p$, and write ϵ as the product of primitive characters ϵ_p of $(\mathfrak{o}/\mathfrak{f}_p)^\times$. Since ψ is Galois-equivariant it follows that $\mathfrak{f}=\bar{\mathfrak{f}}$ and $\mathfrak{f}_p=\bar{\mathfrak{f}}_p$ for all p. For $p \neq 2$, this already implies that $\mathfrak{f}_p$ is the extension of an ideal in $\mathbb{Z}$. We also have, for all p, $\epsilon_p(\bar{\alpha})=\overline{\epsilon_p(\alpha)}$; this implies that ϵ_p vanishes on the subgroup N of $(\mathfrak{o}/\mathfrak{f}_p)^\times$ consisting of elements of the form $\widetilde{\alpha\bar{\alpha}}$, where $\sim$ denotes reduction mod p. If $p \neq 2$, -1 is in N, so $\epsilon_p(-1)=1$. Since $\epsilon(-1)=1$ it follows that $\epsilon_2(-1)=1$. So now we know that ϵ_2 is a Galois-equivariant character of $(\mathfrak{o}/\pi^\nu)^\times$ which vanishes on ± 1, where $\pi=(1+i)$. Equivalently, ϵ_2 vanishes on the subgroup $\pm N$. On the other hand, it is easily seen that the order of $(\mathfrak{o}/\pi^\nu)^\times/\pm N$ is $2^{\nu/2}$ if ν is even and $2^{(\nu-1)/2}$ if ν is odd. So the conductor of ϵ_2 is an even power of π, and since $(\pi^2)=(2)$, this completes the proof.

Now let $k=\mathbb{Q}(\sqrt{-3})$. The proof proceeds as in the case when $k=\mathbb{Q}(\sqrt{-1})$. The only differences are that now $\pi=\sqrt{-3}$, and the order of $(\mathfrak{o}/\pi^\nu)^\times/\pm N$ is $3^{\nu/2}$ if ν is even, $3^{(\nu-1)/2}$ if ν is odd, and $(\pi^2)=(3)$.

This concludes the proof of Damerell's Theorem.

§5. Proof of the Γ-hypothesis

Let $k=\mathbb{Q}(\sqrt{-p})$ be an imaginary quadratic field of odd class number and let E be as in paragraph 3 – in particular if $p>3$ then E is the $\mathbb{Q}$-curve $A(p)$ defined in [G].

Let $R=\mathbb{Z}[G(k/\mathbb{Q})]$. For any Hecke character ψ of type A_0, denote the infinity-type of ψ by $I(\psi)$.

LEMMA 5.1: *Let $T \subset R$ be the subgroup of infinity-types of Jacobi-sum Hecke characters of k, let T' be the subgroup generated by $I(\psi_E)$ and $I(\mathbb{N})$ and let T'' be generated by $I(\bar{\psi}_E)$ and $I(\mathbb{N})$. Then if $k \neq \mathbb{Q}(\sqrt{-2})$*

$$T=T'=T''.$$

PROOF: We first observe that $\mathbb{N}$ is a Jacobi-sum character. (We have $\mathbb{N}=(\chi_{-p}\mathbb{N}_{\mathbb{Q}})\circ\mathbb{N}_{k/\mathbb{Q}}$, since χ_{-p} is the quadratic character of $\mathbb{Q}$ corresponding to k; by Corollary 2.3, $\chi_{-p}\mathbb{N}_{\mathbb{Q}}$ is a Jacobi-sum character of $\mathbb{Q}$.)

Since $\bar{\psi}_E=\mathbb{N}^h\psi_E^{-1}$ it is clear that $T'=T''$, and by Lemma 3.4, $T' \subset T$. $I(\mathbb{N})=e+\rho$ and $I(\psi_E)=he$, so clearly $[R:T']=h$. But by [Si] Theorems 2.1 and 5.3, $[R:T]$ is also equal to h so we may conclude that $T=T'$.

Now let ψ be a Jacobi-sum character of k. Then, if $k \neq \mathbb{Q}(\sqrt{-2})$, by Lemma 5.1 ψ can be written as $\psi=\chi\bar{\psi}_E^{a_0}\mathbb{N}^{-b}$, where χ is a Dirichlet character and a_0 and b are integers. If $k=\mathbb{Q}(\sqrt{-2})$, by Lemma 3.2 we have the same result with the additional information that a_0 is even.

Since ψ, $\bar{\psi}_E$ and $\mathbb{N}$ are all Jacobi-sum characters (by assumption and by Lemma 3.4), so is χ.

Thus to compute $\Gamma_\Sigma(\psi)$ it suffices to compute $\Gamma_\Sigma(\psi_E)$, $\Gamma_\sigma(\chi)$ and $\Gamma_\sigma(\mathbb{N})$. In order to express $L(\psi, s)$ as a product of values of the Γ-function we first apply our version of Damerell's theorem (Theorem 4.1) and then use a refinement of the Chowla-Selberg formula, due to Gross (see Theorem (21.2.2) in [G]), to express the product of periods as a product of values of the Γ-function.

THEOREM 5.2 (*the Chowla-Selberg formula*):

(i) *If* $p > 3$ *then*

$$\prod_{\mathfrak{a}} \Omega_{\mathfrak{a}} \sim \left[\prod_{\substack{0<n<p \\ \kappa(n)=1}} \Gamma\left(\frac{n}{p}\right) \right] (2\pi \mathrm{i})^{-d},$$

where κ *is the quadratic character corresponding to* k *and* $d = \frac{1}{4}(p-1-2h)$.

(ii) *If* p *is* 1, 2 *or* 3 (*or more generally if* $h = 1$) *then we have*

$$\tilde{\omega}^{24} = \left(\frac{2\pi}{m}\right)^{12} \prod_{n=1}^{m-1} \Gamma\left(\frac{n}{m}\right)^{6w\kappa(n)},$$

where $m = |d_k|$, $w =$ *the number of roots of unity in* k, *and* κ *is the quadratic character corresponding to* k.

PROOF:

(i) follows from the proof of Theorem 21.2.2 of [G]. In the proof a special choice of the representatives $\mathfrak{a}$ is made use of, but clearly choosing different $\mathfrak{a}$'s will only alter the periods by an element of k^* and so will not change the formula.

(ii) is [W3], p. 92.

THEOREM 5.3 (*The* Γ-*hypothesis*): *Let* $\psi = \chi \bar{\psi}_E^{a_0} \mathbb{N}^{-b}$ *be a Jacobi-sum character as above and let* $I(\psi) \in C(\Sigma)$. *Then* $\Gamma_\Sigma(\psi) L(\psi, 0)$ *lies in* $\mathbb{Q}$.

NOTE: By Corollary 3.7 we are justified in writing just $\Gamma_\Sigma(\psi)$ rather than $\Gamma_\Sigma(\psi, \theta)$.

PROOF OF THEOREM: In the introduction, a condition is given for when $I(\psi) \in C(\{e\})$. In our case $I(\psi) = (a-b)\rho - be$, and a short computation shows that both *a*) and *b*) of the condition simplify to $b \geqslant 1$ and $a - b \geqslant 0$, or equivalently $1 \leqslant b \leqslant a$. Since $L(\psi, 0) = L(\chi \bar{\psi}_E^{a_0}, b)$ this is exactly the range covered by Damerell's Theorem, from which we obtain (note that by Theorem 1.1b) v) ψ is Galois-equivariant)

$$L(\psi, 0) \sim (\sqrt{p}/\pi)^{c}(\Pi\Omega_{\alpha})^{a_0}U,$$

where $c = a - b \geqslant 0$ and U transforms via $\bar{\chi}$. (Note that in our case $\sqrt{|d_k|} \sim \sqrt{p}$.).

Assume first that $k \neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, or $\mathbb{Q}(\sqrt{-3})$. $i \sim \sqrt{p} \sim (\sqrt{p})^{-1}$, so from Theorem 5.2 i) we obtain

$$L(\psi, 0) \sim (\pi\sqrt{p})^{-a_0 d - c}\left[\prod_{\substack{0<n<p\\ \kappa(n)=1}} \Gamma\left(\frac{n}{p}\right)\right]^{a_0} U.$$

On the other hand since $\bar{\psi}_E = \psi_E^{-1}\mathbb{N}^h = \psi_J^{-1}\mathbb{N}^{d+h}$ (by Lemma 3.4),

$$\begin{aligned}\Gamma_\Sigma(\psi) &= \Gamma_\Sigma\left(\chi\psi_J^{-a_0}\mathbb{N}^{a_0 d + a_0 h - b}\right)\\ &= \Gamma_\Sigma\left(\chi\psi_J^{-a_0}\mathbb{N}^{a_0 d + c}\right)\\ &= \Gamma_\Sigma(\chi)\Gamma_\Sigma(\psi_J)^{-a_0}\Gamma_\Sigma(\mathbb{N})^{a_0 d + c}.\end{aligned}$$

by Definitions 1.5–1.7

$$\Gamma_\Sigma(\psi_J) \sim \prod_{\substack{0<n<p\\ \kappa(n)=1}} \Gamma\left(\frac{n}{p}\right).$$

(Recall that $\Sigma = \{e\}$.)

$\mathbb{N} = (\chi_{-p}\mathbb{N}_{\mathbb{Q}}) \circ \mathbb{N}_{k/\mathbb{Q}}$, so $\Gamma_\Sigma(\mathbb{N}) = \Gamma_{\Sigma'}(\chi_{-p}\mathbb{N}_{\mathbb{Q}})$, Σ' being the set consisting of the one embedding of $\mathbb{Q}$ into $\mathbb{C}$. $\chi_{-p}\mathbb{N}^{(p-1)/2} = J([1]_p, \mathbb{Q})$ and $\mathbb{N}_{\mathbb{Q}}^2 = (\chi_{-3}\mathbb{N}_{\mathbb{Q}})^2 = J(2[1]_3, \mathbb{Q})$, so $\chi_{-p}\mathbb{N}_{\mathbb{Q}} = J([1]_p - ((p-3)/2)[1]_3, \mathbb{Q})$. Hence by Theorem 2.4

$$\Gamma_\Sigma(\mathbb{N}) \sim \pi\sqrt{p}.$$

Hence

$$L(\psi, 0)\Gamma_\Sigma(\psi) \sim (\pi\sqrt{p})^{-a_0 d - c}\left[\prod_{\substack{0<n<p\\ \kappa(n)=1}} \Gamma\left(\frac{n}{p}\right)\right]^{a_0} U\Gamma_\Sigma(\chi)$$

$$\cdot\left[\prod_{\substack{0<n<p\\ \kappa(n)=1}} \Gamma\left(\frac{n}{p}\right)\right]^{-a_0} (\pi\sqrt{p})^{a_0 d + c} \sim U\Gamma_\Sigma(\chi).$$

U transforms via $\bar{\chi}$ and by the generalized Deligne's Theorem (Theorem 3.6) $\Gamma_\Sigma(\chi)$ transforms via χ. It follows that $U\Gamma_\Sigma(\chi) \in k$, hence that $L(\psi, 0)\Gamma_\Sigma(\psi) \in k$. However, both $L(\psi, 0)$ and $\Gamma_\Sigma(\psi)$ are real, so $L(\psi, 0)\Gamma_\Sigma(\psi)$ is in fact rational, and we have proved our theorem when $\Sigma = \{e\}$ and $k \neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, or $\mathbb{Q}(\sqrt{-3})$.

Now assume that $k = \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, or $\mathbb{Q}(\sqrt{-3})$. In these three cases, the class number of k is equal to 1, so $G = \{e\}$; we may put $\Omega_e = \Omega$, and Damerell's theorem becomes

$$L(\psi, 0) \sim \left(\sqrt{p}/\pi\right)^c \Omega^a U,$$

where U transforms via $\bar{\chi}$.

For each of the fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{-3})$, we now compute both $L(\psi, 0)$ and $\Gamma_\Sigma(\psi)$, using the results of paragraph 3.

We begin with $k = \mathbb{Q}(\sqrt{-1})$. We have seen in Lemma 3.4 that $\psi_E = \chi_{2,k}\psi_J$ where $\chi_{2,k}$ is the quadratic character of k corresponding to $k(\sqrt{2})$, and $\psi_J = J([1]_4 + [2]_4 - [3]_4)$. So $\Gamma(\psi_E) = \Gamma(\chi_{2,k})\Gamma(\psi_J)$. But $\Gamma(\chi_{2,k}) = \sqrt{2}$ and

$$\Gamma(\psi_J) = \Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{3}{4}\right)^{-1} = \pi^{1/2}\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right)^{-1},$$

so

$$\Gamma(\psi_E) = \sqrt{2}\,\pi^{1/2}\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right)^{-1}.$$

Since $\psi_E\bar{\psi}_E = \mathbb{N}$, and since $\Gamma(\mathbb{N}) \sim \pi$, we have $\Gamma(\bar{\psi}_E) \sim (2\pi)^{1/2}\Gamma\left(\tfrac{1}{4}\right)^{-1}\Gamma\left(\tfrac{3}{4}\right)$. Since $\psi = \chi\bar{\psi}_E^a\mathbb{N}^{-b}$,

$$\begin{aligned}\Gamma(\psi) &= \Gamma(\chi)\left((2\pi)^{1/2}\Gamma\left(\tfrac{1}{4}\right)^{-1}\Gamma\left(\tfrac{3}{4}\right)\right)^a\pi^{-b} \\ &= V\left((2\pi)^{1/2}\Gamma\left(\tfrac{1}{4}\right)^{-1}\Gamma\left(\tfrac{3}{4}\right)\right)^a\pi^{-b}\end{aligned}$$

where V transforms via χ by Deligne's Theorem.

On the other hand, by Damerell's theorem, $L(\psi, 0) \sim \pi^{b-a}\Omega^a U$, where U transforms via $\bar{\chi}$. By Proposition 3.11 $\Omega = \omega_E = 2^{-1}\tilde{\omega}$, and by Theorem 5.2 ii)

$$\tilde{\omega} = \pi^{1/2}2^{-1/2}\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right)^{-1}$$

(note that since $\tilde{\omega}$ and the $\Gamma(n/m)$ are all positive we are justified in taking 24th roots), so

$$L(\psi, 0) \sim \pi^{b-a}\left(\pi^{1/2}2^{-1/2}\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right)^{-1}\right)^a U,$$

where U transforms via $\bar{\chi}$. Thus $L(\psi, 0)\Gamma_\Sigma(\psi)$ is in k, and hence in $\mathbb{Q}$ since it is real.

Now let $k = \mathbb{Q}(\sqrt{-2})$. We have seen in Lemma 3.4 that $\psi_E^2 = \psi_J \mathbb{N}$, where $\psi_J = J([1]_8 - [5]_8)$. So

$$\Gamma(\psi_E^2) = \Gamma(\psi_J)\Gamma(\mathbb{N}) = \Gamma(\tfrac{1}{8})\Gamma(\tfrac{3}{8})\Gamma(\tfrac{5}{8})^{-1}\Gamma(\tfrac{7}{8})^{-1}\pi\sqrt{2}\,.$$

Hence

$$\Gamma(\bar{\psi}_E^2) = \Gamma(\tfrac{1}{8})^{-1}\Gamma(\tfrac{3}{8})^{-1}\Gamma(\tfrac{5}{8})\Gamma(\tfrac{7}{8})\pi\sqrt{2}\,.$$

Since, by Theorem 3.5, $\psi = \chi\bar{\psi}_E^{2a}\mathbb{N}^{-b}$, we have

$$\Gamma(\psi) = V\left(\Gamma(\tfrac{1}{8})^{-1}\Gamma(\tfrac{3}{8})^{-1}\Gamma(\tfrac{5}{8})\Gamma(\tfrac{7}{8})\pi\sqrt{2}\right)^a(\pi\sqrt{2})^{-b}$$

where V transforms via χ.

On the other hand, by Damerell's theorem,

$$L(\psi, 0) \sim \pi^{b-2a}(\sqrt{2})^{2a-b}\Omega^{2a}U,$$

where U transforms via $\bar{\chi}$. By Proposition 3.11, $\Omega^2 \sim \tilde{\omega}^2 2^{1/2}$, and by Theorem 5.2 ii)

$$\tilde{\omega}^2 = \pi\left(\Gamma(\tfrac{1}{8})\Gamma(\tfrac{3}{8})\Gamma(\tfrac{5}{8})^{-1}\Gamma(\tfrac{7}{8})^{-1}\right),$$

so,

$$L(\psi, 0) \sim \pi^{b-a}(\sqrt{2})^{a-b}\left(\Gamma(\tfrac{1}{8})\Gamma(\tfrac{3}{8})\Gamma(\tfrac{5}{8})^{-1}\Gamma(\tfrac{7}{8})^{-1}\right)^a U,$$

where U transforms via $\bar{\chi}$, so we conclude as above.

Finally, let $k = \mathbb{Q}(\sqrt{-3})$. We have seen in Lemma 3.4 that $\psi_E = \psi_J$, where $\psi_J = J([2]_6 + [3]_6 - [5]_6)$. So

$$\Gamma(\psi_E) = \Gamma(\tfrac{1}{3})\Gamma(\tfrac{1}{2})\Gamma(\tfrac{5}{6})^{-1} = \pi^{1/2}\Gamma(\tfrac{1}{3})\Gamma(\tfrac{5}{6})^{-1}.$$

Hence

$$\Gamma(\bar{\psi}_E) = \pi^{1/2}\sqrt{3}\,\Gamma(\tfrac{1}{3})^{-1}\Gamma(\tfrac{5}{6}),$$

and since $\psi = \chi\bar{\psi}_E^a\mathbb{N}^{-b}$,

$$\Gamma(\psi) = V\left(\pi^{1/2}\sqrt{3}\,\Gamma(\tfrac{1}{3})^{-1}\Gamma(\tfrac{5}{6})\right)^a(\pi\sqrt{3})^{-b}.$$

Using the Gauss multiplication formula and the functional equation for the Γ-function, we find that $\Gamma(\frac{5}{6}) \sim \pi^{3/2}\Gamma(\frac{1}{3})^{-2} \cdot \sqrt{3} \cdot 2^{-2/3}$, so

$$\Gamma(\psi) \sim V\pi^{2a-b}(\sqrt{3})^{-b}2^{-2a/3}\Gamma(\tfrac{1}{3})^{-3a},$$

where V transforms via χ.

By Damerell's theorem, $L(\psi, 0) \sim \pi^{b-a}(\sqrt{3})^{b-a}\Omega^a U$, where U transforms via $\bar{\chi}$. By Proposition 3.11, $\Omega \sim \tilde{\omega} \cdot 3^{1/4} \cdot 2^{-1/3}$, and by Theorem 5.2 (ii)

$$\tilde{\omega} = 2^{1/2} \cdot 3^{-1/2}\Gamma(\tfrac{1}{3})^{3/2}\Gamma(\tfrac{2}{3})^{-3/2}\pi^{1/2},$$

so

$$L(\psi, 0) \sim \pi^{b-a}(\sqrt{3})^{b-a}\left(2^{1/2} \cdot 3^{-1/2}\Gamma(\tfrac{1}{3})^{3/2}\Gamma(\tfrac{2}{3})^{-3/2}\right)^a$$
$$\times \pi^{a/2} \cdot 3^{a/4} \cdot 2^{-a/3}U$$

where U transforms via $\bar{\chi}$. Since $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \pi/\sin(\pi/3) = 2\pi/\sqrt{3}$ this reduces to

$$L(\psi, 0) \sim \pi^{b-2a}(\sqrt{3})^b\Gamma(\tfrac{1}{3})^{3a}2^{-a/3}U,$$

and we conclude as above.

If $\Sigma = \{\rho\}$ then (by a short computation) $b \leqslant 0$ and $b - a \geqslant 1$. Note that if these inequalities are satisfied then the infinity-type of

$$\bar{\psi} = \bar{\chi}\psi_E^{a_0}\mathbb{N}^{-b} = \bar{\chi}\bar{\psi}_E^{-a_0}\mathbb{N}^{a-b}$$

belongs to $C(\{e\})$, so Damerell's Theorem applies and we find that

$$L(\bar{\psi}, 0)\Gamma_{\{e\}}(\psi) \in \mathbb{Q}.$$

But $L(\bar{\psi}, 0) = \overline{L(\psi, 0)} = L(\psi, 0)$ and by Lemma 1.8 $\Gamma_{\{e\}}(\bar{\psi}) = \Gamma_{\{\rho\}}(\psi)$, so we conclude again

$$L(\psi, 0)\Gamma_\Sigma(\psi) \in \mathbb{Q}.$$

Thus we have proved the Γ-hypothesis for all imaginary quadratic fields of odd class number.

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Mary Ingraham Binting Institute
Radcliffe College
10 Garden Street
Cambridge, MA 02138
USA

Department of Mathematics
Cornell University
Ithaca, NY 14853
USA