COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 53, nº 1 (1984), p. 91-104
<http://www.numdam.org/item?id=CM_1984__53_1_91_0>

Compositio Mathematica 53 (1984) 91–104.

REMOVABLE SINGULARITIES OF YANG-MILLS FIELDS IN R^3

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Introduction

We consider the question of removable isolated singularities of Yang-Mills fields in 3-dimensions. In R^4, Uhlenbeck's Theorem [7] states that apparent point singularities in finite action solutions may be removed by a gauge transformation. On the other hand, finite action does not seem to be the right condition in other dimensions. If $n \geqslant 5$, the theorem is false [7], as shown by examples which are in L^p for $2 \leqslant p < \frac{1}{2}n$, but not for $p \geqslant \frac{1}{2}n$. In 3 dimensions, Jaffe and Taubes [4] have shown the only finite action solution in all of space is identically zero. It was conjectured by Uhlenbeck that in dimension n, the relevant norm is the $L^{n/2}$ norm, which is also the conformally invariant one.

In this paper, we shall prove that apparent point singularities in solutions for which the $L^{n/2}$ norm is finite ($n = 3$, 5, 6 or 7) may be removed by a gauge transformation.

The physically interesting dimension is, of course, $n = 3$. The theorem is also hardest to prove in this dimension and requires the use of weighted L^2 norms in which the curvature is estimated. A certain auxilary eigenvalue problem is crucial in obtaining these estimates. For completeness, we also prove the theorem in dimensions 5, 6 and 7. The proof mysteriously breaks down if $n \geqslant 8$ and the reason for this is indicated.

Since the basic geometric framework is well-documented in the literature (see, for example, [1], [3], [4], [6] and [7]) we describe it only briefly.

Let M be a Riemannian manifold of dimension n. Let η be a vector bundle over M with structure group G, called the gauge group, and Ad η the adjoint bundle with fiber $(\mathrm{Ad}\,\eta)_x \simeq \mathfrak{G}$, the Lie algebra of G. We denote exterior differentiation by d and its adjoint by δ. The Lie bracket in $\mathfrak{G}$ is denoted by [,].

A covariant derivative D in η is given by $D = \mathrm{d} + \Gamma$ where Γ, called the connection, is a Lie algebra valued one-form. D maps p-forms φ into

* Research supported in part by NSF grant MCS81-03403 and by Sonderforschungsbereich 72, University of Bonn.

$p+1$-forms as follows:

$$D\varphi = \mathrm{d}\varphi + [\Gamma, \varphi].$$

The curvature Ω of the connection is a Lie algebra valued two-form which satisfies

$$D^2\varphi = [\Omega, \varphi] = \left[\mathrm{d}\Gamma + \tfrac{1}{2}[\Gamma, \Gamma], \varphi\right]$$

for all p-forms with values in $\mathfrak{G}$. The Bianchi identities, $D\Omega = 0$, are automatically satisfied by Ω.

The Yang-Mills equations, which are the Euler-Lagrange equations of the action functional, are:

$$D^*\Omega = \delta\Omega + {}^*[\Gamma, {}^*\Omega] = 0$$

where D^* is the adjoint of D.

Therefore, given a bundle η over M with covariant derivative D defined by a connection Γ, a Yang-Mills field Ω is a Lie algebra valued two-form satisfying

$$\Omega = \mathrm{d}\Gamma + \tfrac{1}{2}[\Gamma, \Gamma] \tag{0.1}$$

$$D^*\Omega = 0. \tag{0.2}$$

Gauge transformations g are sections of Aut η which act on connections and curvature forms according to the transformations:
(a) $\Gamma^g = g^{-1}\Gamma g + g^{-1}\mathrm{d}g$
(b) $\Omega^g = g^{-1}\Omega g$.

The pair (Γ, Ω) is gauge equivalent to $(\overline{\Gamma}, \overline{\Omega})$ if there is a gauge transformation g such that $\overline{\Gamma} = \Gamma^g$ and $\overline{\Omega} = \Omega^g$. Gauge equivalent pairs belong to the same orbit under the gauge group and are physically equivalent if g is smooth.

We consider first the problem of proving, that under some suitable additional hypothesis, a weak Yang-Mills field is smooth. Smoothness properties of the pair (Γ, Ω) vary in different gauges and there are subtle difficulties involved in finding a gauge in which the pair is smooth.

A two-form $\Omega \in L^p(M)$ is called a weak-solution of the Yang-Mills equations (ie, a weak Yang-Mills field) if Ω satisfies (0.1) and

$$\int (D\varphi, \Omega) = 0 \tag{0.2$'$}$$

for every smooth compactly supported one-form φ with values in $\mathfrak{G}$.

If Ω is a weak solution belonging to $L^p(M)$ with $p \geqslant \frac{1}{2}n$, then (Γ, Ω)

is gauge equivalent to $(\bar{\Gamma}, \bar{\Omega})$ with $\delta\bar{\Gamma} = 0$ and $\bar{\Gamma} \in H_1^p(M)$. The gauge transformation g relating the pairs belongs to $H_2^p(M)$. (See [8] for the proof of these results.) Differentiating the equations satisfied by $(\bar{\Gamma}, \bar{\Omega})$ one finds that $\bar{\Gamma}$ is a weak solution of the second order elliptic system

$$(d\delta + \delta d)\bar{\Gamma} + \delta[\bar{\Gamma}, \bar{\Gamma}] + *[\bar{\Gamma}, *d\bar{\Gamma}] + *[\bar{\Gamma}, *[\bar{\Gamma}, \bar{\Gamma}]] = 0 \qquad (0.3)$$

If $p > \frac{1}{2}n$, standard results of Morrey ([5], Chapter 6) imply $\bar{\Gamma}$ (and therefore, $\bar{\Omega}$) smooth. Moreover, since $g \in H_2^p$ with $2p > n$, by Sobolev's lemma, g is continuous. Therefore, the bundle η on which the gauge group acts is unaltered topologically by this "change of gauge".

If $p = \frac{1}{2}n$, gauge transformations may be discontinuous. Although $(\bar{\Gamma}, \bar{\Omega})$ can be shown to be smooth (see [8]), the gauge transformation g may have changed the bundle η in the process. Therefore, the hypothesis that $\Omega \in L^{n/2}(M)$ is not enough to insure that Γ defines a smooth covariant derivative in the bundle η with which we started.

We now restrict our attention to the unit ball B in R^n, punctured at the origin, and suppose η is a bundle over $B - \{0\}$ with gauge group G and covariant derivative $D = d + \Gamma$. We assume that the curvature Ω is smooth except at the origin where it has a possible singularity. Our main result is the following

THEOREM: *Let Ω be a smooth curvature form of a Yang-Mills connection in η over $B - \{0\}$ for which $\int_B |\Omega|^{n/2} dx < \infty$, with $3 \leqslant n \leqslant 7$. Then the pair (Γ, Ω) is gauge equivalent by a continuous gauge transformation to a C^∞ pair $(\bar{\Gamma}, \bar{\Omega})$ on B. The bundle η extends continuously to a bundle over B, in which D is given by $d + \bar{\Gamma}$, and $\bar{\Omega}$ is the curvature form of the connection.*

(Since this result holds for any ball punctured at a point, this theorem and the result of Jaffe and Taubes mentioned in the first paragraph, imply that a finite action solution Ω in a bundle over R^3-{finite number of points} must necessarily be identically zero.)

Briefly, the proof is concerned with getting a good growth estimate on the curvature near the origin. An elementary computation shows that if $\Omega \in L^{n/2}$ with $n \geqslant 3$, then Ω is a weak solution in all of B, even in a neighborhood of the puncture. As seen above, this is not enough to prove the theorem. We will, in fact, show that for some $\delta > 0$, $|x|^{2-\delta}|\Omega(x)|$ is bounded. This then implies that $\Omega \in L^p$ for $p > n/2$. As previously discussed, the results of [8] and standard elliptic theory then apply.

In Section 1 we use scalar elliptic theory to obtain, in any dimension, a preliminary growth estimate on the curvature. The second section is devoted to improving this estimate. Here, a very delicate elliptic inequality of Uhlenbeck's is recalled. This inequality appears to be limited to dimensions greater than or equal to four. However, a modified version of it which is sufficient for our purposes, can be proved in three dimensions, and this is done. In Section 3 all results are combined to prove the theorem.

I am indebted to Karen Uhlenbeck for suggesting this problem to me and for many discussions. Conversations with Josef Dodziuk, Jerry Kazdan, Ed Miller and Cliff Taubes were also invaluable to me.

1. A Sub-elliptic estimate for the curvature

We consider weak solutions Ω of the Yang-Mills equations in $B - \{0\}$ for which $\|\Omega\|_{L^{n/2}(B)} < \infty$.

LEMMA 1.1: *Given any $\gamma > 0$, there exists a metric g_0, conformally equivalent to the Euclidean metric, in which $(\int_B |\Omega|^{n/2} \mathrm{d}x)^{2/n} < \gamma$.*

PROOF: As in ([7], Lemma 4.4) this follows from the invariance of the $L^{n/2}$ norm under scale transformations and from the continuity of the L^p norms.

In the remainder of this paper, we fix g_0 so that γ is sufficiently small for our purposes. Since the size of γ depends only on a finite number of universal constants, this can always be done. We will point out as we go along, the bounds needed for γ in the proof.

We frequently prove estimates in "reference rings" which are regions in R^n bounded by concentric spheres about the origin and we denote these by $V_\rho = \{x | \frac{1}{2}\rho \leqslant |x| \leqslant 2\rho\}$. The Sobolev constant in dimension n is always denoted by C_n. Our first restriction on γ is $\gamma < \gamma_1 = (2n - 4)/(n^2 C_n)$.

The main result of this section is:

THEOREM 1.1: *If $\gamma < \gamma_1$, the function $|x|^2 |\Omega(x)$ is bounded in B, and there is a constant C such that for $|x| = r$,*

$$|x|^2 |\Omega(x)| \leqslant C \|\Omega\|_{L^{n/2}(V_r)}. \tag{1.1}$$

In order to prove Theorem 1.1, we next study Yang-Mills fields in a bundle η over the unit reference ring $V_1 = \{y | \frac{1}{2} \leqslant |y| \leqslant 2\}$, and prove

PROPOSITION 1.1: *If Ω is smooth, and if $\|\Omega\|_{L^{n/2}(V_1)} < \gamma_1$, then there is a constant C such that*

$$|\Omega(y)| \leqslant C \|\Omega\|_{L^{n/2}(V_1)} \tag{1.2}$$

for y belonging to the unit sphere in V_1, $|y| = 1$.

The remainder of this section is devoted to the proof of Proposition 1.1. Before doing this we show

PROPOSITION 1.1 IMPLIES THEOREM 1.1: The transformation $y = x/r$ maps V_r onto V_1 carrying the sphere of radius r onto the unit sphere. Smooth solutions of the Yang-Mills equations remain smooth solutions. By the norm invariance, $\|\Omega\|_{L^{n/2}(V_1)} = \|\Omega\|_{L^{n/2}(V_r)} < \gamma < \gamma_1$. Therefore, $\Omega(y)$ satisfies the hypotheses of Proposition 1.1, and hence, the inequality (1.2). Pulling back to V_r, $|\Omega(y)| = r^2|\Omega(x)|$ and we obtain the inequality (1.1). This proves the theorem, assuming the proposition is true.

The proof of the proposition proceeds through several lemmas to which we now turn our attention. In the following, $B(\xi_0, R) = \{y \mid |y - \xi_0| \leqslant R\}$ denote balls which are always assumed to be strictly contained in V_1.

LEMMA 1.2: *The scalar function $u = |\Omega|$ satisfies the inequality*

$$\Delta u \geqslant -4u^2. \tag{1.3}$$

If $\gamma < \gamma_1$, there is a constant K such that for every ball contained in V_1,

$$\int_{B(y_0,\rho)} |\nabla u^{n/4}|^2 \mathrm{d}y \leqslant \frac{K}{a^2} \int_{B(y_0,\rho+a)} u^{n/2} \mathrm{d}y \tag{1.4}$$

PROOF OF LEMMA 1.2: The inequality (1.3) is discussed in detail in [1] and briefly in [7]. Therefore, we use it without proof. Integrating by parts,

$$\int \nabla u \cdot \nabla \zeta \mathrm{d}y \leqslant 4 \int u^2 \zeta \mathrm{d}y \tag{1.3$'$}$$

for every non-negative $\zeta \in C_0^\infty(V_1)$. For $\epsilon > 0$, $2\beta - 1 > 0$, and $\eta \in C_0^\infty$, the function $\zeta = \eta^2(u + \epsilon)^{2\beta - 1}$ is a non-negative C_0^∞ function. Substituting in (1.3′),

$$\int (2\beta - 1)\eta^2(u+\epsilon)^{2\beta-2}|\nabla u|^2 \mathrm{d}y$$

$$\leqslant \int \eta(u+\epsilon)^{2\beta-1}\nabla u \cdot \nabla\eta \mathrm{d}y + 4\int \eta^2 u^2 (u+\epsilon)^{2\beta-1} \mathrm{d}y$$

Since $2\beta - 1 > 0$, the right hand side converges as ϵ tends to zero and we obtain

$$\int \frac{(2\beta-1)}{\beta^2}\eta^2 |\nabla u^\beta|^2 \mathrm{d}y$$

$$\leqslant \frac{1}{\beta}\int (\eta\nabla u^\beta)\cdot(u^\beta\nabla\eta)\mathrm{d}y + 4\int \eta^2 u^{2\beta+1}\mathrm{d}y = I_1 + 4I_2.$$

Estimating I_1 using Young's inequality, we obtain

$$\left(\frac{2\beta-1}{\beta^2}-\epsilon\right)\int\eta^2|\nabla u^\beta|^2\mathrm{d}y \leqslant C(\epsilon)\int u^{2\beta}|\nabla\eta|^2\mathrm{d}y+4I_2. \tag{1.5}$$

By Sobolev's inequality,

$$I_2 \leqslant \|u\|_{n/2}\left(\int(\eta u^\beta)^{2n/(n-2)}\mathrm{d}y\right)^{(n-2)/n} \leqslant C_n\gamma\left(\int|\nabla(\eta u^\beta)|^2\mathrm{d}y\right)$$

Choosing $\beta = n/4$ and using the fact that $\gamma < \gamma_1$, we obtain with a new constant K,

$$\int\eta^2|\nabla u^{n/4}|^2\mathrm{d}y \leqslant K\int u^{n/2}|\nabla\eta|^2\mathrm{d}y \tag{1.6}$$

$$\text{Choosing } \eta = \begin{cases} 1 & \text{for } y\in B(y_0,\rho) \\ 0 & \text{for } y\notin B(y_0,\rho+a) \end{cases}$$

with $|\nabla\eta| \leqslant 2/a$ completes the proof of Lemma 1.2.

REMARK: Lemma 1.2 and Sobolev's inequality imply that on any compact subdomain $\overline{V}_1$ contained in int V_1,

$$\left(\int_{\overline{V}_1} u^{ns/2}\mathrm{d}y\right)^{1/s} \leqslant k\int_{V_1} u^{n/2}\mathrm{d}y,$$

where $s = n/(n-2)$ and k depends on the distance to the boundary. It will suffice to use this inequality on a fixed subdomain $\overline{V}_1$ in which case, k can be chosen as a fixed constant.

From the remark and Hölder's inequality, we obtain a growth condition on small balls contained in $\overline{V}_1$,

$$\int_{B(y_1,\rho)} u^{n/2}\mathrm{d}y \leqslant k'\rho^2. \tag{1.7}$$

To prove Proposition 1.1, we use a special case of the Morrey-Moser iteration ([5], Theorem 5.3.1) which we state as

LEMMA 1.3: *Let D be an open domain in R^n. Let $U\in H_1^2(D)$ with $U\geqslant 0$, and suppose $W = U^\lambda$ for some λ, $1\leqslant\lambda<2$, satisfies*

$$\int_D(\nabla W\cdot\nabla\zeta+\zeta AW)\mathrm{d}x \leqslant 0, \tag{1.8}$$

for all non-negative $\zeta \in C_0^\infty(D)$, *where* A *satisfies a growth condition of the form* (1.7) *on small balls in* D. (*In Morrey's notation, the exponent of* ρ *in* (1.7) *is* $\mu_1 n/2$, *with* $\mu_1 > 0$.) *Then* U *is bounded on compact subdomains of* D, *and for* $y \in B(y_0, \rho)$,

$$|U(y)|^2 \leqslant \frac{C'}{a^n} \int_{B(y_0, \rho + a)} |U(y)|^2 \mathrm{d}y. \tag{1.9}$$

We want to apply this lemma to $U = u^{n/4}$. In all dimensions, u is a solution of (1.8) with $A = u$, and therefore by (1.7), A satisfies the growth condition we need. If $n = 3$, $W = U^{4/3} = u$ satisfies (1.8) and we are through. If $n > 3$, an easy computation shows that U itself is a solution of (1.8), again with $A = u$. Applying Lemma 1.3, we find

$$|u(y)|^{n/2} \leqslant \frac{C'}{a^n} \int_{B(y_0, \rho + a)} |u(y)|^{n/2} \mathrm{d}y \qquad \text{for} \quad y \in B(y_0, \rho).$$

Now choose a finite number of balls centered on the unit sphere. We obtain with a new constant C, for $|y| = 1$,

$$|u(y)| \leqslant C \|u\|_{L^{n/2}(V_1)}.$$

Since $u = |\Omega|$, this proves Proposition 1.1, and therefore, the theorem.

The techniques in this section will now be used to obtain decay estimates at infinity for Yang-Mills fields defined on exterior domains in R^n. Whether these are best possible is not known. In R^4, curvature decays as $|x|^{-4}$ [7].

THEOREM 1.2: *Let* $E = \{x \,|\, |x| \geqslant 1\} \subset R^n$, *and suppose* $\|\Omega\|_{L^{n/2}(E)} < \infty$. *Then, for* $|x| \geqslant R$,

$$|x|^2 |\Omega(x)| \leqslant K.$$

PROOF: An inequality of the form (1.3) holds in small balls contained in E. From Moser's iteration,

$$|\Omega(y)| \leqslant K \|\Omega\|_{L^{n/2}(E)}$$

for $|y| \geqslant \delta > 1$.

Let $x = (R/\delta)y$. Using invariance, we obtain for $|x| = R$, $R^2 |\Omega(x)| \leqslant K \|\Omega\|_{L^{n/2}(E)}$ which proves Theorem 1.2.

2. An elliptic estimate for the curvature

We again assume Ω is a Yang-Mills field in $B - \{0\}$ with $\|\Omega\|_{n/2} < \gamma < \gamma_2$, where bounds on γ_2 are given later.

THEOREM 2.1: *(a) If* $n=3$, $\int_B |x|^\alpha |\Omega(x)|^2 \mathrm{d}x < \infty$ *for any* $\alpha > 1$. *Moreover, if* α *is sufficiently close to one,*

$$\int_B |x|^\alpha |\Omega(x)|^2 \mathrm{d}x \leqslant K \int_{|x|=1} |\Omega(x)|^2 \mathrm{d}S$$

where K is independent of α.
(b) If $n \geqslant 4$, $\int_B |\Omega(x)|^2 \mathrm{d}x < \infty$ *and*

$$\int_B |\Omega(x)|^2 \mathrm{d}x \leqslant \frac{1}{(1-k\gamma_2)(\nu-\gamma_2)^{1/2}} \int_{|x|=1} |\Omega|^2 \mathrm{d}S \tag{2.2}$$

where ν *is the first eigenvalue of the Laplacian on co-closed one-forms over* S^{n-1}.

We first derive

COROLLARY 2.1: *(a) If* $n=3$,

$$\int_{|x|\leqslant r} |x|^\alpha |\Omega(x)|^2 \mathrm{d}x \leqslant r^{1/K} \int_{|x|\leqslant 1} |x|^\alpha |\Omega(x)|^2 \mathrm{d}x, \tag{2.3}$$

where K is the constant of (2.1).
(b) If $n \geqslant 4$,

$$\int_{|x|\leqslant r} |\Omega(x)|^2 \mathrm{d}x \leqslant r^{(1-k\gamma_2)(\nu-\gamma_2)^{1/2}} \int_{|x|\leqslant 1} |\Omega(x)|^2 \mathrm{d}x \tag{2.4}$$

PROOF OF COROLLARY 2.1: To prove (a), make a change of variables, $y = rx$ in (2.1). We obtain

$$\int_{|y|\leqslant r} |y|^\beta |\Omega(y)|^2 \mathrm{d}y \leqslant Kr \int_{|y|=r} |y|^\beta |\Omega(y)|^2 \mathrm{d}S_y.$$

Denoting the left hand side by $f(r)$, this inequality becomes the differential inequality

$$f(\rho) \leqslant K\rho f'(\rho).$$

Integrating from $\rho = r$ to $\rho = 1$, gives (2.3). The proof of (2.4) is exactly analogous, which proves the corollary.

To prove the theorem, we begin with some lemmas. Let U be the reference domain, $U = \{x | 1 \leqslant |x| \leqslant \tau\}$ where $\tau > 1$ is arbitrary. We consider an eigenvalue problem for one-forms defined on U. Here, ω_s refers to the tangential component of the form ω on the boundary.

PROBLEM: Find ω satisfying in U, the

(a) equations: $\delta\omega = 0$ and $\delta d\omega + \lambda\omega = 0$

(b) boundary conditions: $\delta_s\omega_s = 0$ for $|x| = 1$ and $|x| = \tau$

(c) homology condition on absolute cycles: $\int_{|x|=\rho}({}^*\omega)_s = 0$, $1 \leqslant \rho \leqslant \tau$.

LEMMA 2.1: *Let $n = 3$. The eigenvalues of this problem are of the form $m(m+1)$ where m is a positive integer. The first eigenvalue is greater than or equal to* 2.

PROOF: Express the solution in spherical coordinates, $\omega = \omega_r dr + \omega_\theta d\theta + \omega_\varphi d\varphi$. Computing the coefficient of dr in the one-form $\delta d\omega + \lambda\omega$, and using the fact that $\delta\omega = 0$, we find that

$$\frac{1}{\sin^2\varphi}\frac{\partial^2\omega_r}{\partial\theta^2} + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\omega_r}{\partial\varphi}\right) + \lambda\omega_r = 0 \tag{2.5}$$

Expand the function ω_r in spherical harmonics,

$$\omega_r(r, \theta, \varphi) = \sum_{m=0}^{\infty} a_m(r) Y_m(\varphi, \theta)$$

where Y_m are surface spherical harmonics of degree m, each of which is a solution of (2.5) with $\lambda = m(m+1)$. In order for ω_r to be a solution of (2.5), we must have $\lambda = m(m+1)$ for some m, and $\omega_r = a_m(r)Y_m(\varphi, \theta)$. Since zero is not an eigenvalue (see [7], Corollary 2.9) the first eigenvalue is at least two.

We next state results of Uhlenbeck [7] Theorems 2.5 and 2.8 which demonstrate the existence of Hodge gauges in bundles η over the sphere S^m and the reference ring $U = \{x | 1 \leqslant |x| \leqslant \tau\}$. In the following, $\nu > 0$ is the first eigenvalue of the Laplacian on co-closed one-forms over S^m, and λ is the first eigenvalue of the eigenvalue problem above. Upper bounds, $\gamma_2(S^m)$ and $\gamma_2(U)$, on the $L^{n/2}$ norm of Ω will be needed. Here, we choose $\gamma_2 = \min(\gamma_1, \gamma_2(S^m), \gamma_2(U))$.

LEMMA 2.2: (*Hodge gauges*): *Let η be a bundle over S^m or U, with Ω, the curvature of a Yang-Mills field. There is a constant κ depending on dimension such that if $\|\Omega\|_{L^{n/2}} < \gamma < \gamma_2 < \kappa$, then there exists a gauge for η in which $\delta\Gamma = 0$ and $\|\Gamma\|_\infty \leqslant k\|\Omega\|_\infty$, where k is a constant. The gauge is unique up to multiplication by a constant element of G. Moreover,*

(*i*) *If η is a bundle over S^m,*

$$\int_{S^m} |\Gamma|^2 dS \leqslant \frac{1}{\nu - \gamma_2}\int_{S^m} |\Omega|^2 dS.$$

(ii) *If η is a bundle over U, then Γ satisfies the boundary conditions (b), the homology conditions (c), as well as*

$$\int_U |\Gamma|^2 \mathrm{d}x \leqslant \frac{1}{\lambda - \gamma_2} \int_U |\Omega|^2 \mathrm{d}x.$$

Now let $U^{\prime} = \{x | 1/\tau^{\prime} \leqslant |x| \leqslant 1/\tau^{\prime-1}\}$ and $S^{\prime} = \{x || x| = 1/\tau^{\prime}\}$. The next lemma expresses the existence of "broken" Hodge gauges on $B = \bigcup_{l=1}^{\infty} U_l$.

LEMMA 2.3: *There exist gauges for $\eta | U^{\prime}$ such that*

(*a*) $\delta\Gamma^{\prime} = 0$
(*b*) $\delta_s \Gamma_s^{\prime} = 0$ *on* $S^{\prime}$ *and* $S^{\prime-1}$
(*c*) $\int (*\Gamma^{\prime})_s = 0$ *on absolute cycles*
(*d*) $|\Gamma^{\prime}(x)| \leqslant \tau^{\prime}\gamma_2$
(*e*$_1$) $\int_{U^{\prime}} |\Gamma^{\prime}(x)|^2 \mathrm{d}x \leqslant (1/\tau^{2\prime})(1/\lambda - \gamma_2) \int_{U^{\prime}} |\Omega^{\prime}(x)|^2 \mathrm{d}x$
(*e*$_2$) *If* $n = 3$, *then for any* $\alpha > 1$,

$$\int_{U^{\prime}} |x|^{\alpha} |\Gamma^{\prime}(x)|^2 \mathrm{d}x \leqslant \frac{1}{\tau^{2\prime}} \left(\frac{\tau^{\alpha}}{\lambda - \gamma_2} \right) \int_{U^{\prime}} |x|^{\alpha} |\Omega^{\prime}(x)|^2 \mathrm{d}x$$

(*f*) $\Gamma_s^{\prime} = \Gamma_s^{\prime+1}$ *on boundary spheres* $S^{\prime}$
(*g*) $\int_{S^0} |\Gamma_s^1|^2 \mathrm{d}S \leqslant (1/\nu - \gamma_2) \int_{S^0} |\Omega^1|^2 \mathrm{d}S.$

PROOF OF LEMMA 2.3: All properties are proved in ([7] Theorem 4.6) except (e_2). The idea of the proof is to make the change of variables $y = \tau^{\prime}x$ and pull back the field to U. Apply Lemma 2.2 on U, and then verify that the conditions stated in Lemma 2.3 are satisfied in the original ring $U^{\prime}$. We now verify (e_2) which relates the weighted L^2 norms of $\Gamma^{\prime}$ and $\Omega^{\prime}$:

$$(\lambda - \gamma_2)\tau^{2\prime} \int_{U^{\prime}} |x|^{\alpha} |\Gamma^{\prime}|^2 \mathrm{d}x$$

$$\leqslant \frac{\tau^{\alpha}}{\tau^{\alpha\prime}} \left((\lambda - \gamma_2)\tau^{2\prime} \int_{U^{\prime}} |\Gamma^{\prime}|^2 \mathrm{d}x \right)$$

$$\leqslant \frac{\tau^{\alpha}}{\tau^{\alpha\prime}} \int_{U^{\prime}} |\Omega^{\prime}(x)|^2 \mathrm{d}x$$

$$\leqslant \tau^{\alpha} \int_{U^{\prime}} |x|^{\alpha} |\Omega^{\prime}(x)|^2 \mathrm{d}x.$$

This completes the proof of Lemma 2.3.

We now turn our attention to the proof of Theorem 2.1. Note that the preceding lemmas are valid for arbitrary $\tau > 1$. In higher dimensions, it is customary to choose $\tau = 2$ ([6], [7]). In three dimensions, we will need the restriction that $\tau < 2$. We also make an additional restriction on γ_2: namely, that the quantity

$$\phi(\gamma_2) = \left(\frac{\tau}{2-\gamma_2}\right)^{1/2}\left(1 + \frac{\gamma_2}{2}\right) < 1. \tag{2.6}$$

Since $\phi(0) = \sqrt{\tau/2}$, this can always be arranged.

PROOF OF THEOREM 2.1: The proof of (b) is in Uhlenbeck ([7], Prop. 4.7). We restrict our attention to proving (a). First, we observe that the growth condition established in Theorem 1.1 implies that the weighted L^2-norm of Ω is finite for any $\alpha > 1$. We now integrate by parts over each U',

$$\begin{aligned}\int_{U'}|x|^\alpha|\Omega'(x)|^2\mathrm{d}x &= \int_{U'}(\Gamma', D^*(|x|^\alpha\Omega')) - \int_{U'}\tfrac{1}{2}([\Gamma', \Gamma'], |x|^\alpha\Omega') \\ &\quad + \int_{S'^{-1}} - \int_{S'}\Gamma'_s \wedge |x|^\alpha({}^*\Omega')_S \\ &= I_1 + I_2 + \text{boundary terms}. \end{aligned}\tag{2.7}$$

Since $D^*\Omega = 0$, we find from (e_2),

$$\begin{aligned} I_1 &\leqslant \alpha\int_{U'}|\Gamma'||x|^{\alpha-1}|\Omega'(x)|\mathrm{d}x \\ &\leqslant \alpha\tau'\left(\int_{U'}|x|^\alpha|\Gamma'|^2\mathrm{d}x\right)^{1/2}\left(\int_{U'}|x|^\alpha|\Omega'(x)|^2\mathrm{d}x\right)^{1/2} \\ &\leqslant \alpha\left(\frac{\tau^\alpha}{\lambda-\gamma_2}\right)^{1/2}\int_{U'}|x|^\alpha|\Omega'(x)|^2\mathrm{d}x. \end{aligned}\tag{2.8}$$

From (d) and (e_2),

$$\begin{aligned} I_2 &\leqslant \tfrac{1}{2}\|\Gamma\|_\infty\left(\int_{U'}|x|^\alpha|\Gamma'|^2\mathrm{d}x\right)^{1/2}\left(\int_{U'}|x|^\alpha|\Omega'|^2\mathrm{d}x\right)^{1/2} \\ &\leqslant \frac{\gamma_2}{2}\left(\frac{\tau^\alpha}{\lambda-\gamma_2}\right)^{1/2}\int_{U'}|x|^\alpha|\Omega'(x)|^2\mathrm{d}x. \end{aligned}\tag{2.9}$$

Therefore, since $\lambda \geqslant 2$,

$$\int_{U^i} |x|^\alpha |\Omega^i(x)|^2 \mathrm{d}x \leqslant \left\{ \left(\frac{\tau^\alpha}{2-\gamma_2} \right)^{1/2} \left(\alpha + \frac{\gamma_2}{2} \right) \right\} \int_{U^i} |x|^\alpha |\Omega^i(x)|^2 \mathrm{d}x$$

$$+ \text{ boundary terms.} \tag{2.10}$$

From (2.6). if α is sufficiently close to one, the constant on the right hand side is smaller than some $\sigma < 1$, and

$$(1-\sigma)\int_{U^i} |x|^\alpha |\Omega^i(x)|^2 \mathrm{d}x \leqslant \int_{S^{i-1}} - \int_{S^i} \Gamma_s^i \wedge |x|^\alpha ({}^*\Omega^i)_s. \tag{2.11}$$

Adding these inequalities on all the U^i, we see that intermediate boundary terms cancel, and the boundary terms on S^i tend to zero as i tends to infinity. By gauge invariance, the pointwise norm $|\Omega(x)|^2 = |g^{-1}\Omega^i(x)g|^2 = |\Omega^i(x)|^2$. Therefore,

$$(1-\sigma)\int_B |x|^\alpha |\Omega(x)|^2 \mathrm{d}x \leqslant \int_{S^0} \Gamma_s^1 \wedge ({}^*\Omega)_s$$

$$\leqslant \left(\int_{S^0} |\Gamma_s^1|^2 \mathrm{d}S \right)^{1/2} \left(\int_{S^0} |\Omega|^2 \mathrm{d}S \right)^{1/2}$$

$$\leqslant \frac{1}{(\nu - \gamma_2)^{1/2}} \int_{|x|=1} |\Omega|^2 \mathrm{d}S$$

using (g). This proves theorem 2.1 with $K = 1/((1-\sigma)(\nu-\gamma_2)^{1/2}) > 0$.

3. Proof of the removable singularities theorem

We now prove our main theorem stated in the introduction: an apparent point singularity of a Yang-Mills field with finite $L^{n/2}$ – norm over $R^n (3 \leqslant n \leqslant 7)$ is removable.

It is well-known that, for forms, the first eigenvalue of the Laplacian on S^{n-1} is positive. (This has already been used in the proof of Theorem 2.1). In dimensions $n > 4$, we need the additional result of Gallot-Meyer [2]: the first eigenvalue of the Laplacian on co-closed p-forms over S^m is $(p+1)(m-p)$. Applied to one-forms on S^{n-1}, the first eigenvalue is $2(n-2)$. For $n = 5, 6, 7$ we also require an additional restriction on the bound γ_2 of the $L^{n/2}$ norm of Ω; namely

$$(4-n) + (1-k\gamma_2)(2(n-2)-\gamma_2)^{1/2} > 0. \tag{3.1}$$

(Note that this inequality cannot be satisfied if $n \geqslant 8$.)

PROPOSITION 3.1: *Let* $3 \leqslant n \leqslant 7$. *Then, for some* $\delta > 0$,

$$|x|^{2-\delta}|\Omega(x)| \leqslant C. \tag{3.2}$$

PROOF: Case a: Let $n = 3$. From Theorem 1.1, Hölder's inequality and Corollary 2.1, for $|x| = r$,

$$\begin{aligned} |x|^2|\Omega(x)| &\leqslant C_1\|\Omega\|_{L^{n/2}(V_r)} \\ &\leqslant C_2 r^{(1-\alpha)/2}\left(\int_{|x|\leqslant 2r}|x|^{\alpha}|\Omega(x)|^2\mathrm{d}x\right)^{1/2} \\ &\leqslant C r^{(1-\alpha)/2} r^{1/2K} \end{aligned}$$

for any $\alpha > 1$ and K independent of α. Choosing α sufficiently close to one, proves case *a*.

Case b: Let $n \geqslant 4$. Similarly, by the preceding sections, for $|x| = r$,

$$\begin{aligned} |x|^2|\Omega(x)| &\leqslant C_1\|\Omega\|_{L^{n/2}(V_r)} \\ &\leqslant C_1\left(\|\Omega\|_{\infty(V_r)}\right)^{(n-4)/n}\left(\int_{|x|\leqslant 2r}|\Omega(x)|^2\mathrm{d}x\right)^{2/n} \\ &\leqslant C_2 r^{(2/n)(4-n)}\left(\int_{|x|\leqslant 2r}|\Omega(x)|^2\mathrm{d}x\right)^{2/n} \\ &\leqslant C r^{(2/n)(4-n+(1-k\gamma_2)(2(n-2)-\gamma_2)^{1/2})}. \end{aligned}$$

By assumption (3.1), the exponent on the right is positive, which proves case b.

COROLLARY 3.1: $\Omega \in L^p$ *for* $\frac{1}{2}n \leqslant p < n/(2-\delta)$ *and is a weak solution of the Yang-Mills equations in the full ball B.*

The proof of Corollary 3.1 is elementary and we omit it. The main theorem now follows from the following theorem of Uhlenbeck ([8], Theorem 1.3) applied to solutions with isolated point singularities:

THEOREM: *Let D be a covariant derivative in a bundle over B whose curvature is a weak solution of the Yang-Mills equations on B. If the curvature is in* L^p *for* $p > \frac{1}{2}n$, *then D is gauge equivalent by a continuous gauge transformation to an analytic connection.*

A result analogous to our main theorem has been obtained for the *coupled* Yang-Mills equations (cf. [4] and [6]) in dimension 3, and appears in [9].

Added in proof

An elementary proof of the removable point singularity theorem in dimension $n \geqslant 5$ will appear in a forthcoming paper.

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(Oblatum 22-X-1982 & 15-III-1983)

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