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## HOMOGENEOUS-RATIONAL MANIFOLDS AND UNIQUE FACTORIZATION

Manfred Steinsiek

### Introduction. Statement of the results

All varieties occurring in this note are assumed to be defined over  $\mathbb{C}$ . We call an affine (resp. projective) variety *factorial* if its affine (resp. homogeneous) coordinate ring is a unique factorization domain. It seems that the question whether a given affine or projective variety is factorial, goes back to Felix Klein and Max Noether in the late 19th century (see e.g. [20, p. 32]).

Some well-known examples of factorial projective varieties are, besides the trivial example  $\mathbb{P}_n$ , the nonsingular quadric  $Q_n \subset \mathbb{P}_{n+1}$  for  $n \geq 3$  (Klein), and the Grassmann variety  $G_{n,k}$  of  $k$ -planes in  $\mathbb{P}_n$  considered as a projective variety in  $\mathbb{P}_N$  via Plücker embedding, where  $N = \binom{n+1}{k+1} - 1$  (the question whether  $G_{n,k}$  is factorial was raised by Severi around 1915 and answered in the affirmative by Samuel in the early 1960's (cf. [1] and [20, pp. 37 ff.])).

Since the examples mentioned above are all *homogeneous-rational* manifolds, it is a quite natural question whether any homogeneous-rational manifold is factorial or, more realistically, to decide which of them are. Here, by a homogeneous-rational manifold we mean a compact homogeneous projective-rational complex manifold of positive dimension. Equivalently, a compact complex manifold  $X$  of positive dimension is homogeneous-rational if and only if either there is a connected semisimple complex Lie group  $G$  acting transitively on  $X$  such that  $X = G/H$ , where  $H$  is a proper parabolic subgroup of  $G$ , or  $X$  is homogeneous with vanishing first Betti number and nonvanishing Euler characteristic, or  $X$  is homogeneous and Kähler with  $H^1(X, \mathcal{O}) = 0$  (see [2], [3], [8]).

Now, none of the above equivalent conditions for homogeneous-rationality involves an embedding of  $X$  into  $\mathbb{P}_N$ . Hence the question for factoriality of homogeneous-rational manifolds should be stated more precisely in the following form: *Given a holomorphic embedding  $f: X \rightarrow \mathbb{P}_N$  of a homogeneous-rational manifold  $X$ , under which conditions (on  $X$  and  $f$ ) is  $f(X)$  factorial?*

We first define some rather special embeddings. For this purpose let  $X$  be a homogeneous-rational manifold,  $G$  a connected *simply-connected* semisimple complex Lie group acting transitively on  $X$ . A holomorphic embedding  $f: X \rightarrow \mathbb{P}_N$  is called *homogeneously normal* if  $f$  is  $G$ -equivariant, i.e. if there is a holomorphic representation  $\phi_f: G \rightarrow \mathrm{SL}(N+1, \mathbb{C})$  such that  $\phi_f(g)(f(x)) = f(g(x))$  for all  $g \in G$  and  $x \in X$ . It is not difficult to see that this definition is independent of  $G$ , i.e. if  $G^*$  is another connected simply-connected semisimple complex Lie group acting likewise transitively on  $X$ , then a holomorphic embedding  $f: X \rightarrow \mathbb{P}_N$  is  $G$ -equivariant if and only if  $f$  is  $G^*$ -equivariant (cf. [22, Kap. II, Sect. 2.3]). A holomorphic embedding  $f: X \rightarrow \mathbb{P}_N$  is called *homogeneously minimal* if it is homogeneously normal and if  $N$  is minimal, i.e.  $N \leq M$  for any homogeneously normal embedding  $f^*: X \rightarrow \mathbb{P}_M$ . Then we have the following result which is a special case of a theorem of Tits ([24, III.D]):

**THEOREM T:** *There exists a homogeneously minimal embedding of  $X$ , and this is unique up to an automorphism of the ambient projective space.*

Note that it is necessary to assume the group  $G$  to be simply-connected in order to obtain homogeneously minimal embeddings which everybody would expect (for instance, there is no  $\mathrm{PGL}(2, \mathbb{C})$ -equivariant embedding of  $\mathbb{P}_1$  in  $\mathbb{P}_1$ ). In the case  $X = G_{n,k}$ , the homogeneously minimal embedding of  $X$  is just the Plücker embedding. It should be pointed out that, in contrast to the name “minimal”, this  $N$  is not necessarily so small: For instance, if  $X = G/B$ , where  $B$  is a Borel subgroup of  $G$ , then  $N = 2^{\dim X} - 1$ . On the other hand, it is well-known that any projective-algebraic manifold of dimension  $d$  admits an embedding into  $\mathbb{P}_{2d+1}$  (cf. [9, p. 173]).

Let  $X$  be a homogeneous-rational manifold and  $G$  a connected semisimple complex Lie group acting transitively on  $X$ . Write  $X = G/H$  with  $H$  a proper parabolic subgroup of  $G$ , and denote by  $H'$  the commutator group of  $H$ . We define the *rank*<sup>(1)</sup> of  $X$ , written  $\mathrm{rk}(X)$ , as the dimension of the complex Lie group  $H/H'$ . Equivalently,  $\mathrm{rk}(X)$  is the number of maximal parabolic subgroups of  $G$  which contain  $H$ . Using this description of the rank and a theorem of Remmert-van de Ven ([19, Satz (2.2)]), it is easy to see that the rank of  $X$  depends only on  $X$ , but not on the group  $G$ . One can also show that  $\mathrm{rk}(X) = b_2(X)$ , where  $b_2(X)$  denotes the 2<sup>nd</sup> Betti number of  $X$  (cf. [5, p. 245] and [23, Remark in § 3]). Obviously,  $\mathrm{rk}(X) = 1$  if and only if  $H$  is a maximal parabolic subgroup of  $G$  and, by [18], this is equivalent to the condition that each holomorphic map  $h: X \rightarrow Y$  of  $X$  into a complex space  $Y$  of dimension  $< \dim X$  be

<sup>(1)</sup> This definition of the rank has nothing whatsoever to do with the rank of a symmetric (in particular hermitian symmetric) space in differential geometry. However, our notation seems to be rather familiar, see e.g. [25, p. 114].

constant. In particular, every rank 1-homogeneous-rational manifold is *irreducible*. (A homogeneous-rational manifold  $X$  is called irreducible if  $\text{Aut}(X)$  is a simple complex Lie group, reducible otherwise. Evidently, a homogeneous-rational manifold  $X$  is reducible if and only if there are homogeneous-rational manifolds  $X_1, X_2$  such that  $X \cong X_1 \times X_2$ .)

Let us look at some examples: The homogeneous-rational manifolds of rank 1 are projective spaces, quadrics of dimension  $\geq 3$ , Grassmannians, “Grassmannians” of linear subspaces of  $\mathbb{P}_{n+1}$  which lie on the quadric  $Q_n \subset \mathbb{P}_{n+1}$  (cf. [24, II.C.7, II.C.11]), “Grassmannians” of linear subspaces of  $\mathbb{P}_n$ ,  $n \geq 5$  odd, which are totally isotropic with respect to a nullcorrelation (cf. [24, II.C.7, II.C.11]), and finally 24 pairwise non-isomorphic rank 1-homogeneous-rational manifolds, whose automorphism groups are exceptional simple complex Lie groups (cf. [23]). In higher rank, the probably best-known examples are, besides direct products of rank 1-homogeneous-rational manifolds, the flag manifolds of  $\mathbb{P}_n$ ,  $n \geq 2$ , of rank  $n$ , the simplest being the rank 2-homogeneous-rational manifold  $F_2 = \{(x, L) \in \mathbb{P}_2 \times \mathbb{P}_2^*; x \in L\}$  of dimension 3.

Now we are in position to state our main results.

**THEOREM 1 (FACTORIALITY CRITERION):** *The following statements about a holomorphic embedding  $f: X \rightarrow \mathbb{P}_N$  of a homogeneous-rational manifold  $X$  are equivalent:*

- (i)  $f(X)$  is factorial;
- (ii) (a)  $\text{rk}(X) = 1$  and  
(b) there is a linear  $k$ -plane  $\mathbb{P}_k \subset \mathbb{P}_N$  such that  $f(X) \subset \mathbb{P}_k$  and  $f: X \rightarrow \mathbb{P}_k$  is homogeneously minimal.

The proof of this theorem is carried out in Section 2 by inspecting the divisor class group  $\text{Cl}(X)$  of  $X$  and using a criterion of factoriality which is due to Samuel. Fundamental for the proof is the following Normality Criterion which is proved in Section 1.

**THEOREM 2 (NORMALITY CRITERION):** *The following statements about a holomorphic embedding  $f: X \rightarrow \mathbb{P}_N$  of a homogeneous-rational manifold  $X$  are equivalent:*

- (i)  $f(X)$  is projectively normal;
- (ii)  $f$  is homogeneously normal.

Recall that a projective variety is called *projectively normal* if its homogeneous coordinate ring is a normal domain. We suspect that (at least) part of the Normality Criterion is known, but we do not know any adequate reference (except for the case  $X = G_{n,k}$ ,  $f =$  Plücker embedding: Severi showed in 1915 that  $f(X)$  is projectively normal (cf. [21, p. 100], see also [13])).

Let  $X$  be a homogeneous-rational manifold homogeneously minimally embedded in  $\mathbb{P}_N$ . We define an *affine kernel*  $X_a$  of  $X$  to be the comple-

ment of a general hyperplane section in  $X$ . Thus  $X_a$  is an affine variety, and we ask for the divisor class group of  $X_a$ . This question was raised by Remmert around 1965. We give a complete answer to this question:

**THEOREM 3:** *If  $X$  is a homogeneous-rational manifold, then the divisor class group  $\text{Cl}(X_a)$  of an affine kernel  $X_a$  of  $X$  is isomorphic to  $\mathbb{Z}^{\text{rk}(X)-1}$ . In particular,  $X_a$  is factorial if and only if  $\text{rk}(X) = 1$ .*

The proof of this theorem is similar to that of Theorem 1 and is also given in Section 2. It is done by investigating the canonical surjective mapping  $\text{Cl}(X) \rightarrow \text{Cl}(X_a)$  between the divisor class groups of  $X$  and  $X_a$ .

In Section 3, we give two applications of Theorem 1. First, *the homogeneous coordinate ring  $S$  of a homogeneously minimally embedded rank 1-homogeneous manifold  $X$  is Gorenstein* (for Grassmannians, this has been shown by Hochster in [11]). This is proved by first showing that  $S$  is Cohen-Macaulay and then, of course, applying Murthy's Theorem ([17]). For the second application, let  $X$  be a rank 1-homogeneous-rational manifold, homogeneously minimally embedded in  $\mathbb{P}_N$ , and let  $R$  be the local ring of the vertex of the affine cone over  $X$  in affine  $(N + 1)$ -space. Using a result of Danilov ([6]), we prove that, unless  $X$  is isomorphic to a projective space (in which case  $R$  is regular),  $R$  is a non-regular local unique factorization domain whose completion  $\hat{R}$  is again factorial.

It should be noted that the proofs of the theorems as well as the applications, though not being very complicated, depend on an interplay of several mathematical fields: from representation theory of semisimple complex Lie algebras and Lie groups, we use Tits' embedding theorem and the Borel-Weil Theorem; from complex analysis, we use Bott's Theorem and results of Remmert-van de Ven; from algebraic geometry and commutative algebra, we use Samuel's Criterion of Factoriality, Murthy's Theorem, results of Danilov on the divisor class group of a complete local ring, etc.

It seems that most of our results carry over to varieties  $G/H$  over more general algebraically closed ground fields  $K$ , at least for  $\text{char } K = 0$ .

It is the author's pleasure to thank Prof. R. Remmert for bringing the above mentioned problems to his attention as well as for many helpful conversations during the preparation of this paper.

## 1. Proof of Theorem 2

We first discuss the Borel-Weil Theorem, which will turn out to be crucial for the proof of our Normality Criterion. Let always  $X = G/H$  be a homogeneous-rational manifold, where  $G$  is a connected simply-connected semisimple complex Lie group acting transitively on  $X$  and  $H$  is a proper parabolic subgroup of  $G$ . We further denote by  $\text{Cl}(X)$  the divisor class group of  $X$ . We begin with the following simple

LEMMA 1:  $Cl(X) \cong H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{rk(X)}$ .

PROOF: The first isomorphism follows from [10, p. 145]. Next, since  $H^q(X, \mathcal{O}) = 0$  for  $q \geq 1$  (cf. [5, Lemma 14.2]), from the exact cohomology sequence belonging to the short exact exponential sequence we obtain  $H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z})$ . Finally, since  $b_1(X) = 0$ ,  $H^2(X, \mathbb{Z})$  is torsion-free, whence  $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{rk(X)}$  because of  $b_2(X) = rk(X)$ .  $\square$

Now let  $\mathcal{L}$  be a line bundle on  $X$ ,  $\mathcal{L} = \mathcal{O}(D)$  with  $D$  a divisor on  $X$ . Let  $|D| \cong \mathbb{P}(H^0(X, \mathcal{L}))$  be the corresponding linear system (which may be empty). Since  $G$  is connected,  $G$  acts trivially on  $H^2(X, \mathbb{Z})$ , hence on  $Cl(X)$ . Thus, for  $D^* \in |D|$ ,  $g \in G$ ,  $g(D^*) \in |D|$ , and hence  $G$  acts on  $|D| \cong \mathbb{P}(H^0(X, \mathcal{L}))$ . Since  $G$  is simply-connected, this action lifts to a linear action of  $G$  on  $H^0(X, \mathcal{L})$ . In particular,  $H^0(X, \mathcal{L})$  is a  $G$ -module. Now we state the following special case of the Borel-Weil Theorem ([4]):

THEOREM BW: *If  $\mathcal{L}$  is a very ample line bundle on  $X$ , then the  $G$ -module  $H^0(X, \mathcal{L})$  is irreducible.*

PROOF OF THEOREM 2: Denote by  $S = \sum_{n \geq 0} S_n$  the homogeneous coordinate ring of  $f(X)$ , and let  $\tilde{S}$  be the integral closure of  $S$ . Then we have (cf. [10, Ch. II, Ex. 5.14]):  $\tilde{S} = \sum_{n \geq 0} H^0(f(X), \mathcal{O}(n))$ . Here,  $\mathcal{O}(1)$  denotes the twisting sheaf of Serre (cf. [10, p. 117]), i.e. considered as a line bundle on  $X$ ,  $\mathcal{O}(1) = f^*\mathcal{H}$ , where  $\mathcal{H}$  is the hyperplane section bundle on  $\mathbb{P}_N$ , and  $\mathcal{O}(n) = \mathcal{O}(1)^n$ . Then, by Theorem BW, for all  $n \geq 0$ ,  $H^0(f(X), \mathcal{O}(n))$  is an irreducible  $G$ -module.

Now let  $f$  be homogeneously normal. Then the linear action  $\phi_f: G \rightarrow SL(N + 1, \mathbb{C})$  induces a natural action  $\tilde{\phi}_f$  of  $G$  on  $S$ . This action preserves the grading of  $S$ , i.e.  $\tilde{\phi}_f(G)S_n \subset S_n$  for all  $n \geq 0$ , and, in fact, furnishes  $S_n$  with the structure of a  $G$ -submodule of  $H^0(f(X), \mathcal{O}(n))$ . But  $H^0(f(X), \mathcal{O}(n))$  is an irreducible  $G$ -module. Hence, for a fixed  $n \geq 0$ , we have either  $S_n = 0$  which is clearly impossible, or  $S_n = H^0(f(X), \mathcal{O}(n))$ . Thus we obtain  $S = \tilde{S}$ , and  $f(X)$  is projectively normal.

Finally, if  $f$  is not homogeneously normal, then evidently  $S_1 \subsetneq H^0(f(X), \mathcal{O}(1))$ , and hence  $f(X)$  is not projectively normal.  $\square$

EXAMPLE: As a special case of Theorem 2, for  $X = \mathbb{P}_1$ , we obtain the following well-known facts:

- (1) The  $d$ -uple embedding of  $\mathbb{P}_1$  in  $\mathbb{P}_d$  (this embedding is given by the monomials in two variables of degree  $d$ ) is projectively normal (cf. [10, Ch. IV, Ex. 3.4]).
- (2) The twisted quartic curve in  $\mathbb{P}_3$  (this is given by the (non-homogeneously normal) embedding  $[z_0 : z_1] \rightarrow [z_0^4 : z_0^3 z_1 : z_0 z_1^3 : z_1^4]$  of  $\mathbb{P}_1$  in  $\mathbb{P}_3$ ) is not projectively normal (cf. [10, Ch. I, Ex. 3.18]).

### 2. Proofs of Theorem 1 and Theorem 3

We begin with some useful remarks concerning very ample line bundles on homogeneous-rational manifolds. In this whole section,  $X = G/H$  is a homogeneous-rational manifold, where, as usual,  $G$  is a connected simply-connected semisimple complex Lie group acting transitively on  $X$  and  $H$  a proper parabolic subgroup of  $G$ . We further let  $r = \text{rk}(X)$ .

By Lemma 1,  $H^1(X, \mathcal{O}^*) \cong \mathbb{Z}^r$ , so it is reasonable to speak of line bundles of type  $(n_1, \dots, n_r) \in \mathbb{Z}^r$  on  $X$ . However, since a line bundle of type  $(1, \dots, 1)$  should be “positive”, this has to be rendered precise: Let  $B$  be a Borel subgroup of  $G$ , and consider the  $B$ -action on  $X$ . Then there is an open  $B$ -orbit  $U$  in  $X$  (the open “Bruhat-cell”), and the complement  $X - U$  is a divisor on  $X$  consisting of  $r$  irreducible components  $D_1, \dots, D_r$  (see [15] for details in the case  $X = G/B$ ). Now, for  $(n_1, \dots, n_r) \in \mathbb{Z}^r$ , define the line bundle of type  $(n_1, \dots, n_r)$  to be the line bundle belonging to the divisor  $n_1 D_1 + \dots + n_r D_r$ . Equivalently, line bundles of type  $(n_1, \dots, n_r)$  may be defined as follows: Let  $R = T \cdot S$  be the reductive part of  $H$ , where  $T \cong (\mathbb{C}^*)^r$  and  $S$  is semisimple. Then the groups  $T^*$  and  $H^*$  of holomorphic characters of  $T$  and  $H$  are isomorphic,  $T^* \cong H^*$ . Obviously,  $T^* \cong \mathbb{Z}^r$ , the isomorphism being given by  $\mathbb{Z}^r \ni (n_1, \dots, n_r) \rightarrow \chi_{(n_1, \dots, n_r)} \in T^*$ , where  $\chi_{(n_1, \dots, n_r)}(z_1, \dots, z_r) = z_1^{n_1} \cdot \dots \cdot z_r^{n_r}$ . Now the line bundle of type  $(n_1, \dots, n_r)$  on  $X$  is the homogeneous line bundle given by the character  $\chi_{(n_1, \dots, n_r)} \in T^* \cong H^*$ . Furthermore, line bundles of type  $(n_1, \dots, n_r)$  on  $X$  may be described in the following way: Let  $P_1, \dots, P_r$  the maximal parabolic subgroups of  $G$  which contain  $H$ , let  $X_i = G/P_i$ , and let  $\pi_i: X \rightarrow X_i$  the natural fibrations. Since  $\text{rk}(X_i) = 1$ ,  $H^1(X_i, \mathcal{O}^*) \cong \mathbb{Z}$ . Now take positive generators  $\mathcal{L}_i$  of  $H^1(X_i, \mathcal{O}^*)$ ,  $i = 1, \dots, r$ . Then the line bundle  $\mathcal{L}$  of type  $(n_1, \dots, n_r)$  on  $X$  is given by  $\mathcal{L} = \pi_1^*(\mathcal{L}_1^{n_1}) \otimes \dots \otimes \pi_r^*(\mathcal{L}_r^{n_r})$ .

Now, very ample line bundles on  $X$  can be easily characterised:

**REMARK:** *A line bundle  $\mathcal{L}$  of type  $(n_1, \dots, n_r)$  on  $X$  is very ample if and only if  $n_i > 0$ ,  $i = 1, \dots, r$ . In particular, a holomorphic embedding  $f: X \rightarrow \mathbb{P}_N$  is homogeneously minimal if and only if  $f$  is given by a base  $s_0, \dots, s_N$  of  $H^0(X, \mathcal{L})$ , where  $\mathcal{L}$  is a line bundle of type  $(1, \dots, 1)$  on  $X$ .*

**PROOF:** The first part is contained in [4, §4], the second assertion follows from Tits’ Theorem (see [22, Korollar 2.2.2]). □

We now come to the proof of Theorem 1. We employ Samuel’s

**CRITERION OF FACTORIALITY** (cf. [10, Ch. II, Ex. 6.3]): *A projective variety  $V$  is factorial if and only if (1)  $V$  is projectively normal, and (2) the divisor class group  $\text{Cl}(V)$  of  $V$  is isomorphic to  $\mathbb{Z}$  and is generated by the class of a (suitable) hyperplane section.*

PROOF OF THEOREM 1: By Samuel's Criterion and Lemma 1, clearly  $\text{rk}(X) = 1$  if  $f(X)$  is factorial. So assume  $\text{rk}(X) = 1$ . Take a line bundle  $\mathcal{L}_0$  of type (1) on  $X$ . Hence  $\mathcal{L}_0$  generates  $H^1(X, \mathcal{O}^*) \cong \mathbb{Z}$ . By the Remark,  $f$  is given by sections  $s_0, \dots, s_N \in H^0(X, \mathcal{L})$ , where  $\mathcal{L}$  is a line bundle of type  $(n)$  on  $X$ ,  $n \geq 1$ , i.e.  $\mathcal{L} = \mathcal{L}_0^n$  with  $n \geq 1$ . Let  $V^*$  be a hyperplane in  $\mathbb{P}_N$  not containing  $f(X)$  and  $V$  the corresponding hyperplane section. Then we have  $\text{Cl}(f(X))/\langle V \rangle \cong H^1(X, \mathcal{O}^*)/\langle \mathcal{L} \rangle = (\mathcal{L}_0)/\langle \mathcal{L} \rangle \cong \mathbb{Z}/n\mathbb{Z}$ . Now, by Samuel's Criterion,  $f(X)$  is factorial if and only if  $n = 1$  and  $f(X)$  is projectively normal. Hence the assertion follows from Theorem 2 and the Remark.  $\square$

PROOF OF THEOREM 3: Let  $f: X \rightarrow \mathbb{P}_N$  be a homogeneously minimal embedding of  $X$ , and let an affine kernel  $X_a$  of  $X$  be given by  $X_a = f(X) - Z$ , where  $Z$  is a general (i.e. smooth) hyperplane section of  $f(X)$ . We consider the exact sequence  $\mathbb{Z} \xrightarrow{i} \text{Cl}(X) \rightarrow \text{Cl}(X_a) \rightarrow 0$ , where the map  $i$  is given by  $1 \rightarrow 1 \cdot Z$  (cf. [10, Ch. II, Prop. 6.5]). Since the group  $\text{Cl}(X)$  is torsion-free, the map  $i$  is injective. We have to determine the image  $\text{Im}(i)$  of  $i$  in  $\text{Cl}(X)$ . First,  $\text{Cl}(X) \cong H^1(X, \mathcal{O}^*) \cong \mathbb{Z}^r$ . Next, by the Remark,  $f$  is given by a base  $s_0, \dots, s_N$  of  $H^0(X, \mathcal{L}_0)$ , where  $\mathcal{L}_0$  is a line bundle of type  $(1, \dots, 1)$  in  $H^1(X, \mathcal{O}^*)$ . Hence, for the divisor class group of  $X_a$  we obtain:  $\text{Cl}(X_a) \cong \text{Cl}(X)/\text{Im}(i) \cong H^1(X, \mathcal{O}^*)/\langle \mathcal{L}_0 \rangle \cong \mathbb{Z}^r / \langle (1, \dots, 1) \rangle \cong \mathbb{Z}^{r-1}$ , whence the assertion. In particular, if  $r = 1$ , then  $\text{Cl}(X_a) = 0$ , and hence  $X_a$  is factorial (cf. [10, Ch. II, Prop. 6.2]).

### 3. Applications

In this section, we give two applications of Theorem 1 and the following special case of a theorem of Bott (cf. [5, Thm. IV']):

THEOREM B: *If  $\mathcal{L}$  is a very ample line bundle on a homogeneous-rational manifold  $X$ , then  $H^q(X, \mathcal{L}) = 0$  for  $q \geq 1$  and  $H^q(X, \mathcal{L}^{-1}) = 0$  for  $q < \dim X$ .*

For the first application, recall that a noetherian ring  $A$  is called *Cohen-Macaulay* (Gorenstein, resp.) if, for every maximal ideal  $\mathfrak{m}$  of  $A$ , the local ring  $A_{\mathfrak{m}}$  is Cohen-Macaulay (Gorenstein, resp.), i.e.  $\dim A_{\mathfrak{m}} = \text{depth } A_{\mathfrak{m}}$  (the injective dimension of  $A_{\mathfrak{m}}$  is finite, resp.). For generalities on Cohen-Macaulay and Gorenstein rings, see e.g. [14]. Now we can state:

COROLLARY 1: *The homogeneous coordinate ring  $S$  of a homogeneously minimally embedded rank 1-homogeneous-rational manifold  $X$  is Gorenstein.*

PROOF: According to a theorem of Murthy ([17], see also [7, Thm. 12.3]), a factorial Cohen-Macaulay factor ring of a Gorenstein ring is Gorenstein. Hence, by Theorem 1, it is sufficient to show that  $S$  is Cohen-Macaulay. In fact, we have quite generally

PROPOSITION: *If  $f: X \rightarrow \mathbb{P}_N$  is a homogeneously normal embedding of a homogeneous-rational manifold  $X$ , then the homogeneous coordinate ring  $S(f(X))$  of  $f(X)$  is Cohen-Macaulay.*

PROOF: It is well-known that the homogeneous coordinate ring  $S(V)$  of a nonsingular projectively normal projective variety  $V$  is Cohen-Macaulay provided  $H^q(V, \mathcal{O}(n)) = 0$  for all  $n \in \mathbb{Z}$  and  $1 \leq q \leq \dim V - 1$  (this is e.g. a special case of Prop. B, p. 131, and Prop. 5.1 of [12]). Applying this theorem to our situation, it follows from Theorem 2 and Theorem B that  $S(f(X))$  is Cohen-Macaulay.  $\square$

For the second application, let  $X$  be a homogeneous-rational manifold, and let  $f: X \rightarrow \mathbb{P}_N$  be a homogeneously minimal embedding of  $X$ . Denote by  $V(X)$  the affine cone over  $f(X)$  in affine  $(N + 1)$ -space. Let  $P$  be the vertex of  $V(X)$  and  $R = \mathcal{O}_{V(X), P}$  the local ring of  $P$  on  $V(X)$ . Thus, by Theorem 2,  $R$  is a normal domain, and, unless  $X$  is isomorphic to  $\mathbb{P}_n$  for some  $n$ ,  $R$  is not regular.

Now assume additionally  $\text{rk}(X) = 1$ . Then, by Theorem 1,  $f(X)$  is factorial, and hence  $R$  is a unique factorization domain (cf. [7, Cor. 10.3]). One may ask whether the completion  $\hat{R}$  of  $R$  with respect to its maximal ideal is again factorial. In general, if  $A$  is a local noetherian Krull domain and  $\hat{A}$  its completion, then, by Mori's Theorem (cf. [7, Cor. 6.12]),  $A$  is factorial if  $\hat{A}$  is, but the converse is false in general (see [7, Example 19.9] for a counterexample). In our case, however, we have

COROLLARY 2: *The ring  $\hat{R}$  is a unique factorization domain.*

PROOF: We use the following result of Danilov (cf. [6, Theorem in §2 and Prop. 8], see also [16, p. 532 f.]): If  $V$  is a nonsingular projectively normal projective variety,  $A$  the local ring of the vertex of the affine cone over  $V$ , and  $\hat{A}$  the completion of  $A$ , then the divisor class groups of  $A$  and  $\hat{A}$  are isomorphic if and only if  $H^1(V, \mathcal{O}(n)) = 0$  for all  $n \geq 1$ . Applying this theorem to our situation, by Theorem B we obtain  $\text{Cl}(\hat{R}) = \text{Cl}(R) = 0$ , whence the assertion.  $\square$

## References

- [1] A. ANDREOTTI and P. SALMON: Anelli con unica decomponibilità in fattori primi ed un problema di intersezione complete. *Monatshefte Math. Physik* 61 (1957) 97–142.
- [2] A. BOREL: Kählerian coset spaces of semi-simple Lie groups. *Proc. Nat. Acad. Sci. USA* 40 (1954) 1147–1151.

- [3] A. BOREL and R. REMMERT: Über kompakte homogene Kählersche Mannigfaltigkeiten. *Math. Ann.* 145 (1962) 429–439.
- [4] A. BOREL and A. WEIL: Représentations linéaires et espaces homogènes Kählériens des groupes de Lie compacts. Sém. Bourbaki 1954, no. 100 (Exposé par J.-P. Serre).
- [5] R. BOTT: Homogeneous vector bundles. *Ann. Math. (2)* 66 (1957) 203–248.
- [6] V.I. DANILOV: The group of ideal classes of a completed ring. *Math. USSR Sb.* 6 (1968) 493–500.
- [7] R.M. FOSSUM: The divisor class group of a Krull domain. *Erg. der Math.* 74, Springer (1973).
- [8] M. GOTO: On algebraic homogeneous spaces. *Amer. J. Math.* 76 (1954) 811–818.
- [9] P.A. GRIFFITHS and J. HARRIS: *Principles of algebraic geometry*. New York: Wiley (1978).
- [10] R. HARTSHORNE: Algebraic geometry. *GTM 52*, Springer (1977).
- [11] M. HOCHSTER: Grassmannians and their Schubert varieties are arithmetically Cohen-Macaulay. *J. Algebra* 25 (1973) 40–57.
- [12] M. HOCHSTER and J.L. ROBERTS: Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. *Advances in Math.* 13 (1974) 115–175.
- [13] J. IGUSA: On the arithmetic normality of the Grassmann variety. *Proc. Nat. Acad. Sci. USA* 40 (1954) 309–313.
- [14] I. KAPLANSKY: *Commutative rings*. Boston: Allyn and Bacon (1970).
- [15] M. LAZARD: Groupes semi-simples: Structure de B et de G/B. Sém. Chevalley 1956–1958: Classification des groupes de Lie algébriques. Exposé 13.
- [16] J. LIPMAN: Unique factorization in complete local rings. *AMS Proc. Symp. Pure Math.* 29 (1975) 531–546.
- [17] M. MURTHY: A note on factorial rings. *Arch. Math.* 15 (1964) 418–420.
- [18] R. REMMERT and T. VAN DE VEN: Über holomorphe Abbildungen projektiv-algebraischer Mannigfaltigkeiten auf komplexe Räume. *Math. Ann.* 142 (1961) 453–486.
- [19] R. REMMERT and A. VAN DE VEN: Zur Funktionentheorie homogener komplexer Mannigfaltigkeiten. *Topology* 2 (1963) 137–157.
- [20] P. SAMUEL: Lectures on unique factorization domains. *Tata Inst. Fund. Res.*, No. 30, Bombay (1964).
- [21] F. SEVERI: Sulla varietà che rappresenta gli spazi subordinati di data dimensione, immersi in uno spazio lineare. *Ann. di Mat. (3)* 24 (1915) 89–120.
- [22] M. STEINSIEK: Über homogen-rationale Mannigfaltigkeiten. *Schriftenr. Math. Inst. Univ. Münster*, 2. Serie, Bd. 23 (1982).
- [23] M. STEINSIEK: Transformation groups on homogeneous-rational manifolds. *Math. Ann.* 260 (1982) 423–435.
- [24] J. TITS: Sur certaines classes d'espaces homogènes de groupes de Lie. *Mém. Acad. Roy. Belg.* 29 (1955) 1–268.
- [25] J. TITS: Espaces homogènes complexes compacts. *Comment. Math. Helv.* 37 (1962/63) 111–120.

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