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HOMOGENEOUS-RATIONAL MANIFOLDS AND UNIQUE FACTORIZATION

Manfred Steinsiek

Introduction. Statement of the results

All varieties occurring in this note are assumed to be defined over $\mathbb{C}$. We call an affine (resp. projective) variety *factorial* if its affine (resp. homogeneous) coordinate ring is a unique factorization domain. It seems that the question whether a given affine or projective variety is factorial, goes back to Felix Klein and Max Noether in the late 19th century (see e.g. [20, p. 32]).

Some well-known examples of factorial projective varieties are, besides the trivial example $\mathbb{P}_n$, the nonsingular quadric $Q_n \subset \mathbb{P}_{n+1}$ for $n \geqslant 3$ (Klein), and the Grassmann variety $G_{n,k}$ of k-planes in $\mathbb{P}_n$ considered as a projective variety in $\mathbb{P}_N$ via Plücker embedding, where $N = \binom{n+1}{k+1} - 1$ (the question whether $G_{n,k}$ is factorial was raised by Severi around 1915 and answered in the affirmative by Samuel in the early 1960's (cf. [1] and [20, pp. 37 ff.])).

Since the examples mentioned above are all *homogeneous-rational* manifolds, it is a quite natural question whether any homogeneous-rational manifold is factorial or, more realistically, to decide which of them are. Here, by a homogeneous-rational manifold we mean a compact homogeneous projective-rational complex manifold of positive dimension. Equivalently, a compact complex manifold X of positive dimension is homogeneous-rational if and only if either there is a connected semisimple complex Lie group G acting transitively on X such that $X = G/H$, where H is a proper parabolic subgroup of G, or X is homogeneous with vanishing first Betti number and nonvanishing Euler characteristic, or X is homogeneous and Kähler with $H^1(X, \mathcal{O}) = 0$ (see [2], [3], [8]).

Now, none of the above equivalent conditions for homogeneous-rationality involves an embedding of X into $\mathbb{P}_N$. Hence the question for factoriality of homogeneous-rational manifolds should be stated more precisely in the following form: *Given a holomorphic embedding f: $X \to \mathbb{P}_N$ of a homogeneous-rational manifold X, under which conditions (on X and f) is $f(X)$ factorial?*

We first define some rather special embeddings. For this purpose let X be a homogeneous-rational manifold, G a connected *simply-connected* semisimple complex Lie group acting transitively on X. A holomorphic embedding $f\colon X \to \mathbb{P}_N$ is called *homogeneously normal* if f is G-equivariant, i.e. if there is a holomorphic representation $\phi_f\colon G \to \mathrm{SL}(N+1, \mathbb{C})$ such that $\phi_f(g)(f(x)) = f(g(x))$ for all $g \in G$ and $x \in X$. It is not difficult to see that this definition is independent of G, i.e. if G^* is another connected simply-connected semisimple complex Lie group acting likewise transitively on X, then a holomorphic embedding $f\colon X \to \mathbb{P}_N$ is G-equivariant if and only if f is G^*-equivariant (cf. [22, Kap. II, Sect. 2.3]). A holomorphic embedding $f\colon X \to \mathbb{P}_N$ is called *homogeneously minimal* if it is homogeneously normal and if N is minimal, i.e. $N \leqslant M$ for any homogeneously normal embedding $f^*\colon X \to \mathbb{P}_M$. Then we have the following result which is a special case of a theorem of Tits ([24, III.D]):

THEOREM T: *There exists a homogeneously minimal embedding of X, and this is unique up to an automorphism of the ambient projective space.*

Note that it is necessary to assume the group G to be simply-connected in order to obtain homogeneously minimal embeddings which everybody would expect (for instance, there is no $\mathrm{PGL}(2, \mathbb{C})$-equivariant embedding of $\mathbb{P}_1$ in $\mathbb{P}_1$). In the case $X = G_{n,k}$, the homogeneously minimal embedding of X is just the Plücker embedding. It should be pointed out that, in contrast to the name "minimal", this N is not necessarily so small: For instance, if $X = G/B$, where B is a Borel subgroup of G, then $N = 2^{\dim X} - 1$. On the other hand, it is well-known that any projective-algebraic manifold of dimension d admits an embedding into $\mathbb{P}_{2d+1}$ (cf. [9, p. 173]).

Let X be a homogeneous-rational manifold and G a connected semisimple complex Lie group acting transitively on X. Write $X = G/H$ with H a proper parabolic subgroup of G, and denote by H' the commutator group of H. We define the *rank* (1) of X, written $\mathrm{rk}(X)$, as the dimension of the complex Lie group H/H'. Equivalently, $\mathrm{rk}(X)$ is the number of maximal parabolic subgroups of G which contain H. Using this description of the rank and a theorem of Remmert-van de Ven ([19, Satz (2.2)]), it is easy to see that the rank of X depends only on X, but not on the group G. One can also show that $\mathrm{rk}(X) = b_2(X)$, where $b_2(X)$ denotes the 2nd Betti number of X (cf. [5, p. 245] and [23, Remark in § 3]). Obviously, $\mathrm{rk}(X) = 1$ if and only if H is a maximal parabolic subgroup of G and, by [18], this is equivalent to the condition that each holomorphic map $h\colon X \to Y$ of X into a complex space Y of dimension $< \dim X$ be

(1) This definition of the rank has nothing whatsoever to do with the rank of a symmetric (in particular hermitian symmetric) space in differential geometry. However, our notation seems to be rather familiar, see e.g. [25, p. 114].

constant. In particular, every rank 1-homogeneous-rational manifold is *irreducible.* (A homogeneous-rational manifold X is called irreducible if $\mathrm{Aut}(X)$ is a simple complex Lie group, reducible otherwise. Evidently, a homogeneous-rational manifold X is reducible if and only if there are homogeneous-rational manifolds X_1, X_2 such that $X \cong X_1 \times X_2$.)

Let us look at some examples: The homogeneous-rational manifolds of rank 1 are projective spaces, quadrics of dimension $\geqslant 3$, Grassmannians, "Grassmannians" of linear subspaces of $\mathbb{P}_{n+1}$ which lie on the quadric $Q_n \subset \mathbb{P}_{n+1}$ (cf. [24, II.C.7, II.C.11]), "Grassmannians" of linear subspaces of $\mathbb{P}_n$, $n \geqslant 5$ odd, which are totally isotropic with respect to a nullcorrelation (cf. [24, II.C.7, II.C.11]), and finally 24 pairwise non-isomorphic rank 1-homogeneous-rational manifolds, whose automorphism groups are exceptional simple complex Lie groups (cf. [23]). In higher rank, the probably best-known examples are, besides direct products of rank 1-homogeneous-rational manifolds, the flag manifolds of $\mathbb{P}_n$, $n \geqslant 2$, of rank n, the simplest being the rank 2-homogeneous-rational manifold $F_2 = \{(x, L) \in \mathbb{P}_2 \times \mathbb{P}_2^*;\ x \in L\}$ of dimension 3.

Now we are in position to state our main results.

THEOREM 1 (FACTORIALITY CRITERION): *The following statements about a holomorphic embedding f: $X \to \mathbb{P}_N$ of a homogeneous-rational manifold X are equivalent*:

(i) *$f(X)$ is factorial*;

(ii) *(a) $\mathrm{rk}(X) = 1$ and*

(b) *there is a linear k-plane $\mathbb{P}_k \subset \mathbb{P}_N$ such that $f(X) \subset \mathbb{P}_k$ and f: $X \to \mathbb{P}_k$ is homogeneously minimal.*

The proof of this theorem is carried out in Section 2 by inspecting the divisor class group $\mathrm{Cl}(X)$ of X and using a criterion of factoriality which is due to Samuel. Fundamental for the proof is the following Normality Criterion which is proved in Section 1.

THEOREM 2 (NORMALITY CRITERION): *The following statements about a holomorphic embedding f: $X \to \mathbb{P}_N$ of a homogeneous-rational manifold X are equivalent*:

(i) *$f(X)$ is projectively normal*;

(ii) *f is homogeneously normal.*

Recall that a projective variety is called *projectively normal* if its homogeneous coordinate ring is a normal domain. We suspect that (at least) part of the Normality Criterion is known, but we do not know any adequate reference (except for the case $X = G_{n,k}$, $f =$ Plücker embedding: Severi showed in 1915 that $f(X)$ is projectively normal (cf. [21, p. 100], see also [13])).

Let X be a homogeneous-rational manifold homogeneously minimally embedded in $\mathbb{P}_N$. We define an *affine kernel* X_a of X to be the complement of a general hyperplane section in X. Thus X_a is an affine variety, and we ask for the divisor class group of X_a. This question was raised by Remmert around 1965. We give a complete answer to this question:

THEOREM 3: *If X is a homogeneous-rational manifold, then the divisor class group* $\mathrm{Cl}(X_a)$ *of an affine kernel X_a of X is isomorphic to $\mathbb{Z}^{\mathrm{rk}(X)-1}$. In particular, X_a is factorial if and only if* $\mathrm{rk}(X) = 1$.

The proof of this theorem is similar to that of Theorem 1 and is also given in Section 2. It is done by investigating the canonical surjective mapping $\mathrm{Cl}(X) \to \mathrm{Cl}(X_a)$ between the divisor class groups of X and X_a.

In Section 3, we give two applications of Theorem 1. First, *the homogeneous coordinate ring S of a homogeneously minimally embedded rank 1-homogeneous manifold X is Gorenstein* (for Grassmannians, this has been shown by Hochster in [11]). This is proved by first showing that S is Cohen-Macaulay and then, of course, applying Murthy's Theorem ([17]). For the second application, let X be a rank 1-homogeneous-rational manifold, homogeneously minimally embedded in $\mathbb{P}_N$, and let R be the local ring of the vertex of the affine cone over X in affine $(N+1)$-space. Using a result of Danilov ([6]), we prove that, unless X is isomorphic to a projective space (in which case R is regular), *R is a non-regular local unique factorization domain whose completion $\hat{R}$ is again factorial.*

It should be noted that the proofs of the theorems as well as the applications, though not being very complicated, depend on an interplay of several mathematical fields: from representation theory of semisimple complex Lie algebras and Lie groups, we use Tits' embedding theorem and the Borel-Weil Theorem; from complex analysis, we use Bott's Theorem and results of Remmert-van de Ven; from algebraic geometry and commutative algebra, we use Samuel's Criterion of Factoriality, Murthy's Theorem, results of Danilov on the divisor class group of a complete local ring, etc.

It seems that most of our results carry over to varieties G/H over more general algebraically closed ground fields K, at least for char $K = 0$.

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1. Proof of Theorem 2

We first discuss the Borel-Weil Theorem, which will turn out to be crucial for the proof of our Normality Criterion. Let always $X = G/H$ be a homogeneous-rational manifold, where G is a connected simply-connected semisimple complex Lie group acting transitively on X and H is a proper parabolic subgroup of G. We further denote by $\mathrm{Cl}(X)$ the divisor class group of X. We begin with the following simple

LEMMA 1: $\mathrm{Cl}(X) \cong H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\mathrm{rk}(X)}$.

PROOF: The first isomorphism follows from [10, p. 145]. Next, since $H^q(X, \mathcal{O}) = 0$ for $q \geqslant 1$ (cf. [5, Lemma 14.2]), from the exact cohomology sequence belonging to the short exact exponential sequence we obtain $H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z})$. Finally, since $b_1(X) = 0$, $H^2(X, \mathbb{Z})$ is torsion-free, whence $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\mathrm{rk}(X)}$ because of $b_2(X) = \mathrm{rk}(X)$. □

Now let $\mathscr{L}$ be a line bundle on X, $\mathscr{L} = \mathcal{O}(D)$ with D a divisor on X. Let $|D| \cong \mathbb{P}(H^0(X, \mathscr{L}))$ be the corresponding linear system (which may be empty). Since G is connected, G acts trivially on $H^2(X, \mathbb{Z})$, hence on $\mathrm{Cl}(X)$. Thus, for $D^* \in |D|$, $g \in G$, $g(D^*) \in |D|$, and hence G acts on $|D| \cong \mathbb{P}(H^0(X, \mathscr{L}))$. Since G is simply-connected, this action lifts to a linear action of G on $H^0(X, \mathscr{L})$. In particular, $H^0(X, \mathscr{L})$ is a G-module. Now we state the following special case of the Borel-Weil Theorem ([4]):

THEOREM BW: *If $\mathscr{L}$ is a very ample line bundle on X, then the G-module $H^0(X, \mathscr{L})$ is irreducible.*

PROOF OF THEOREM 2: Denote by $S = \Sigma_{n \geqslant 0} S_n$ the homogeneous coordinate ring of $f(X)$, and let $\tilde{S}$ be the integral closure of S. Then we have (cf. [10, Ch. II, Ex. 5.14]): $\tilde{S} = \Sigma_{n \geqslant 0} H^0(f(X), \mathcal{O}(n))$. Here, $\mathcal{O}(1)$ denotes the twisting sheaf of Serre (cf. [10, p. 117]), i.e. considered as a line bundle on X, $\mathcal{O}(1) = f^*\mathscr{H}$, where $\mathscr{H}$ is the hyperplane section bundle on $\mathbb{P}_N$, and $\mathcal{O}(n) = \mathcal{O}(1)^n$. Then, by Theorem BW, for all $n \geqslant 0$, $H^0(f(X), \mathcal{O}(n))$ is an *irreducible* G-module.

Now let f be homogeneously normal. Then the linear action ϕ_f: $G \to \mathrm{SL}(N+1, \mathbb{C})$ induces a natural action $\tilde{\phi}_f$ of G on S. This action preserves the grading of S, i.e. $\tilde{\phi}_f(G) S_n \subset S_n$ for all $n \geqslant 0$, and, in fact, furnishes S_n with the structure of a G-submodule of $H^0(f(X), \mathcal{O}(n))$. But $H^0(f(X), \mathcal{O}(n))$ is an irreducible G-module. Hence, for a fixed $n \geqslant 0$, we have either $S_n = 0$ which is clearly impossible, or $S_n = H^0(f(X), \mathcal{O}(n))$. Thus we obtain $S = \tilde{S}$, and $f(X)$ is projectively normal.

Finally, if f is not homogeneously normal, then evidently $S_1 \subsetneqq H^0(f(X), \mathcal{O}(1))$, and hence $f(X)$ is not projectively normal. □

EXAMPLE: As a special case of Theorem 2, for $X = \mathbb{P}_1$, we obtain the following well-known facts:

(1) The d-uple embedding of $\mathbb{P}_1$ in $\mathbb{P}_d$ (this embedding is given by the monomials in two variables of degree d) is projectively normal (cf. [10, Ch. IV, Ex. 3.4]).

(2) The twisted quartic curve in $\mathbb{P}_3$ (this is given by the (non-homogeneously normal) embedding $[z_0 : z_1] \to [z_0^4 : z_0^3 z_1 : z_0 z_1^3 : z_1^4]$ of $\mathbb{P}_1$ in $\mathbb{P}_3$) is not projectively normal (cf. [10, Ch. I, Ex. 3.18]).

2. Proofs of Theorem 1 and Theorem 3

We begin with some useful remarks concerning very ample line bundles on homogeneous-rational manifolds. In this whole section, $X = G/H$ is a homogeneous-rational manifold, where, as usual, G is a connected simply-connected semisimple complex Lie group acting transitively on X and H a proper parabolic subgroup of G. We further let $r = \mathrm{rk}(X)$.

By Lemma 1, $H^1(X, \mathcal{O}^*) \cong \mathbb{Z}^r$, so it is reasonable to speak of line bundles of type $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ on X. However, since a line bundle of type $(1, \ldots, 1)$ should be "positive", this has to be rendered precise: Let B be a Borel subgroup of G, and consider the B-action on X. Then there is an open B-orbit U in X (the open "Bruhat-cell"), and the complement $X - U$ is a divisor on X consisting of r irreducible components $D_1, \ldots, D_r$ (see [15] for details in the case $X = G/B$). Now, for $(n_1, \ldots, n_r) \in \mathbb{Z}^r$, define the line bundle of type $(n_1, \ldots, n_r)$ to be the line bundle belonging to the divisor $n_1 D_1 + \ldots + n_r D_r$. Equivalently, line bundles of type $(n_1, \ldots, n_r)$ may be defined as follows: Let $R = T \cdot S$ be the reductive part of H, where $T \cong (\mathbb{C}^*)^r$ and S is semisimple. Then the groups T^* and H^* of holomorphic characters of T and H are isomorphic, $T^* \cong H^*$. Obviously, $T^* \cong \mathbb{Z}^r$, the isomorphism being given by $\mathbb{Z}^r \ni (n_1, \ldots, n_r) \to \chi_{(n_1, \ldots, n_r)} \in T^*$, where $\chi_{(n_1, \ldots, n_r)}(z_1, \ldots, z_r) := z_1^{n_1} \cdot \ldots \cdot z_r^{n_r}$. Now the line bundle of type $(n_1, \ldots, n_r)$ on X is the homogeneous line bundle given by the character $\chi_{(n_1, \ldots, n_r)} \in T^* \cong H^*$. Furthermore, line bundles of type $(n_1, \ldots, n_r)$ on X may be described in the following way: Let $P_1, \ldots, P_r$ the maximal parabolic subgroups of G which contain H, let $X_i := G/P_i$, and let $\pi_i\colon X \to X_i$ the natural fibrations. Since $\mathrm{rk}(X_i) = 1$, $H^1(X_i, \mathcal{O}^*) \cong \mathbb{Z}$. Now take positive generators $\mathcal{L}_i$ of $H^1(X_i, \mathcal{O}^*)$, $i = 1, \ldots, r$. Then the line bundle $\mathcal{L}$ of type $(n_1, \ldots, n_r)$ on X is given by $\mathcal{L} = \pi_1^*(\mathcal{L}_1^{n_1}) \otimes \ldots \otimes \pi_r^*(\mathcal{L}_r^{n_r})$.

Now, very ample line bundles on X can be easily characterised:

Remark: *A line bundle $\mathcal{L}$ of type $(n_1, \ldots, n_r)$ on X is very ample if and only if $n_i > 0$, $i = 1, \ldots, r$. In particular, a holomorphic embedding $f\colon X \to \mathbb{P}_N$ is homogeneously minimal if and only if f is given by a base $s_0, \ldots, s_N$ of $H^0(X, \mathcal{L})$, where $\mathcal{L}$ is a line bundle of type $(1, \ldots, 1)$ on X.*

Proof: The first part is contained in [4, §4], the second assertion follows from Tits' Theorem (see [22, Korollar 2.2.2]). □

We now come to the proof of Theorem 1. We employ Samuel's

Criterion of Factoriality (cf. [10, Ch. II, Ex. 6.3]): *A projective variety V is factorial if and only if (1) V is projectively normal, and (2) the divisor class group $\mathrm{Cl}(V)$ of V is isomorphic to $\mathbb{Z}$ and is generated by the class of a (suitable) hyperplane section.*

PROOF OF THEOREM 1: By Samuel's Criterion and Lemma 1, clearly $\mathrm{rk}(X)=1$ if $f(X)$ is factorial. So assume $\mathrm{rk}(X)=1$. Take a line bundle $\mathscr{L}_0$ of type (1) on X. Hence $\mathscr{L}_0$ generates $H^1(X, \mathcal{O}^*) \cong \mathbb{Z}$. By the Remark, f is given by sections $s_0, \ldots, s_N \in H^0(X, \mathscr{L})$, where $\mathscr{L}$ is a line bundle of type (n) on X, $n \geqslant 1$, i.e. $\mathscr{L} = \mathscr{L}_0^n$ with $n \geqslant 1$. Let V^* be a hyperplane in $\mathbb{P}_N$ not containing $f(X)$ and V the corresponding hyperplane section. Then we have $\mathrm{Cl}(f(X))/(V) \cong H^1(X, \mathcal{O}^*)/(\mathscr{L}) = (\mathscr{L}_0)/(\mathscr{L}) \cong \mathbb{Z}/n\mathbb{Z}$. Now, by Samuel's Criterion, $f(X)$ is factorial if and only if $n=1$ and $f(X)$ is projectively normal. Hence the assertion follows from Theorem 2 and the Remark. □

PROOF OF THEOREM 3: Let $f\colon X \to \mathbb{P}_N$ be a homogeneously minimal embedding of X, and let an affine kernel X_a of X be given by $X_a = f(X) - Z$, where Z is a general (i.e. smooth) hyperplane section of $f(X)$. We consider the exact sequence $\mathbb{Z} \xrightarrow{i} \mathrm{Cl}(X) \to \mathrm{Cl}(X_a) \to 0$, where the map i is given by $1 \to 1 \cdot Z$ (cf. [10, Ch. II, Prop. 6.5]). Since the group $\mathrm{Cl}(X)$ is torsion-free, the map i is injective. We have to determine the image $\mathrm{Im}(i)$ of i in $\mathrm{Cl}(X)$. First, $\mathrm{Cl}(X) \cong H^1(X, \mathcal{O}^*) \cong \mathbb{Z}^r$. Next, by the Remark, f is given by a base $s_0, \ldots, s_N$ of $H^0(X, \mathscr{L}_0)$, where $\mathscr{L}_0$ is a line bundle of type $(1, \ldots, 1)$ in $H^1(X, \mathcal{O}^*)$. Hence, for the divisor class group of X_a we obtain: $\mathrm{Cl}(X_a) \cong \mathrm{Cl}(X)/\mathrm{Im}(i) \cong H^1(X, \mathcal{O}^*)/(\mathscr{L}_0) \cong \mathbb{Z}^r/\langle(1, \ldots, 1)\rangle \cong \mathbb{Z}^{r-1}$, whence the assertion. In particular, if $r=1$, then $\mathrm{Cl}(X_a)=0$, and hence X_a is factorial (cf. [10, Ch. II, Prop. 6.2]).

3. Applications

In this section, we give two applications of Theorem 1 and the following special case of a theorem of Bott (cf. [5, Thm. IV′]):

THEOREM B: *If $\mathscr{L}$ is a very ample line bundle on a homogeneous-rational manifold X, then $H^q(X, \mathscr{L}) = 0$ for $q \geqslant 1$ and $H^q(X, \mathscr{L}^{-1}) = 0$ for $q < \dim X$.*

For the first application, recall that a noetherian ring A is called *Cohen-Macaulay* (*Gorenstein*, resp.) if, for every maximal ideal $\mathfrak{m}$ of A, the local ring $A_{\mathfrak{m}}$ is Cohen-Macaulay (Gorenstein, resp.), i.e. $\dim A_{\mathfrak{m}} = \operatorname{depth} A_{\mathfrak{m}}$ (the injective dimension of $A_{\mathfrak{m}}$ is finite, resp.). For generalities on Cohen-Macaulay and Gorenstein rings, see e.g. [14]. Now we can state:

COROLLARY 1: *The homogeneous coordinate ring S of a homogeneously minimally embedded rank 1-homogeneous-rational manifold X is Gorenstein.*

Proof: According to a theorem of Murthy ([17], see also [7, Thm. 12.3]), a factorial Cohen-Macaulay factor ring of a Gorenstein ring is Gorenstein. Hence, by Theorem 1, it is sufficient to show that S is Cohen-Macaulay. In fact, we have quite generally

Proposition: *If f: $X \to \mathbb{P}_N$ is a homogeneously normal embedding of a homogeneous-rational manifold X, then the homogeneous coordinate ring $S(f(X))$ of $f(X)$ is Cohen-Macaulay.*

Proof: It is well-known that the homogeneous coordinate ring $S(V)$ of a nonsingular projectively normal projective variety V is Cohen-Macaulay provided $H^q(V, \mathcal{O}(n)) = 0$ for all $n \in \mathbb{Z}$ and $1 \leqslant q \leqslant \dim V - 1$ (this is e.g. a special case of Prop. B, p. 131, and Prop. 5.1 of [12]). Applying this theorem to our situation, it follows from Theorem 2 and Theorem B that $S(f(X))$ is Cohen-Macaulay. □

For the second application, let X be a homogeneous-rational manifold, and let f: $X \to \mathbb{P}_N$ be a homogeneously minimal embedding of X. Denote by $V(X)$ the affine cone over $f(X)$ in affine $(N+1)$-space. Let P be the vertex of $V(X)$ and $R = \mathcal{O}_{V(X),P}$ the local ring of P on $V(X)$. Thus, by Theorem 2, R is a normal domain, and, unless X is isomorphic to $\mathbb{P}_n$ for some n, R is *not* regular.

Now assume additionally $\mathrm{rk}(X) = 1$. Then, by Theorem 1, $f(X)$ is factorial, and hence R is a unique factorization domain (cf. [7, Cor. 10.3]). One may ask whether the completion $\hat{R}$ of R with respect to its maximal ideal is again factorial. In general, if A is a local noetherian Krull domain and $\hat{A}$ its completion, then, by Mori's Theorem (cf. [7, Cor. 6.12]), A is factorial if $\hat{A}$ is, but the converse is false in general (see [7, Example 19.9] for a counterexample). In our case, however, we have

Corollary 2: *The ring $\hat{R}$ is a unique factorization domain.*

Proof: We use the following result of Danilov (cf. [6, Theorem in §2 and Prop. 8], see also [16, p. 532 f.]): If V is a nonsingular projectively normal projective variety, A the local ring of the vertex of the affine cone over V, and $\hat{A}$ the completion of A, then the divisor class groups of A and $\hat{A}$ are isomorphic if and only if $H^1(V, \mathcal{O}(n)) = 0$ for all $n \geqslant 1$. Applying this theorem to our situation, by Theorem B we obtain $\mathrm{Cl}(\hat{R}) = \mathrm{Cl}(R) = 0$, whence the assertion. □

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