COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 52, n° 2 (1984), p. 203-210
<http://www.numdam.org/item?id=CM_1984__52_2_203_0>

Compositio Mathematica 52 (1984) 203–210

CONVEX POLYTOPES AS MATRIX INVARIANTS

Gerard Sierksma and Klaas de Vos

Abstract

For a convex polytope P which is the convex hull of a finite number of points, the set $\pi(P)$ consists of all real square matrices A such that $AP \subset P$, i.e. that leave P invariant. In this paper the extremals of $\pi(P)$ are characterized for P being a convex simplex, and the number of its extremes is determined.

1. Introduction

In Berman and Plemmons [2] the first chapter deals with matrices that leave a cone invariant, i.e. $\pi(K) = \{A \in \mathbb{R}^{d\times d} \mid AK \subset K\}$ with K a cone in $\mathbb{R}^d$. An extensive bibliography on properties of $\pi(K)$ can be found in this book. In e.g. Tam [8] it is shown that $\pi(K)$ is a polyhedral cone if K is a polyhedral cone. One of the main problems is to characterize the extremals of such a polyhedral cone $\pi(K)$; see e.g. Adin [1]. Instead of taking a cone as matrix invariant we consider in this paper convex polytopes, with a convex polytope being the convex hull of a finite nonempty set of points in $\mathbb{R}^d$; see e.g. Eggleston [4] and Sierksma [6]. In Valentine [9] the term convex polyhedron is used. In a recent paper by Elsner [3] is also deviated from the idea of using cones; here so-called nontrivial convex sets are used as matrix invariants. In this paper we restrict ourselves mainly to convex simplices S_0 with one vertex at the origin. By a convex simplex P in $\mathbb{R}^d$ we mean the convex hull of $d+1$ points in $\mathbb{R}^d$ with nonempty interior. We shall characterize the extremes of $\pi(S_0)$ and calculate its number. In general, we define for $X \subset \mathbb{R}^d$ the set of matrices

$$\pi(X) = \{A \in \mathbb{R}^{d\times d} \mid AX \subset X\}.$$

Note that if $X = \{0\}$, then $\pi(X) = \mathbb{R}^{d\times d}$. If $X = \{x\}$ with $x \neq 0$, then $\pi(X)$ consists of all (d, d)-matrices with eigenvector x and eigenvalue 1. Before restricting ourselves to convex polytopes we give the following result for arbitrary sets. Note that if X is convex then so is $\pi(X)$. By cone X we mean the convex cone generated by X, i.e. all nonnegative linear combinations of X. The set cone X is also denoted by X^G; see [2].

THEOREM 1: *Let $X \subset \mathbb{R}^d$. Then the following holds*
(a) cone $\pi(X) \subset \pi(\text{cone } X)$;
(b) cone $\pi(X) = \pi(\text{cone } X)$ *if X is compact, convex and contains* 0.

PROOF: (a) Take any $A \in \text{cone } \pi(X)$. Then there are matrices $A_1, \ldots, A_n \in \pi(X)$ such that $A = \sum_{i=1}^n \lambda_i A_i$ with $\lambda_i \geqslant 0$ for all i. Furthermore let $x = \sum_{i=1}^k \mu_i x_i \in \text{cone } X$ with $\mu_i \geqslant 0$ and $x_i \in X$ for all i. Then it follows that $Ax = A(\sum_{i=1}^k \mu_i x_i) = \sum_{i=1}^k \mu_i A x_i = \sum_{i=1}^k \mu_i (\sum_{j=1}^n \lambda_j A_j x_i) = \sum_{i,j} \lambda_j \mu_i A_j x_i \subset \text{cone } X$. Hence, $A \subset \pi(\text{cone } X)$.
(b) Take any $A \in \pi(\text{cone } X)$ and let $X \neq \{0\}$. Hence $A(\text{cone } X) \subset \text{cone } X$. As $0 \in X$ and X convex it follows that for each $x \in X$ there are λ, $\mu > 0$ and $y \in X$ such that

$$A(\lambda x) = \mu y,$$

or that $(\lambda/\mu)Ax \in X$. Let λ^* be the infimum of all λ/μ over x. Then $\lambda^* \neq 0$, and $(\lambda^* A)X \subset X$. This means that $\lambda^* A \in \pi(X)$. As $\pi(X)$ is convex and contains 0, it follows that $A \in \text{cone } \pi(X)$.

In Theorem 1(b) we have a sufficient condition in order to obtain equality in (a). Note that we also have equality if $X = \text{cone } X$, because in that case both X and $\pi(X)$ are convex cones; see e.g. Berman and Plemmons [2]. The following example shows that equality does not hold in general in Theorem 1. Take $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geqslant 0, x_1 + x_2 = 1\}$. Then cone $X = \mathbb{R}^2_+$, and $\pi(\text{cone } X)$ consists of all nonnegative (2,2)-matrices. On the other hand $\pi(X)$ consists of all matrices

$$\begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$$

with $0 \leqslant a, b \leqslant 1$, so cone $\pi(X)$ consists of all nonnegative multiples of these matrices. Hence $\pi(\text{cone } X) \neq \text{cone } \pi(X)$.

In the following chapters we replace "cone" by "conv" and "extr", so we consider $\pi(\text{conv } X)$, conv $\pi(X)$ and $\pi(\text{extr } X)$, extr $\pi(X)$.

2. Polytopes and simplices as matrix invariants

The main purpose of this chapter is to study the commutativity of π and conv, i.e. $\pi(\text{conv } X) = \text{conv } \pi(X)$ for X a polytope. Clearly, if X is convex so is $\pi(X)$, and in that case we have conv $\pi(X) = \pi(\text{conv } X)$.

THEOREM 2: *For each X in* $\mathbb{R}^d$ *the following holds*

$$\text{conv } \pi(X) \subset \pi(\text{conv } X).$$

PROOF: Take any $X \subset \mathbb{R}^d$. Clearly $\pi(\text{conv } X)$ is convex in $\mathbb{R}^{d\times d}$. So we only have to show that $\pi(X) \subset \pi(\text{conv } X)$. Take any $A \in \pi(X)$. Then $AX \subset X$. Now let $x \in A(\text{conv } X)$. Then there are $x_1,\dots,x_s \in X$ and $\lambda_1,\dots,\lambda_s \geqslant 0$ with $\lambda_1 + \dots + \lambda_s = 1$, such that $x = A(\sum_{i=1}^s \lambda_i x_i) = \sum_{i=1}^s \lambda_i A x_i$. As $Ax_i \in X$ for each i we have $x \in \text{conv } X$, and it follows that $A(\text{conv } X) \subset \text{conv } X$. Hence, $A \in \pi(\text{conv } X)$.

Equality does not hold in general in the above theorem as is shown by the following example. Take $X = \{(1,0), (0,1), (\frac{1}{2},0), (1,1), (0,0)\}$. Then

$$\pi(X) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then $\text{conv } X = \{(x_1, x_2) \mid 0 \leqslant x_1, x_2 \leqslant 1\}$, and $\pi(\text{conv } X)$ consists of all non-negative matrices with row sums $\leqslant 1$. On the other hand the (2,1)-th element of each matrix in $\text{conv } \pi(X)$ is zero. Equality also does not hold, in general, in case X consists of the extremals of a convex cone X, i.e. $X = \text{extr } K$.

It is well-known that $\text{conv } X = \text{conv}(\text{extr } K) = K$ (the Krein-Milman theorem). However, in general, $\pi(\text{conv } X) = \pi(K) \neq \text{conv } \pi(\text{extr } K)$. In order for $\pi(K)$ being equal to $\text{conv } \pi(\text{extr } K)$ it is therefore necessary that $\text{extr } \pi(K) \subset \pi(\text{extr } K)$ which is a conjecture of Loewy and Schneider [5].

In the following we shall write $0 \cup S$ instead of $\{0\} \cup S$. In the next theorem $S_0 = \text{conv}(0 \cup S)$ is a convex simplex i.e. a convex simplex with one vertex at the origin and $|S| = d$. The theorem shows the commutativity of π and conv for $0 \cup S$. The proof of it is after Theorem 6.

THEOREM 3: *Let $0 \cup S$ be the vertices of a convex simplex in $\mathbb{R}^d$. Then the following holds*

$$\pi(\text{conv}(0 \cup S)) = \text{conv } \pi(0 \cup S).$$

In order to prove this theorem we need a nonsingular transformation $T: \mathbb{R}^d \to \mathbb{R}^d$ such that

$$T(e_1,\dots,e_d) = S$$

with S as in the above theorem. We denote $E_d = \{e_1,\dots,e_d\}$, i.e. the set of unit vectors in $\mathbb{R}^d$.

THEOREM 4: *The following assertions are equivalent*:

(i) $A \in \pi(\text{conv}(0 \cup E_d))$;

(ii) $Ae_1,\dots,Ae_d \in \text{conv}(0 \cup E_d)$;

(iii) $A \geqslant 0$, column sums of $A \leqslant 1$;

PROOF: We shall show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).
(i) $\Rightarrow$ (ii): Let $A \in \pi(\text{conv}(0 \cup E_d))$, so $A(\text{conv}(0 \cup E_d)) \subset \text{conv}(0 \cup E_d)$. As $e_i \in \text{conv}(0 \cup E_d)$, it follows that $Ae_i \in \text{conv}(0 \cup E_d)$ for each $i = 1,\dots,d$.
(ii) $\Rightarrow$ (iii): As $Ae_j \in \text{conv}(0 \cup E_d)$ for each j, it follows that the j-th column of A is equal to $\sum_{i=1}^d \lambda_{ij} e_i$ with $\lambda_{1j},\dots,\lambda_{dj} \geqslant 0$ and $\lambda_{1j} + \dots + \lambda_{dj} \leqslant 1$, so the j-th column sum of A is equal to $\sum_{i=1}^d \lambda_{ij}$ and is $\leqslant 1$. The matrix A is nonnegative, because all λ_{ij}'s are nonnegative.
(iii) $\Rightarrow$ (i): Take any $x \in \text{conv}(0 \cup E_d)$. Then there are scalars $\alpha_1,\dots,\alpha_d \geqslant 0$ with $\alpha_1 + \dots + \alpha_d \leqslant 1$ such that $x = \sum_{i=1}^d \alpha_i e_i$. Hence, $Ax = A(\sum_{i=1}^d \alpha_i e_i) = \sum_{i=1}^d \alpha_i Ae_i$. As the column sums of A are $\leqslant 1$, it follows directly that $Ae_i \in \text{conv}(0 \in E_d)$ for each $i = 1,\dots,d$, and therefore we have $Ax \in \text{conv}(0 \cup E_d)$, and hence $A \in \pi(\text{conv}(0 \cup E_d)$.

Theorem 4 implies that all matrices in $\pi(0 \cup E_d)$ have *Perron-Frobenius eigen-value* $\leqslant 1$; this is the well-known Minkowski-theorem, see e.g. Sierksma [7].

LEMMA 5: *Let* $T: \mathbb{R}^d \to \mathbb{R}^d$ *be a nonsingular transformation and let* $X \subset \mathbb{R}^d$. *Then the following assertions hold*:

(a) $\text{conv}(TX) = T(\text{conv } X)$,
(b) $\pi(TX) = T\pi(X)T^{-1}$;
(c) $\text{conv } \pi(TX) = T[\text{conv } \pi(X)]T^{-1}$.

PROOF: (a) is left to the reader. In order to prove (b), take any $A \in \pi(TX)$. Hence, $A(TX) \subset TX$, and this implies that $(T^{-1}AT)(X) \subset X$, so that $T^{-1}AT \in \pi(X)$, or $A \in T[\pi(X)]T^{-1}$. The converse inclusion is shown similarly. To prove (c), take any $A \in \text{conv } \pi(TX) = \text{conv}[T\pi(X)T^{-1}]$. Then there are matrices $B_1,\dots,B_k \in \pi(X)$, and scalars $\lambda_1,\dots,\lambda_k \geqslant 0$ with $\lambda_1 + \dots + \lambda_k = 1$ such that $A = \sum_{i=1}^k \lambda_i (TB_i T^{-1}) = T(\sum_{i=1}^k \lambda_i B_i)T^{-1}$. As $\sum_{i=1}^k \lambda_i B_i \in \text{conv } \pi(X)$, it follows that $A \in T[\text{conv } \pi(X)]T^{-1}$. The other conclusion is also shown similarly.

The following theorem gives the commutativity of π and conv for $0 \cup E_d$.

THEOREM 6: $\pi(\text{conv}(0 \cup E_d)) = \text{conv } \pi(0 \cup E_d)$.

PROOF: According to Theorem 2, we only have to show that $\pi(\text{conv}(0 \cup E_d)) \subset \text{conv } \pi(0 \cup E_d)$. Take any $A \in \pi(\text{conv}(0 \cup E_d))$. Then $A(\text{conv}(0 \cup E_d)) \subset \text{conv}(0 \cup E_d)$. Theorem 4 then gives that $A \geqslant 0$ and that all column sums of A are $\leqslant 1$. We must show now that A can be written as a convex combination of matrices from $\pi(0 \cup E_d)$. To show this, we first define $A = A_1 = \{a_{ij}^{(1)}\}$. Moreover, we define

$$\lambda_1 = \min_j \max_i a_{ij}^{(1)}.$$

and I_1 is the matrix with precisely one 1 in the j-th column in the (i, j)-th position if a_{ij} is the maximum in the j-th column of A_1 (if there are more maxima in the j-th column choose one!) and zeroes otherwise ($j = 1, \ldots, d$). Then consider the matrix

$$A_2 = A_1 - \lambda_1 I_1,$$

with $A_2 = \{a_{ij}^{(2)}\}$ and define

$$\lambda_2 = \min \max a_{ij}^{(2)}.$$

Also define $A_3 = A_2 - \lambda_2 I_2 = A_1 - \lambda_1 I_1 - \lambda_2 I_2$, where I_2 is defined for the matrix A_2 in the same way as I_1 for A_1. Continuing this process we obtain, after at most d^2 steps, the zero-matrix. So we obtain

$$0 = A_{d^2+1} = A - \lambda_1 I_1 - \lambda_2 I_2 - \ldots - \lambda_{d^2} I_{d^2}.$$

Hence, $A = \sum_{i=1}^{d^2} \lambda_i I_i$. Clearly, $A \geqslant 0$, and each $\lambda_i \geqslant 0$. Note that a column of I_i becomes zero if the corresponding column of A_i is zero. If after $d^2 - 1$ steps there still is some nonzero element in A_{d^2-1} we have, say in the j-th column,

$$a_{ij} + \ldots + a_{dj} - (\lambda_1 + \ldots + \lambda_{d^2}) = 0,$$

or $\sum_{i=1}^{d^2} \lambda_i = \sum_{i=1}^{d} a_{ij} \leqslant 1$. And this means that in fact $A \in \operatorname{conv} \pi(0 \cup E_d)$.

The number of steps in the proof of the above theorem is, in general $\leqslant d^2$. Question: under what conditions is the number of steps equal to d^2?

Proof of Theorem 3: First note that $0 \cup S$ is the set of vertices of a simplex, so $|S| = d$. We must show that

$$\pi(\operatorname{conv}(0 \cup S)) \subset \operatorname{conv} \pi(0 \cup S).$$

Clearly, there is a nonsingular transformation $T: \mathbb{R}^d \to \mathbb{R}^d$ such that $S = TE_d = T(e_1, \ldots, e_d)$. According to Lemma 5 and Theorem 6 we have $\pi(\operatorname{conv}(0 \cup S)) = \pi(\operatorname{conv}(0 \cup TE_d)) = \pi(\operatorname{conv} T(0 \cup E_d)) = \pi(T\operatorname{conv}(0 \cup E_d)) = T(\pi(\operatorname{conv}(0 \cup E_d)))T^{-1} = T(\operatorname{conv} \pi(0 \cup E_d))T^{-1} = \operatorname{conv} \pi(T(0 \cup E_d)) = \operatorname{conv} \pi(0 \cup TE_d) = \operatorname{conv} \pi(0 \cup S)$.

3. The extremes of $\pi(S_0)$

In this chapter we shall characterize the extreme vertices of $\pi(S_0)$ and determine their number. If P is a convex polytope then, in general, $\pi(P)$ is not a polytope. For instance if we take the two points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

then $P = \text{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ is convex but $\pi(P)$ is not: in $\pi(P)$ there are matrices $A = (a_{ij})$ with $a_{11} = -a_{12}$, so a_{11} can be as large as possible which means that $\pi(P)$ is not bounded, so certaintly is not a convex polytope. The following theorem gives a sufficient condition in order to save the boundedness of $\pi(P)$.

THEOREM 7: *If X is a bounded set in $\mathbb{R}^d$ with* $\text{intconv}(0 \cup X) \neq \emptyset$ *then $\pi(X)$ is bounded.*

PROOF: Suppose, to the contrary, that $\pi(X)$ is not bounded. Then there is a sequence of matrices A_k in $\pi(X)$ such that one of the elements, say the (i,j)-th element, of the A_k's goes to infinity. As the interior of $\text{conv}(0 \cup X)$ is nonempty, there is an element $y \in \text{intconv}(0 \cup X)$ with $y_i \neq 0$. Then $y = \sum_{i=1}^{s} \lambda_i x_i$ with $x_i \in X$, $\lambda_i \geqslant 0$, and $\lambda_1 + \ldots + \lambda_s \leqslant 1$. Then $(A_k y)_j \to \infty$ for $k \to \infty$, As $A_k y = \sum_{i=1}^{s} \lambda_i A_k x_i$ is a finite sum, we have $A_k x_i \to \infty$ for $k \to \infty$ and for some i. As $A_k x_i \in X$, it follows that X is not bounded which is a contradiction. Therefore we have in fact that $\pi(X)$ is bounded.

It is open question whether $\pi(X)$ is a convex polytope in case X is a convex polytope in $\mathbb{R}^d$ with $\text{intconv}(0 \cup X) \neq \emptyset$. Question: Is the number of extreme vertices of $\pi(X)$ less then or equal to $(d+1)^d$ (see the following theorem)?

THEOREM 8: *Let $0 \cup S$ be the vertices of a convex simplex. Then the following holds*:

$$|\pi(0 \cup S)| = |\pi(0 \cup E_d)| = (d+1)^d.$$

PROOF: There is a nonsingular transformation $T: \mathbb{R}^d \to \mathbb{R}^d$ such that $0 \cup S = T(0 \cup E_d)$, so we have directly that $|\pi(0 \cup S)| = |\pi(0 \cup E_d)|$. We only have to show that

$$|\pi(0 \cup E_d)| = (d+1)^d.$$

First note that $A0 = 0$ for each $A \in \pi(0 \cup E_d)$. The problem of determining the number of matrices in $\pi(0 \cup E_d)$ is therefore equivalent to the problem of finding the number of bipartite graphs (G, H) on $2(d+1)$ vertices with $|G| = |H| = d+1$, with one edge fixed, and such that the degree of each vertex in G is 1. Let there be a fixed edge between $a \in G$ and $b \in H$. Then there are for each vertex $\neq a$ in G precisely $d+1$ possibilities in H. This holds for all of the vertices in G that are $\neq a$. So the number of such bipartite graphs is equal to

$$\underbrace{(d+1) \times \ldots \times (d+1)}_{d \text{ times}}.$$

Hence, $|\pi(0 \cup E_d)| = (d+1)^d$.

To illustrate the above theorem we consider the following example. Let $d=2$ and $S_0 = \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \operatorname{conv}(0 \cup E_2)$. Then we have

$$\pi(0 \cup S) = \left\{\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\right\}, \text{ and}$$

$$\pi(S_0) = \left\{\begin{pmatrix}a & b\\c & d\end{pmatrix} \mid a, b, c, d \geqslant 0, a+c \leqslant 1, b+d \leqslant 1\right\}.$$

Note that $|\pi(0 \cup S)| = 3^2 = 9$. In order to characterize the extreme vertices of S_0 we need the following two lemmas.

Lemma 9: *Let $X \subset \mathbb{R}^{d \times d}$ be convex and $T: \mathbb{R}^d \to \mathbb{R}^d$ be nonsingular. Then the following holds*:

$$\operatorname{extr}(TXT^{-1}) = T(\operatorname{extr} X)T^{-1}$$

Proof: Take any $A \in \operatorname{extr}(TXT^{-1})$. Hence, $A \notin \operatorname{conv}(TXT^{-1})\backslash\{A\}$. Suppose, to the contrary, that $A \notin T(\operatorname{extr} X)T^{-1}$. We shall show first that $T^{-1}AT \in \operatorname{extr} X$, or that $T^{-1}AT \notin \operatorname{conv} X\backslash\{T^{-1}AT\}$. Taking $T^{-1}AT \in \operatorname{conv} X\backslash\{T^{-1}AT\}$, there should exist matrices $B_1, \ldots, B_k \in X$, all $\neq T^{-1}AT$, and scalars $\lambda_1, \ldots, \lambda_k \geqslant 0$ with $\lambda_1 + \ldots + \lambda_k = 1$, such that $T^{-1}AT = \sum_{i=1}^k \lambda_i B_i$, or $A = \sum_{i=1}^k \lambda_i TB_iT^{-1}$, and $B_i \neq T^{-1}AT$ for all i. Because $TB_iT^{-1} \in \operatorname{conv}(TXT^{-1})$ for all i, we have $A \in \operatorname{conv}(TXT^{-1})\backslash \{A\}$, hence $A \notin \operatorname{extr}(TAT^{-1})$, and this is a contradiction. Therefore we have, $T^{-1}AT \in \operatorname{extr} X$, and this means that $A \in T(\operatorname{extr} X)T^{-1}$. The converse can be shown similarly.

Lemma 10: extrconv $\pi(0 \cup E_d) = \pi(0 \cup E_d)$.

Proof: As all columns of the matrices in $\pi(0 \cup E_d)$ consists of zero or unit vectors, no such a matrix can be written as a convex combination of the other ones in $\pi(0 \cup E_d)$.

The next theorem characterizes the extreme vertices of $\pi(S_0) = \pi(\operatorname{conv}(0 \cup S))$; they are precisely the matrices that leave the vertices invariant.

Theorem 11: extr $\pi(S_0) = \operatorname{extr} \pi(\operatorname{conv}(0 \cup S)) = \pi(0 \cup S)$.

Proof: Let $T: \mathbb{R}^d \to \mathbb{R}^d$ be a nonsingular transformation such that

$0 \cup S = T(0 \cup E_d)$. Then we have

$$\begin{aligned}\operatorname{extr} \pi(S_0) &= \operatorname{extr} \pi(\operatorname{conv}(0 \cup S)) = \operatorname{extr} \pi(\operatorname{conv}(T(0 \cup E_d)))\\ &= \operatorname{extrconv} \pi(T(0 \cup E_d)) = \operatorname{extrconv}\left[T\pi(0 \cup E_d)T^{-1}\right]\\ &= \operatorname{extr}\left(T\left[\operatorname{conv} \pi(0 \cup E_d)\right]T^{-1}\right)\\ &= T\left[\operatorname{extrconv} \pi(0 \cup E_d)\right]T^{-1} = T\left[\pi(0 \cup E_d)\right]T^{-1}\\ &= \pi(T(0 \cup E_d)) = \pi(0 \cup S).\end{aligned}$$

Acknowledgement

I would like to thank Evert Jan Bakker for his helpful suggestions.

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(Oblatum 24-VIII-1982 & 10-II-1983

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