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BASE CURVES OF MULTICANONICAL SYSTEMS ON THREEFOLDS

P.M.H. Wilson

Introduction

Let V be a smooth complex projective threefold of general type and K_V the canonical divisor on V. In this paper we consider in detail the case when there exists an integer $m > 0$ such that the m-canonical system $|mK_V|$ has no fixed components, and the corresponding rational map ϕ_{mK_V} is generically finite; in particular we study the base locus of this linear system.

Using the theory of Hilbert schemes, it is easy to see that in this case K_V is arithmetically effective (denoted a.e.); i.e. $K_V \cdot C \geqslant 0$ for all curves C on V. In [12] (Theorem 6.2) we saw that in the case when K_V is a.e., the canonical ring $R(V) = \oplus_{n \geqslant 0} H^0(V, nK_V)$ is finitely generated as an algebra over the complex numbers if and only if the linear system $|nK_V|$ has no fixed points for some $n > 0$.

It is easy to see that if $K_V \cdot C > 0$ for every base curve C of $|mK_V|$, then such an n does exist (1.2). Thus the interesting base curves C are those with $K_V \cdot C = 0$. After this paper was written, the author was informed that Kawamata has now proved that such an n exists in the case when K_V is a.e. Thus we see that the curves C with $K_V \cdot C = 0$ are precisely those curves that are contracted down on the canonical model (see [9]). We shall however not assume Kawamata's result in the proofs of this paper.

In Section 2 we therefore study the base curves C of $|mK_V|$ with $K_V \cdot C = 0$. We shall see that in this case C must be isomorphic to $\mathbb{P}^1$ (2.3) and that (2.6) its normal bundle $N_{C/V}$ must be one of $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$, $\mathcal{O}_C(-2) \oplus \mathcal{O}_C$ or $\mathcal{O}_C(-3) \oplus \mathcal{O}_C(1)$.

The case of base curves C with $K_V \cdot C = 0$ is closely connected with the problem of curves homologous to zero on an analytic threefold. Analogous results to (2.3) and (2.6) have been obtained by Pinkham in this case, using a different method [7].

In the cases $N_{C/V} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ or $\mathcal{O}_C(-2) \oplus \mathcal{O}_C$ above, we essentially have all the information we want about C. In the remaining case, we need to know more about the infinitesimal neighbourhoods of C.

Blowing C up, say f_1: $V_1 \to V$ with exceptional surface E_1 (isomorphic to the ruled surface $\mathbb{F}_4$), we let C_1 denote the minimal section of E_1 (hence on E_1 we have $C_1^2 = -4$). Now blow C_1 up, obtaining f_2, V_2, E_2 and C_2. In this way we obtain a sequence of normal bundles $N_{C/V}$, N_{C_1/V_1}, $N_{C_2/V_2}, \dots$. We discover that the sequence of normal bundles so obtained must be $(-3, 1)$, $(-3, 0), \dots, (-3, 0)$, $(-2, -1)$, $(-1, -1)$ (with the obvious notation and where there are a finite number, say $t \geqslant 0$, of $(-3, 0)$'s in the sequence).

As an illustration, we apply these results (in Section 3) to the case of a base curve C with $K_V \cdot C = 0$ that is isolated in the base locus of $|mK_V|$. We discover here that C is not then a base curve of $|(2m+1)K_V|$. As a corollary, we can deduce for instance that the canonical ring is finitely generated in the case when for any two base curves C, C' with $K_V \cdot C = 0 = K_V \cdot C'$, C and C' do not meet.

Let Z_0 denote the cycle of base curves C of $|mK_V|$ with $K_V \cdot C = 0$. In Section 2 we obtained detailed information concerning the individual curves of Z_0; we now consider the question of the possible configurations of curves in Z_0.

This we consider in Section 4. We show that any two curves of Z_0 meet in at most one point (where they meet transversely), and that any given point of V is contained in at most three curves of Z_0 (which meet normally there). Moreover we find that there are no "closed cycles" of curves in Z_0 (apart from three curves meeting at a point).

We then take into account the possible normal bundles (found in Section 2). We show that a curve C in Z_0 with normal bundle $(-3, 1)$ cannot meet another such curve, and also cannot meet a triple point. Similarly, we find that a curve C in Z_0 with normal bundle $(-2, 0)$ meets at most one $(-3, 1)$-curve, meets at most one triple point, and cannot meet both. Finally we consider the case of a curve in Z_0 with normal bundle $(-1, -1)$.

The results and methods of this paper are also closely connected with a conjecture of Reid. He conjectures that if we have a finite collection of curves $Z_0 = \bigcup C_i$ on a smooth projective threefold V with $K_V \cdot C_i = 0$ for all i and $H^1(\mathcal{O}_Z) = 0$ for all schemes Z supported on Z_0, and such that Z_0 is isolated among cycles of this type, then Z_0 is contractible (by a morphism not contracting anything else). An easy modification of the methods of Section 2 shows that in this case also, any C_i (trivially isomorphic to $\mathbb{P}^1$ here) has one of the above three normal bundles, and that on blowing up, the sequence of normal bundles obtained is as described above. The methods of sections 3 and 4 are therefore relevant in this case also.

The author would like to thank Henry Pinkham for the benefit of his valuable comments on both the content and presentation of this paper.

1. Preliminaries

Throughout this paper V will denote a smooth complex projective threefold of general type, and m a positive integer such that $|mK_V|$ has no fixed components and the rational map ϕ_{mK_V} is generically finite.

PROPOSITION 1.1: *K_V is arithmetically effective.*

PROOF: If there exists a curve C on V with $K_V \cdot C < 0$, then by the theory of Hilbert schemes (see for instance [5] Section 1), we deduce that C moves in an algebraic family. This then would yield a fixed component of $|mK_V|$ (one can of course say far more; see [6]). □

PROPOSITION 1.2: *If $K_V \cdot C > 0$ for all base curves C of $|mK_V|$, then for some n the linear system $|nK_V|$ is without fixed points. If C is a base curve with $K_V \cdot C = 0$, then C has arithmetic genus $p_a(C) \leqslant 1$.*

PROOF: Let $D, D' \in |mK_V|$ be general elements, and $Z = D'|D$ denote the scheme theoretic intersection. For $n \geqslant 1$, we have the exact sequence of sheaves

$$0 \to \mathcal{O}_D(nD) \to \mathcal{O}_D((n+1)D) \to \mathcal{O}_Z((n+1)D) \to 0.$$

From the Kawamata-Viehweg form of Kodaira vanishing ([4] or [11]), we see that $h^i(V, nD) = 0$ for $i > 0$ and $n > 0$. Thus from the exact sequence

$$0 \to \mathcal{O}_V(nD) \to \mathcal{O}_V((n+1)D) \to \mathcal{O}_D((n+1)D) \to 0$$

we deduce that $h^i(D, \mathcal{O}_D(nD)) = 0$ for $i > 0$ and $n > 2$. Thus from the former exact sequence, we see that $h^1(Z, \mathcal{O}_Z(nD)) = 0$ for $n > 3$.

Suppose first that C is a base curve with $K_V \cdot C = 0$. We have an epimorphism $\mathcal{O}_Z(nD) \to \mathcal{O}_C(nD)$ of sheaves supported on Z, and hence we deduce that $h^1(C, \mathcal{O}_C(nD)) = 0$ for $n > 3$. Since $D \cdot C = 0$, we see that $\chi(C, \mathcal{O}_C) = \chi(C, \mathcal{O}_C(nD)) \geqslant 0$. Thus $p_a(C) \leqslant 1$ as required.

Suppose therefore that $K_V \cdot C > 0$ for all base curves C of $|mK_V|$. We see then that for all the curves C underlying Z, we have $D \cdot C > 0$. Hence D is ample on Z (Propositions 4.2 and 4.3 of [2]). Thus for n sufficiently large, $\mathcal{O}_Z(nD)$ is generated by its global sections.

From the first exact sequence and the fact that $h^1(\mathcal{O}_D(nD)) = 0$ for large n, we deduce that $\mathcal{O}_D(nD)$ is also generated by its global sections for n sufficiently large. Hence, using the second exact sequence, we see that the same is also true of the sheaf $\mathcal{O}_V(nD)$. Thus the linear system $|nD|$ is fixed point free for large n. □

We see therefore that it is vital to consider further those base curves C with $K_V \cdot C = 0$. This we do in detail in the next section.

2. Base curves with $K_V \cdot C = 0$

Suppose that C is a base curve of $|mK_V|$ with $K_V \cdot C = 0$, and let f_1: $V_1 \to V$ denote the blow up of C. Note here that in the case when C is singular, it has at worst a node or a cusp (by (1.2)), and thus V_1 has only one singular point (cf. [8]), which in the terminology of [9] is a compound Du Val point (in particular it is Gorenstein and rational). Let E_1 denote the exceptional divisor on V_1; note that E_1 is a Cartier divisor. Now on V_1 we have $K_{V_1} = f_1^* K_V + E_1$.

PROPOSITION 2.1:

$$\chi(V, (sm+1)K_V) - \chi(V_1, K_{V_1} + s(mf_1^* K_V - E_1))$$

$$= \tfrac{1}{12} s(s-1)(2s-1) E_1^3 \textit{ for any integer } s \geq 1.$$

PROOF: First suppose that C is smooth. By Riemann-Roch,

$$\chi(V, (sm+1)K_V)$$

$$= \tfrac{1}{12} sm(sm+1)(2sm+1) K_V^3 - (1+2sm)\chi(\mathcal{O}_V)$$

$$\chi(V_1, K_{V_1} + s(mf_1^* K_V - E_1))$$

$$= \tfrac{1}{12}(f_1^*(sm+1)K_V - (s-1)E_1)(smf_1^* K_V - sE_1)$$

$$\times (f_1^*(2sm+1)K_V - (2s-1)E_1)$$

$$-(1+2sm)\chi(\mathcal{O}_{V_1}) - \tfrac{1}{6} sE_1 \cdot c_2(V_1).$$

Now note that $\chi(2K_V) = \chi(K_{V_1} + f_1^* K_V)$ (using the Kawamata-Viehweg form of Kodaira vanishing and the birational invariance of plurigenera). But by Riemann-Roch

$$\chi(2K_V) = \tfrac{1}{2} K_V^3 + \tfrac{1}{12} K_V \cdot c_2(V) - \chi(\mathcal{O}_V) \text{ and}$$

$$\chi(K_{V_1} + f_1^* K_V) = \tfrac{1}{2} K_V^3 + \tfrac{1}{12} f_1^* K_V \cdot c_2(V_1) - \chi(\mathcal{O}_{V_1}).$$

Thus $f_1^* K_V \cdot c_2(V_1) = K_V \cdot c_2(V)$. In particular

$$-24\chi(\mathcal{O}_V) = K_V \cdot c_2(V) = (K_{V_1} - E_1) \cdot c_2(V_1)$$

$$= -24\chi(\mathcal{O}_{V_1}) - E_1 \cdot c_2(V_1).$$

Since the arithmetic genus is birationally invariant, we deduce that $E_1 \cdot c_2(V_1) = 0$. (The fact that $E_1 \cdot c_2(V_1) = -K_V \cdot C$ for the blow up of a smooth curve C holds in general, and is a standard computation with Chern classes.)

Combining the above formulae now yields the result.

The case when C is singular is essentially the same: consider a desingularization $h\colon \tilde{V}_1 \to V_1$ of V_1, and apply Riemann-Roch on $\tilde{V}_1$. Since the singularity on V_1 is rational, we know that $R^i h_* \mathcal{O}_{\tilde{V}_1} = 0$ for all $i > 0$, and that $h_* \mathcal{O}_{\tilde{V}_1} = \mathcal{O}_{V_1}$.

Thus, considering $\chi(\tilde{V}_1, h^*(K_{V_1} + f_1^* K_V))$ on $\tilde{V}_1$, we deduce that $h^* E_1 \cdot c_2(\tilde{V}_1) = 0$, using essentially the same argument as above together with the Leray spectral sequence. We then deduce (again similarly to the case when C is smooth) that

$$\chi(V, (sm+1)K_V) - \chi\big(\tilde{V}_1, K_{\tilde{V}_1} + sh^*(mf_1^*K_V - E_1)\big)$$
$$= \tfrac{1}{12} s(s-1)(2s-1) E_1^3.$$

However, using the fact that $R^i h_* \mathcal{O}_{\tilde{V}_1} = 0$ for $i > 0$, and hence that the Leray spectral sequence degenerates, we also have that

$$\chi\big(\tilde{V}_1, K_{\tilde{V}_1} + h^* s(mf_1^*K_V - E_1)\big)$$
$$= -\chi\big(\tilde{V}_1, -h^* s(mf_1^*K_V - E_1)\big)$$
$$= -\chi\big(V_1, -s(mf_1^*K_V - E_1)\big) = \chi\big(V_1, K_{V_1} + s(mf_1^*K_V - E_1)\big).$$

Thus the result is now proved in general. □

LEMMA 2.2: *Suppose that X is a Gorenstein threefold with only finitely many singularities, and Δ is an effective Cartier divisor on X such that $\Delta^3 > 0$ and $\Delta \cdot B \geqslant 0$ for all but finitely many curves B on X. Then $h^i(X, K_X + \Delta) = 0$ for $i > 1$.*

PROOF: Choose a smooth very ample divisor H on X not containing any of the finite number of curves or any of the finite number of singularities, such that $\Delta + H$ is ample and $\Delta^2 \cdot H > 0$. In particular, $h^1(H, K_H + \Delta | H) = 0$. Since by Kodaira vanishing $H^2(V, K_V + \Delta + H) = 0$, the result follows using the exact sequence

$$0 \to \mathcal{O}_X(K_X + \Delta) \to \mathcal{O}_X(K_X + \Delta + H) \to \mathcal{O}_H(K_H + \Delta) \to 0. \qquad \square$$

By assumption C is fixed in $|mK_V|$. Thus for some $r > 0$, $|f_1^* mK_V| = |D_1| + rE_1$, where $|D_1|$ is the mobile part of the linear system. We say that the base curve C has multiplicity r in $|mK_V|$. Clearly $\Delta = D_1$ satisfies the conditions of (2.2), and therefore so too does $r(mf_1^*K_V - E_1) \sim (r-1)mf_1^*K_V + D_1$, and hence also $(mf_1^*K_V - E_1)$. (Throughout this paper, $\sim$ denotes linear equivalence.)

PROPOSITION 2.3: *If $K_V \cdot C = 0$ for a base curve C of $|mK_V|$, then C is isomorphic to $\mathbb{P}^1$.*

PROOF: From (1.2) we know that $p_a(C) \leqslant 1$; let us assume that $p_a(C) = 1$ and obtain a contradiction. Then, even if C is singular, we know that it is regularly immersed in V, and that it is Gorenstein with dualizing line bundle $\omega_C \simeq \mathcal{O}_C$. Thus, if $\mathcal{I}_C$ denotes the sheaf of ideals defining C, we have that $\mathcal{I}_C/\mathcal{I}_C^2$ on C is locally free of rank 2. Using the generalized adjunction formula ([1], Chapter 1, Theorem 4.5) we know that the line bundle $\Lambda^2(\mathcal{I}_C/\mathcal{I}_C^2)$ on C has degree 0.

With the notation above, we have $V_1 = \mathrm{Proj}_V(\mathcal{O}_V \oplus \mathcal{I}_C \oplus \mathcal{I}_C^2 \oplus \ldots)$, and that $\mathcal{O}_{V_1}(-E_1) \simeq \mathcal{O}_{V_1}(1)$ where the right hand side denotes the natural relatively ample bundle defined by $\mathcal{I}_C$ (see [3], Chapter II, Proposition 7.13). Let Y denote the Cartier divisor on E_1 corresponding to the line bundle $\mathcal{O}_{E_1}(1)$ (which can be interpreted as either the restriction of $\mathcal{O}_{V_1}(1)$ to E_1, or else the natural relatively ample bundle on $E_1 \simeq \mathbb{P}_C(\mathcal{I}_C/\mathcal{I}_C^2)$). Thus $E_1^3 = -Y^2$.

I claim that for any rank 2 bundle $\mathcal{F}$ on C with $E = \mathbb{P}_C(\mathcal{F})$, and Y the natural relatively ample divisor, we have $Y^2 = \deg(\Lambda^2\mathcal{F})$. We can assume without loss of generality that Y is effective (by [3], Chapter II, Lemm 7.9 and Proposition 7.10). The claim now follows by [2], Chapter 1, §10 in the case when C is smooth, and the same proof works also in the case when C is singular.

Thus we have deduced that $E_1^3 = 0$ on V_1. Note however that for sufficiently large s,

$$h^0\big(V_1, K_{V_1} + s(mf_1^*K_V - E_1)\big) < h^0\big(V_1, (sm+1)f_1^*K_V\big)$$

$$= h^0\big(V, (sm+1)K_V\big)$$

since by Theorem 2.2 of [12], the multiplicity of E_1 in $|nf_1^*K_V|$ is bounded as $n \to \infty$.

Since however we have that $h^i(V, (sm+1)K_V) = 0$ for $i > 0$, and by (2.2), that $h^i(V_1, K_{V_1} + s(mf_1^*K_V - E_1)) = 0$ for $i > 1$, we obtain an immediate contradiction from (2.1).

Thus $p_a(C) = 0$ as required. □

REMARK 2.4: The results (2.1) to (2.3) provide the prototype for a number of the proofs which follow. I shall therefore explain at this stage the ideas behind these proofs.

Let Z_0 denote the cycle of those base curves C of $|mK_V|$ such that $K_V \cdot C = 0$. Suppose now that ϕ: $\tilde{V} \to V$ is any birational morphism (with $\tilde{V}$ smooth) whose exceptional locus lies over Z_0, and let the effective divisor $\mathscr{E}$ on $\tilde{V}$ be defined by $K_{\tilde{V}} = \phi^*K_V + \mathscr{E}$. The proof of (2.1) goes over unchanged to show that

$$\chi\big(V, (sm+1)K_V\big) - \chi\big(\tilde{V}, K_{\tilde{V}} + s(\phi^*mK_V - \mathscr{E})\big)$$
$$= \tfrac{1}{12}s(s-1)(2s-1)\big(K_{\tilde{V}}^3 - K_V^3\big).$$

We note also in passing that if B is a smooth rational curve on a threefold W, h: $\tilde{W} \to W$ the blow up of B, then

$$K_{\tilde{W}}^3 = K_W^3 - 2\big(K_W \cdot B - 1\big).$$

PROPOSITION 2.5: *With the notation above, suppose that $\mathscr{E}$ is fixed in $|m\phi^*K_V|$ (e.g. the case when ϕ is the composite of blow ups of base curves). Suppose also that $K_{\tilde{V}} \cdot B \leqslant 0$ for all but finitely many curves B contained in $\mathscr{E}$ and that $K_{\tilde{V}}^3 \leqslant K_V^3$. Then we have a contradiction.*

PROOF: Since $\mathscr{E}$ is fixed in $|m\phi^*K_V|$, $(m\phi^*K_V - \mathscr{E}) \cdot B \geqslant 0$ for all but finitely many curves B not contained in $\mathscr{E}$. However since $(m\phi^*K_V - \mathscr{E})$ is numerically equivalent to $-K_{\tilde{V}}$ on $\mathscr{E}$, the second assumption now implies that $(m\phi^*K_V - \mathscr{E}) \cdot B \geqslant 0$ for all but finitely many curves B on $\tilde{V}$. Now since $\mathscr{E}^3 = K_{\tilde{V}}^3 - K_V^3 \leqslant 0$, we have that $(m\phi^*K_V - \mathscr{E})^3 = m^3K_V^3 - \mathscr{E}^3 > 0$. Hence by (2.2), $h^i(\tilde{V}, K_{\tilde{V}} + s(m\phi^*K_V - \mathscr{E})) = 0$ for $i > 1$.

Moreover, as in (2.3) we know that

$$h^0\big(\tilde{V}, K_{\tilde{V}} + s(m\phi^*K_V - \mathscr{E})\big) < h^0\big(\tilde{V}, (sm+1)\phi^*K_V\big)$$
$$= h^0\big(V, (sm+1)K_V\big)$$

for s sufficiently large, and that $h^i(V, (sm+1)K_V) = 0$ for $i > 0$. Hence

$$\chi\big(V, (sm+1)K_V\big) - \chi\big(\tilde{V}, K_{\tilde{V}} + s(\phi^*mK_V - \mathscr{E})\big) > 0$$

for s sufficiently large; this then provides the required contradiction using (2.4). □

Returning now to our base curve C of $|mK_V|$ with $K_V \cdot C = 0$, we know that C is isomorphic to $\mathbb{P}^1$, and so the normal bundle is decomposable; say $N_{C/V} = \mathcal{O}_C(e-a) \oplus \mathcal{O}_C(-a)$ where $e \geqslant 0$. From the adjunction formula, $e - 2a = \deg(N_{C/V}) = -2$. In particular we note that $a > 0$. With this notation, E_1 is the ruled surface $\mathbb{F}_e$. In this case when $e \neq 0$, let C_1 denote the minimal section of $E_1(C_1^2 = -e)$, and f_2: $V_2 \to V_1$ the blow up of C_1, with E_2 the corresponding exceptional surface. If the composite

$f_1 \circ f_2$ is denoted f: $V_2 \to V$, then $|f^*mK_V| = |D_2| + rf_2^*E_1 + r_1E_2$, where $|D_2|$ is the mobile part, and $r_1 \leqslant r$ denotes the multiplicity of C_1 in $|D_1|$.

PROPOSITION 2.6: *The normal bundle* $N_{C/V}$ *is one of* $(-1, -1)$, $(-2, 0)$ *or* $(-3, 1)$, *where* $(-1, -1)$ *denotes* $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$, *etc.*

PROOF: We note first that $D_1 \cdot C_1 = -rE_1 \cdot C_1 = -r(e-a) = -r(a-2)$. Thus $D_1 \cdot C_1 \geqslant 0$ if and only if $N_{C/V} = (-1, -1)$ or $(-2, 0)$. Moreover, we note that the normal bundles listed in the Proposition correspond to $a = 1$, 2 and 3 respectively.

Suppose therefore that $D_1 \cdot C_1 < 0$ and hence $r_1 > 0$. Consider f: $V_2 \to V$ as described above. An easy calculation using (2.4) shows that $K_{V_2}^3 - K_V^3 = -2(a-4)$. Thus clearly the conditions of (2.5) are satisfied, and a contradiction is obtained. □

The curves $N_{C/V} = (-1, -1)$ or $(-2, 0)$ are what Reid in [10] calls (-2)-curves, and essentially we have all the information about their neighbourhoods in V that we want (see [10], §5). Let us now concentrate on the case when $N_{C/V} = (-3, 1)$. Blowing up minimal sections of exceptional surfaces, we obtain V_i, E_i, C_i as before, and a sequence of normal bundles $N_{C/V}$, N_{C_1/V_1}, etc.

PROPOSITION 2.7: *The only possible sequence of normal bundles in the case when* $N_{C/V} = (-3, 1)$ *is*:

$$(-3, 1), (-3, 0), \ldots, (-3, 0), (-2, -1), (-1, -1)$$

(*where there are a finite number*, *say* $t \geqslant 0$, *of* $(-3, 0)$*'s in the sequence*).

PROOF: On C_1 we have an exact sequence of sheaves

$$0 \to \mathcal{O}_{C_1}(-4) \to N_{C_1/V_1} \to \mathcal{O}_{C_1}(1) \to 0.$$

Hence $N_{C_1/V_1} = (-4, 1)$, $(-3, 0)$ or $(-2, -1)$. I claim that $(-4, 1)$ cannot occur.

To see this, blow up C_2, giving f_3: $V_3 \to V_2$, and let ϕ: $V_3 \to V$ denote the composite $f_1 \circ f_2 \circ f_3$. If $N_{C_1/V_1} = (-4, 1)$, we see that $D_2 \cdot C_2 = -(r + r_1)$, and using (2.4) that $K_{V_3}^3 = K_V^3$. The other conditions of (2.5) are now easily checked, and thus a contradiction is obtained.

Therefore $N_{C_1/V_1} = (-3, 0)$ or $(-2, -1)$. If for some i, $N_{C_i/V_i} = (-2, -1)$, we deduce from the exact sequence on C_{i+1}

$$0 \to \mathcal{O}_{C_{i+1}}(-1) \to N_{C_{i+1}/V_{i+1}} \to \mathcal{O}_{C_{i+1}}(-1) \to 0$$

that $N_{C_{i+1}/V_{i+1}} = (-1, -1)$. If however for some i, $N_{C_i/V_i} = (-3, 0)$, we deduce from the exact sequence on C_{i+1}

$$0 \to \mathcal{O}_{C_{i+1}}(-3) \to N_{C_{i+1}/V_{i+1}} \to \mathcal{O}_{C_{i+1}} \to 0$$

that $N_{C_{i+1}/V_{i+1}} = (-3, 0)$ or $(-2, -1)$.

Finally we note that if $|D_i|$ denotes the mobile part of $|mK_{V_i}|$, and $N_{C_j/V_j} = (-3, 0)$ for $j = 1, \dots, i-1$, an easy calculation shows that $D_i \cdot C_i = -r$. Hence C_i is a base curve of $|D_i|$. This then shows that an infinite sequence of $(-3, 0)$'s cannot occur, since then C_i would be a base curve of $|D_i|$ for all $i > 0$ (i.e. we cannot resolve the base locus), which is clearly in contradiction to the results say of [13]. □

3. Isolated base curves

As an illustration of the above results, let us consider the case when C has $K_V \cdot C = 0$ and is isolated in the base locus of $|mK_V|$ (we shall note later that a trivial modification of the argument then applies to a slightly more general case).

PROPOSITION 3.1: *If C is isolated in the base locus, and $N_{C/V} = (-1, -1)$ or $(-2, 0)$, then C is not a base curve of $|(2m+1)K_V|$.*

PROOF: We noted in the proof of (2.6) that in this case $-E_1|E_1$ is a.e. on E_1. Using the fact that C is isolated in the base locus of $|mK_V|$, we deduce that $mf_1^*K_V - E_1$ is a.e. on V_1. The Kawamata-Viehweg form of Kodaira vanishing then gives

$$h^i\big(V_1, K_{V_1} + 2(mf_1^*K_V - E_1)\big) = 0 \qquad \text{for} \quad i > 0.$$

Thus (2.1) implies that

$$h^0\big(V, (2m+1)K_V\big) - h^0\big(V_1, (2m+1)f_1^*K_V - E_1\big) = 1.$$

In particular we see that E_1 is not a fixed component of $|(2m+1)f_1^*K_V|$ on V_1, and hence that C is not a base curve of $|(2m+1)K_V|$. □

We consider therefore the case when $N_{C/V} = (-3, 1)$. By (2.7), there exists an integer $t \geqslant 0$ such that $N_{C_i/V_i} = (-3, 0)$ for $1 \leqslant i \leqslant t$, and $N_{C_{t+1}/V_{t+1}} = (-2, -1)$.

PROPOSITION 3.2: *If C is isolated in the base locus, and $N_{C/V} = (-3, 1)$, then C is not a base curve of $|(2m+1)K_V|$.*

PROOF: Suppose first that $t = 0$. Consider the divisor $\mathscr{E} = f_2^*E_1 + E_2$ on V_2. An easy check confirms that $-\mathscr{E} \cdot B \geqslant 0$ for all curves B on $\mathscr{E}$. Since C is isolated in the base locus of $|mK_V|$, we deduce that $mf^*K_V - \mathscr{E}$ is a.e.

on V_2 (where as before f denotes the composite $f_1 \circ f_2$). In particular we have that $h^i(V_2, K_{V_2} + 2(mf^*K_V - \mathscr{E})) = 0$ for $i > 0$. Hence using Remark 2.4, we see that

$$h^0(V, (2m+1)K_V) - h^0(V_2, (2m+1)f^*K_V - \mathscr{E}) = 1;$$

i.e. $\mathscr{E}$ is not fixed in $|(2m+1)f^*K_V|$.

If however C is fixed in $|(2m+1)K_V|$, then we note that C_1 is also fixed in $|D_1|$ on V_1 (since $-E_1 \cdot C_1 < 0$), and hence that $\mathscr{E}$ is fixed in $|(2m+1)f^*K_V|$; we conclude therefore that C is not fixed in $|(2m+1)K_V|$.

The case when $t > 0$ is slightly more complicated. We consider the threefold V_{t+2}. For $i < t+2$, we shall denote by h_i: $V_{t+2} \to V_i$ the obvious composite morphism, and in particular h: $V_{t+2} \to V$. Let us consider first the case when $t = 1$ (where the salient features of the general case are already present). Here we have $N_{C/V} = (-3, 1)$, $N_{C_1/V_1} = (-3, 0)$ and $N_{C_2/V_2} = (-2, -1)$.

If E_1' denotes the proper transform of E_1 under f_2, an easy check shows that on V_2 the minimal section C_2 of E_2 meets $C_1' = E_1' \cap E_2$ in one point. Thus when we blow up C_2, we obtain a configuration on V_3 as in Fig. 1.

In Fig. 1, ℓ is the curve on V_3 corresponding on V_2 to the fibre of the ruling on E_1' containing the point $E_1' \cap C_2$. As surfaces, E_1'' is $\mathbb{F}_4$ blown up in a point on the minimal section, E_2' is $\mathbb{F}_3$ and E_3 is $\mathbb{F}_1$.

On V_3 therefore we consider the divisor $\mathscr{E} = h_1^*E_1 + h_2^*E_2 + E_3$. An easy check shows that $-\mathscr{E} \cdot B \geqslant 0$ for all curves B on $\mathscr{E}$ except for ℓ; also $-\mathscr{E} \cdot \ell = -1$. We note moreover that $N_{\ell/V_3} = (-2, -1)$.

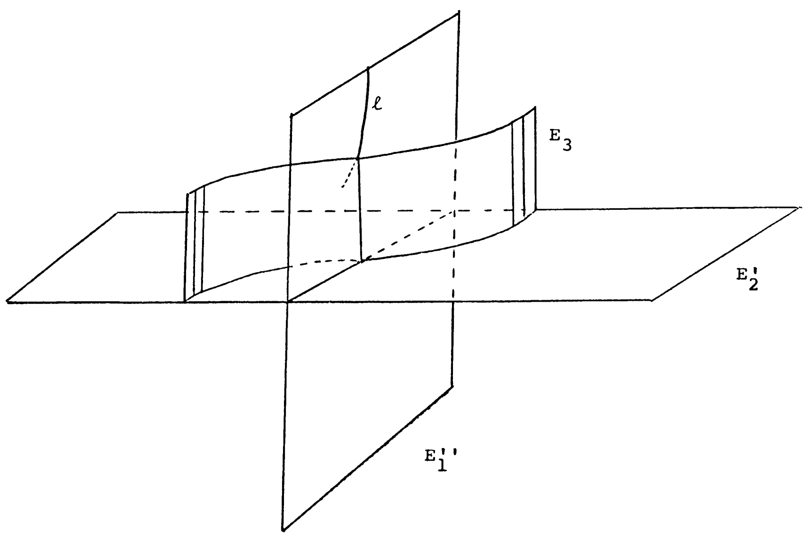

Figure 1.

We now blow up ℓ, obtaining $g\colon \tilde{V}_3 \to V_3$ and exceptional surface E say. On $\tilde{V}_3$ we consider the divisor $\tilde{\mathscr{E}} = g^*\mathscr{E} + E$. It is now straightforward to check that $-\tilde{\mathscr{E}} \cdot B \geqslant 0$ for all curves B on $\tilde{\mathscr{E}}$. Thus as before, using the fact that C is isolated in the base locus of $|mK_V|$, we deduce that $h^i(\tilde{V}_3, K_{\tilde{V}_3} + 2(m\tilde{h}^*K_V - \tilde{\mathscr{E}})) = 0$ for $i > 0$ (where $\tilde{h}$ is the composite $h \circ g$).
Using Remark 2.4 however, we now have

$$h^0(V, (2m+1)K_V) - h^0(\tilde{V}_3, (2m+1)\tilde{h}^*K_V - \tilde{\mathscr{E}}) = 1,$$

and hence that $\tilde{\mathscr{E}}$ is not fixed in $|(2m+1)\tilde{h}^*K_V|$. As before therefore, we discover that C is not a base curve of $|(2m+1)K_V|$ on V.

The case $t > 1$ is similar. Note first that for all $3 \leqslant i \leqslant t+2$, an easy calculation shows that the minimal section C_i of E_i does not meet the proper transform E'_{i-1} of E_{i-1}. Secondly, we note that in Fig. 1 (considered now for arbitrary $t > 1$), the minimal section C_3 of E_3 cannot meet ℓ. For if it did meet ℓ, then on V_4 we would have $K_{V_4} \cdot \ell' = 2$ (where ℓ' on V_4 corresponds to ℓ on V_3). Hence if we blow up ℓ' on V_4, obtaining g': $\tilde{V}_4 \to V_4$, we deduce via Remark 2.4 that $K^3_{\tilde{V}_4} = K^3_V$. Letting $\phi\colon \tilde{V}_4 \to V$ denote the obvious composition of maps, we see that the conditions of (2.5) are satisfied, and hence a contradiction is obtained.

We now consider the divisor $\mathscr{E} = h_1^*E_1 + h_2^*E_2 + \ldots + E_{t+2}$ on V_{t+2}; bearing in mind the above two observations, it is a straightforward check that $-\mathscr{E} \cdot B \geqslant 0$ for all curves B on $\mathscr{E}$ except for the curve ℓ^* on V_{t+2} corresponding to the curve ℓ on V_3; for this curve $-\mathscr{E} \cdot \ell^* = -1$. Now blow up ℓ^* obtaining $g\colon \tilde{V}_{t+2} \to V_{t+2}$, and proceed precisely as in the $t = 1$ case. □

We have thus seen that for any isolated base curve C of $|mK_V|$ with $K_V \cdot C = 0$, C is not a base curve of $|(2m+1)K_V|$. However, the only place that we use the fact that C is isolated in the above is that in order to check that $m\tilde{h}^*K_V - \tilde{\mathscr{E}}$ is a.e., we need to check it not only on the curves in $\tilde{\mathscr{E}}$, but also on the curves of $\tilde{V}_{t+2}$ correspond to the other base curves of $|mK_V|$. Clearly, the only such curves that we need to worry about are those that meet C. If however any other base curve C' which meets C has $K_V \cdot C' > 0$, we can then merely choose N a sufficiently large multiple of m so that, repeating the above argument for $|NK_V|$, the positivity is satisfied on $\tilde{V}_{t+2}$.

COROLLARY 3.3: *If no two base curves C, C' of $|mK_V|$ with $K_V \cdot C = 0 = K_V \cdot C'$ meet, then for some n the linear system $|nK_V|$ is without fixed points.*

PROOF: As above, we choose N sufficiently large so that we can then deduce that the base curves C of $|NK_V|$ with $K_V \cdot C = 0$ are no longer base curves of $|(2N+1)K_V|$. Hence we see that for some n, $|nK_V|$ has no fixed components and the only base curves C' of $|nK_V|$ have $K_V \cdot C' > 0$. Now apply (1.2). □

In Section 2, we obtained detailed information concerning the individual base curves C of $|mK_V|$ with $K_V \cdot C = 0$. We now wish to ask which configurations of such curves can occur. We consider this question in the next section.

4. Configuration of curves in the base locus

With the notation as before, let Z_0 denote the cycle of base curves of $|mK_V|$ with $K_V \cdot C = 0$; in particular any component of Z_0 is isomorphic to $\mathbb{P}^1$. We investigate the question of which configurations can appear in Z_0. In this section, we shall again make extensive use of (2.5).

THEOREM 4.1:

(*a*) *Any two curves in Z_0 meet in at most one point, where they meet transversely.*

(*b*) *At most three curves of Z_0 meet at any given point, where they meet normally.*

(*c*) *There are no "closed cycles" of curves in Z_0 (i.e. curves $B_1, \ldots, B_k$ in Z_0 with B_i meeting B_{i+1} for each i, where B_{k+1} is understood as B_1) apart from three curves meeting at a point.*

PROOF:

(a) Let B_1 and B_2 be any two curves in Z_0. We blow up one of them, say B_2, obtaining a morphism h: $V' \to V$ and exceptional divisor E. If B_1' denotes the curve on V' corresponding to B_1 on V, suppose that $E \cdot B_1' = d$. We need to show that $d \leqslant 1$.

We note however that $K_{V'}^3 = K_V^3 + 2$. Hence if we blow up B_1' on V' and let ϕ: $\tilde{V} \to V$ denote the composite of this map and h, then by (2.4) $K_{\tilde{V}}^3 - K_V^3 = -2(d-2)$. The result now follows easily using (2.5).

(b) Let $B_1, \ldots, B_r$ be curves of Z_0 that meet at some point P on V. Let us blow up one of the curves, say B_1. Let h: $V' \to V$ denote this blow up and E the exceptional divisor. If B_i' $(1 < i \leqslant r)$ denotes the curve on V' corresponding to B_i on V, then the B_i' all meet the same fibre F in the ruling of E over B_1. Note also that $K_{V'}^3 = K_V^3 + 2$, and that $K_{V'} \cdot B_i' = 1$ for $1 < i \leqslant r$.

Thus we note that blowing up a B_i' does not alter K^3. Suppose now that two of the B_i' meet, say B_2' meets B_3'. Blowing up B_2', say g: $V^* \to V'$, we have that $K_{V^*}^3 = K_{V'}^3$ and that $K_{V^*} \cdot B_3'' = 2$ (where B_3'' on V^* corresponds to B_3' on V'). Thus if we now blow up B_3'' and let ϕ: $\tilde{V} \to V$ be the composite of this morphism with the other two blow ups, then using (2.4) we have $K_{\tilde{V}}^3 = K_V^3$. A contradiction now follows easily from (2.5).

Thus no two of $B_2', \ldots, B_r'$ meet on V'. We need to show that $r \leqslant 3$.

Suppose not; let us then blow up the curves B_2', B_3' and B_4'; say h: $V^* \to V'$. Note that by (2.4) $K_{V^*}^3 = K_{V'}^3$.

If we denote by F' the curve on V^* corresponding to F on V', we observe that $K_{V^*} \cdot F' = 2$. Thus if we blow up F' and let ϕ: $\tilde{V} \to V$ be the composite of this morphism with the other blow ups, then using (2.4) we have $K_{\tilde{V}}^3 = K_V^3$. A contradiction again follows easily using (2.5).

(c) The fact that one cannot have any "closed cycles" follows similarly; we blow up one of the curves, say B_1, and K^3 increases by 2. If we now blow up the base curve corresponding to B_2, K^3 will remain unchanged. We continue this procedure until we reach the curve corresponding to B_k. Blowing this curve up now decreases K^3 by 2, and so the total effect has been to leave K^3 unchanged. A contradiction now follows easily using (2.5). □

We now need a couple of lemmas.

LEMMA 4.2: *Suppose that B and C are curves of Z_0 which meet, and that C has normal bundle $(-3, 1)$. Let f_1: $V_1 \to C$ denote the blow up of C, E_1 the exception divisor (isomorphic to $\mathbb{F}_4$) and C_1 the minimal section. If we denote by B' the curve on V_1 corresponding to B, then B' does not meet C_1.*

PROOF: If B' meets C_1, blow up B' and then blow up the curve corresponding to C_1. If ϕ: $\tilde{V} \to V$ denotes the composition of these morphisms, we find that $K_{\tilde{V}}^3 = K_V^3$, and that a contradiction follows using (2.5). □

As in (4.2), suppose that B and C are curves of Z_0 which meet, and suppose that C has normal bundle $(-3, 1)$. However, let us now blow up B first, say h: $V' \to V$ with exceptional divisor E. Then blow up the curve C' on V' corresponding to C, obtaining a threefold V^* and an exceptional divisor E^*. We therefore obtain a configuration as in Fig. 2, where E' is the proper transform of E, and ℓ is the curve on V^* corresponding to the fibre of E' meeting C'.

LEMMA 4.3: *E^* is isomorphic to $\mathbb{F}_3$ and the minimal section C^* meets ℓ.*

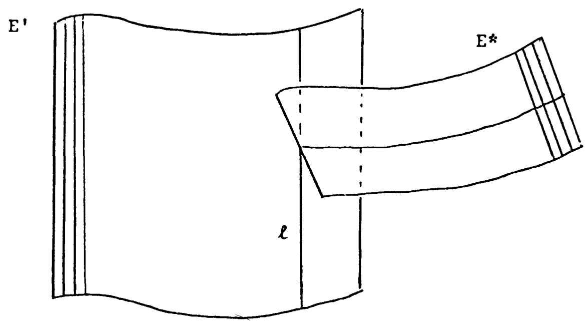

Figure 2.

PROOF: Note that ℓ has normal bundle $(-1, -1)$, and that blowing B and C up in the other order corresponds to making an elementary transformation on ℓ. By an elementary transformation on ℓ, we mean the operation of blowing up ℓ (obtaining an exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$), and then contracting down along the other ruling; see [7] and [10].

If E^* is isomorphic to $\mathbb{F}_{e'}$, then we deduce that $e' = 3$ or 5 according to whether C^* does, or does not, meet ℓ (since if we blow up C on V, we obtain an exceptional divisor E_1 isomorphic to $\mathbb{F}_4$). By inspection of the geometry however, we see that the case when C^* does not meet ℓ corresponds (when we blow B and C up in the other order) to the case when B' meets C_1 in (4.2). Hence the result follows. □

REMARK 4.4: In (4.3) we note that C' has normal bundle $(-3, 0)$. Thus we deduce via (2.4) that the operation of blowing up C' and then blowing up C^* does not alter the value of K^3. I shall refer to this operation as taking the double blow up of C'.

THEOREM 4.5:

(*a*) *A curve of* Z_0 *with normal bundle* $(-3, 1)$ *cannot meet another such curve, and also cannot meet a triple point.*

(*b*) *A curve* Z_0 *with normal bundle* $(-2, 0)$ *meets at most one* $(-3, 1)$*-curve, meets at most one triple point, and cannot meet both.*

PROOF: These results will all follow from (2.5). For brevity I shall therefore just give the sequence of blow ups needed, and leave the reader to verify that they work.

(a) Suppose B and C in Z_0 both have normal bundle $(-3, 1)$. Blow up C obtaining exceptional divisor E_1 isomorphic to $\mathbb{F}_4$ and minimal section C_1. If B' is the curve corresponding to B and F is the fibre of E_1 meeting B', then we take the double blow up of B', blow up C_1 and then blow up the curve F' corresponding to F. We then have a contradiction.

Suppose now that a curve C in Z_0 with normal bundle $(-3, 1)$ has a point in common with two other curves B_1 and B_2 of Z_0. Blow up C as before, obtaining E_1 with minimal section C_1. If B_i' denotes the curve corresponding to B_i, there is a fibre F of E_1 meeting both B_1' and B_2'. Now blow up C_1, B_1' and B_2', and then blow up the curve F' corresponding to F. A contradiction is obtained.

(b) Similar to (a); left as an exercise. □

Let us consider now a curve B in Z_0 with normal bundle $(-1, -1)$. Suppose that B meets a curve C of Z_0 with normal bundle $(-3, 1)$. Let us make an elementary transformation on B; using (4.3), we discover that the curve $\bar{C}$ corresponding to C under this transformation has normal bundle $(-2, 0)$.

Moreover, it is not difficult to show using (2.5) that if B meets a $(-3, 1)$-curve, then the transformed curve $\bar{B}$ does not meet such a curve.

Thus by making an elementary transformation, we can assume that B does not meet any $(-3, 1)$-curve. The question therefore arises as to whether, by also making the transformations described in [10] on $(-2, 0)$-curves, we can eliminate all non-isolated $(-3, 1)$-curves. The isolated $(-3, 1)$-curves we know about from Section 3.

Finally, consider any curve C of Z_0, and let $B_1, \ldots, B_k$ be those curves of Z_0 different from but meeting C. Let us blow up C, and then blow up all the B_i' in some order. If the resulting variety is $\tilde{V}$, and $|\tilde{D}|$ is the mobile part of $|mK_{\tilde{V}}|$, we know that the general element of $|\tilde{D}|$ cuts out an effective divisor on the proper transform of the exceptional divisor over C. This gives us an inequality relating the multiplicities in $|D| = |mK_V|$ of C and of the B_i. We can now repeat this argument on the B_i.

We leave it as an exercise for the reader to show, using the above method, that C meets at most nine other curves of Z_0.

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