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ON GENERALIZED WHITNEY MAPPINGS

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Abstract

Some particular types of maps from a hyperspace into [0,1] are being investigated. We give criteria for their existence in terms of certain countability conditions. We also examine the existence of derived Whitney maps on quotient spaces.

0. Introduction

Let X be a compact space. Its hyperspace [4] will be denoted by $H(X)$. By a *Whitney map* for X is meant a continuous monotonic function

$$w: H(X) \to [0,1],$$

such that $w(\{x\}) = 0$ for each $x \in X$, and such that $w(A) < w(B)$ whenever A is properly included in B, [4]. For a generalization to compact partially ordered spaces, see [6].

In [5] it is shown that "remote" maximal linked systems can be constructed with the aid of functions satisfying somewhat weaker conditions than Whitney maps do. Our present purpose is to give equivalent conditions for the existence of such functions, and to investigate the relationship between these functions and Whitney maps on certain metric quotients.

We note that remote n-linked systems can also be constructed with a method given in [1], but for compact spaces and 2-linked systems, the method of [5] is more general.

1. Existence of generalized Whitney maps

1.1. Preliminaries

A collection $\mathcal{O}$ of nonempty open sets of a space X is called a *pseudo-base* for X if every nonempty open set in X includes a member of $\mathcal{O}$. Then X is

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of *countable pseudo-weight* if it admits a countable pseudo-base, [2, p.5].

Let A, B be subsets of a space X. We say that A is *strongly contained* in B if $\operatorname{int}_X (B \setminus A) \neq \emptyset$ and $A \subset B$. For instance, each proper closed subset of X is strongly contained in X.

By a *Whitney-like* map for X we will understand a continuous monotonic function

$$(*) w: H(X) \to [0,1],$$

such that $w(\{x\}) = 0$ for each $x \in X$, and such that $w(A) < w(B)$ whenever A is strongly contained in B.

We will also be interested in two weaker properties of a function w as in (*) above:

(I) w is continuous, monotonic, and $w(A) < w(X)$ for each proper closed subset $A \in H(X)$;

(II) w is continuous, monotonic, and $w(A) < w(X)$ for each nowhere dense set $A \in H(X)$.

It is clear that

$$\text{Whitney-like} \Rightarrow (\mathrm{I}) \Rightarrow (\mathrm{II}).$$

1.2. THEOREM: *Let X be a compact Hausdorff space. Then the following assertions are equivalent:*

(1) X has a countable pseudo-weight;

(2) X has a Whitney-like mapping;

(3) there is a map w: $H(X) \to [0,1]$ with property (I).

PROOF OF (1) $\Rightarrow$ (2): (Let $\{O_n | n \in \mathbb{N}\}$ be a countable pseudo-base for X. By normality, there exist mappings

$$f_n: X \to [0,1], n \in \mathbb{N},$$

with $f_n(x) = 1$ if $x \notin O_n$ and $f_n(x) < 1$ for some $x \in O_n$. This yields a map

$$f: X \to Q = [0,1]^{\omega_0},$$

defined by $f(x) = (f_n(x))_{n \in \mathbb{N}}$, $x \in X$. Let

$$w_0: H(f(X)) \to [0,1]$$

be a Whitney map for the compact metric space $f(X)$ (cf. [4], [7]). Then define

$$w: H(X) \to [0,1]$$

by $w(A) = w_0 f(A)$, $A \in H(X)$. Then w is obviously continuous, monotone and $w(\{x\})=0$ for each $x \in X$. Let A, $B \in H(X)$ with A strongly contained in B. Then there is an $n \in \mathbb{N}$ with $O_n \subset B \setminus A$, and we can find an $x \in O_n$ with $f_n(x) \notin f_n(A)$. Consequently, $f(x) \notin f(A)$, whence $f(A)$ is properly contained in $f(B)$. It follows that

$$w(A)=w_0 f(A)<w_0 f(B)=w(B)$$

Note that the map $f\colon X \to f(X) \subset Q$ is irreducible by the above argument, in agreement with a result in [3].

PROOF OF (2) $\Rightarrow$ (3): obvious.

PROOF OF (3) $\Rightarrow$ (1): Let $w\colon H(X) \to [0,1]$ have property (I). We let $\mathscr{F}$ denote the net of all finite subsets of X, which converges to $X \in H(X)$. Hence $(w(F))_{F \in \mathscr{F}}$ converges to $w(X)$, and we can pick a sequence $(F_n)_{n \in \mathbb{N}}$ in $\mathscr{F}$ with $w(F_n)_{n \in \mathbb{N}}$ converging to $w(X)$. Now $(F_n)_{n \in \mathbb{N}}$ and $(w(F_n))_{n \in \mathbb{N}}$ converge to, respectively,

$$A = Cl_X\left(\bigcup_n F_n\right) \text{ and } w(A).$$

It follows from property (I) that $A = X$, and that $D = \cup_n F_n$ is a countable dense subset of X which we fix for the rest of the proof.

For each $G \in H(X)$ and for each $n \in \mathbb{N}$ we put

$$O(G,n)=\left\{x \,|\, w(G \cup \{x\})-w(G)>\frac{1}{n}\right\}.$$

The set $O(G, n)$ is open by the continuity of w and of the union operator. We let $\mathscr{O}$ denote the collection of all sets of type

$$O(G,n) \neq \emptyset, \quad G \subset D \text{ finite}, n \in \mathbb{N},$$

or of type

$$\{x\}, x \text{ an isolated point of } X.$$

X being separable, there can be at most countably many isolated points, whence $\mathscr{O}$ is a countable family. We show that it is a pseudo-base for X.

Let A be a proper closed subset of X. If $X \setminus A$ is finite, then it contains an isolated point, and hence some $O \in \mathscr{O}$ satisfies $O \cap A = \emptyset$. Assuming $X \setminus A$ to be infinite, we find that $D \cap (X \setminus A)$ is an infinite set, which we can arrange as a sequence $(x_n)_{n \in \mathbb{N}}$. Put

$$A_0=A; \quad A_n=A \cup \{x_1,\dots,x_n\}, n \in \mathbb{N}.$$

Then each A_n is a proper closed subset of X, and $(A_n)_{n=0}^{\infty}$ converges to X.

Hence $w(A_n) < w(X)$ for each $n \geq 0$ and $w(A_n)$ converges to $w(X)$. Consequently, there is an $n_0 \in \mathbb{N}$ with

$$w\left(A_{n_0-1}\right) < w\left(A_{n_0}\right).$$

By the continuity of w, there is a neighbourhood $\mathscr{U}$ of $A_{n_0-1} \in H(X)$ such that for all $B \in \mathscr{U}$, $w(B) < w(A_{n_0})$. As w is monotonic, we may assume that $\mathscr{U}$ is of type

$$\langle O \rangle = \{ B \in H(X) | B \subset O \},$$

where $O \subset X$ is open. Fix an open set P of X with

$$A_{n_0-1} \subset P \subset \bar{P} \subset O,$$

and fix $k \in \mathbb{N}$ with

$$(*) \quad \frac{1}{k} \leq w\left(A_{n_0}\right) - w(\bar{P}) \leq w\left(\bar{P} \cup \{ x_{n_0} \}\right) - w(\bar{P}).$$

Choose an increasing sequence $(G_n)_{n \in \mathbb{N}}$ of finite subsets of $D \cap P$ converging to $\bar{P}$. Then $(w(G_n))_{n \in \mathbb{N}}$ converges to $w(\bar{P})$ and $(w(G_n \cup \{x_{n_0}\}))_{n \in \mathbb{N}}$ converges to $w(\bar{P} \cup \{x_{n_0}\})$. Hence there is an $n_1 \in \mathbb{N}$ such that

$$\forall n \geq n_1 : w(\bar{P}) - w(G_n) < \frac{1}{2k},$$

and, by (*), there is an $n_2 \in \mathbb{N}$ with

$$(**) \quad \forall n \geq n_2 : w\left(G_n \cup \{ x_{n_0} \}\right) - w(G_n) > \frac{1}{2k}.$$

Let $n = \max\{n_1, n_2\}$. For $x \in \bar{P}$,

$$w(G_n \cup \{x\}) - w(G_n) \leq w(\bar{P}) - w(G_n) < \frac{1}{2k},$$

whence $O(G_n, 2k) \cap \bar{P} = \emptyset$. Also, $x_{n_0} \in O(G_n, 2k)$ by (**), whence $O(G_n, 2k) \in \mathcal{O}$.

1.3. Theorem: *Let X be compact and Hausdorff. Then the following assertions are equivalent:*

(1) there is a countable collection $\{O_n | n \in \mathbb{N}\}$ of nonempty open sets such that each dense open set of X contains some O_n;

(2) there is a map $w: H(X) \to [0,1]$ with property (II).

Proof of (1) $\Rightarrow$ (2): Fix a sequence of maps

$$f_n: X \to [0,1], n \in \mathbb{N},$$

with $f_n(x) = 1$ for $x \notin O_n$, and $f_n(x) < 1$ for some $x \in O_n$. Then define

$$f: X \to Q \text{ and } w: H(X) \to [0,1]$$

by $f(x) = (f_n(x))_{n \in \mathbb{N}}$ and by $w(A) = w_0 f(A)$, where w_0 is a Whitney map for $f(X)$. If $A \in H(X)$ is nowhere dense then there exists an $x \in X \setminus A$ and an $n \in \mathbb{N}$ with $f_n(x) \notin f_n(A)$. Consequently, $f(A)$ is a proper closed subset of $f(X)$, proving that $w(A) < w(X)$.

Proof of (2) $\Rightarrow$ (1): We shall largely borrow from the proof of (3) $\Rightarrow$ (1) in the previous theorem, with some extra complications to be solved. Let w: $H(X) \to [0,1]$ be a map with property (II). Let F be the net of all finite subsets of X. Fix a subsequence $(F_n)_{n \in \mathbb{N}}$ with $(w(F_n))_{n \in \mathbb{N}}$ converging to $w(X)$. Then $(F_n)_{n \in \mathbb{N}}$ converges to

$$K = Cl\Big(\bigcup_n F_n\Big),$$

and it follows from property (II) that $N = \operatorname{int} K \neq \emptyset$. Note that $D = \cup_n F_n$ is a countable dense subset of K, and that $w(K) = w(X)$. Also, if X has an isolated point x, then each dense subset of X contains $\{x\}$, whence X satisfies (1). We assume in the sequel that X has no isolated points.

Define a new function

$$w': H(X) \to [0,1]$$

by $w'(A) = w(A \cup (K \setminus N))$. Then w' is obviously continuous and monotonic, and if $A \in H(X)$ is nowhere dense, then so is $A \cup (K \setminus N)$, whence $w'(A) < w'(X)$. We may therefore assume that the original map w satisfies in addition the following property: if $A \in H(X)$ and if $x \in K \setminus N$, then $w(A) = w(A) \cup \{x\})$.

Let $\mathcal{O}$ denote the countable collection consisting of N and of all nonempty sets of type

$$N \cap O(G, n), \quad G \subset D \text{ finite}, \quad n \in \mathbb{N}$$

(notation as in Theorem 1.2). Let $A \in H(X)$ be nowhere dense. If $A \cap K = \emptyset$, then A does not meet $N \in G$. So assume that $A \cap K \neq \emptyset$. Then $K \setminus A$ is infinite (otherwise $\operatorname{int} K = \emptyset$ since X has no isolated points) and relatively open in K. Therefore, $D \cap (K \setminus A)$ is an infinite dense subset of $K \setminus A$ which we arrange as a sequence $(x_n)_{n \in \mathbb{N}}$. Writing

$$A_0 = A \cap K, \quad A_n = (A \cap K) \cup \{x_1, \ldots, x_n\},$$

we find that each A_n is nowhere dense in X and that $(A_n)_{n\in\mathbb{N}}$ converges to K. Hence, $w(A_n) < w(X) = w(K)$ and $(w(A_n))_{n\in\mathbb{N}}$ converges to $w(K)$, and we can find an $n_0 \in \mathbb{N}$ such that

$$w\left(A_{n_0-1}\right) < w\left(A_{n_0}\right).$$

By the continuity and monotony of w, there is an open set $O \supset A_{n_0-1}$ of X such that for each $B \in H(X)$ with $B \subset O$, $w(B) < w(A_{n_0})$. Choose a relatively open set P of K with

$$A_{n_0-1} \subset P \subset \overline{P} \subset O,$$

and choose $k \in \mathbb{N}$ such that

$$\frac{1}{k} \le w\left(A_{n_0}\right) - w(\overline{P}) \le w\left(\overline{P} \cup \{x_{n_0}\}\right) - w(\overline{P}).$$

As P is open in K, we can find an increasing sequence $(G_n)_{n\in\mathbb{N}}$ of finite subsets of $D \cap P$ converging to $\overline{P}$ in $H(X)$. Fix $n_1 \in \mathbb{N}$ such that

$$\forall n \ge n_1 : w(\overline{P}) - w(G_n) < \frac{1}{2k}.$$

As $(G_n \cup \{x_{n_0}\})_{n\in\mathbb{N}}$ converges to $\overline{P} \cup \{x_{n_0}\}$, we can find an $n_2 \in \mathbb{N}$ such that

$$\forall n \ge n_2 : w\left(G_n \cup \{x_{n_0}\}\right) - w(G_n) > \frac{1}{2k}.$$

For $n = \max\{n_1, n_2)$, we obtain the following : if $x \in \overline{P}$, then

$$w(G_n \cup \{x\}) - w(G_n) \le w(\overline{P}) - w(G_n) < \frac{1}{2k},$$

whence

$$O(G_n, 2k) \cap N \cap A \subset O(G_n, 2k) \cap \overline{P} = \emptyset,$$

whereas $x_{n_0} \in O(G_n, 2k)$. By the extra assumption on w, we also find that $x_{n_0} \notin K \setminus N$, whence $O(G_n, 2k) \cap N \in \mathcal{O}$. This shows that X admits a collection of open sets as required in (1).

2. Relationship with Whitney maps

In Theorems 1.2 and 1.3, the mapping w: $H(X) \to [0,1]$ is obtained by using a Whitney map on a metric quotient of the original space. This naturally leads to the question whether or not every Whitney-like map, or every map with property (I) or (II), arises in this way. A necessary condition is, of course, that $w(\{x\}) = 0$ for each point x. It appears that this "norm" conditions is rather irrelevant:

Let X be compact T_1, and let w: $H(X) \to [0,1]$ be continuous and monotonic. Then there exists a continuous monotonic map w': $H(X) \to [0,1]$ such that

(1) $\forall x \in X$: $w'(\{x\}) = 0$

(2) $\forall A \subset B$ in $H(X)$: $w(A) < w(B) \Rightarrow w'(A) < w'(B)$.

In particular, w' has property (I) (or (II)) if w does. Just define a map by:

$$w'(A) = w(A) - \inf\{w(\{x\}) | x \in A\}, \quad A \in H(X).$$

2.1. Definition: Let X and Y be compact Hausdorff spaces, and let

$$\mu : H(X) \to [0,1], \quad \nu : H(Y) \to [0,1]$$

be continuous functions. A *morphism* f: $(X, \mu) \to (Y, \nu)$ is a continuous map f: $X \to Y$ such that $\mu(A) = \nu f(A)$ for each $A \in H(X)$.

We say that (X, μ) *is obtained from a Whitney map* if there exists an onto morphism $(X, \mu) \to (Y, \nu)$ with ν a Whitney mapping.

2.2. Theorem: *Let X be compact Hausdorff, and let w: $H(X) \to [0,1]$ be continuous. Then there is a quotient map f: $X \to \tilde{X}$ onto a metric space $\tilde{X}$, together with a continuous function $\tilde{w}$: $H(\tilde{X}) \to [0,1]$, such that the following are true.*

(1) f is a morphism $(X, w) \to (\tilde{X}, \tilde{w})$;

(2) The pair $(\tilde{X}, \tilde{w})$ is universal in the following sense. For each pair (Y, μ) and for each surjective morphism g: $(X, w) \to (Y, \mu)$ there is a unique morphism h: $(Y, \mu) \to (\tilde{X}, \tilde{w})$ with $hg = f$;

(3) If (X, w) can be obtained from a Whitney map, then $\tilde{w}$ is a Whitney map, isomorphic to the given one.

Proof: Two points x_1, x_2 of X are called *w-equivalent* if

$$\forall A \in H(X) : w(A \cup \{x_1\}) = w(A \cup \{x_2\}).$$

We let $f(x)$ denote the equivalence class of x, and we denote the resulting quotient map by

$$f : X \to \tilde{X} = \{f(x) | x \in X\}.$$

The saturation $f^{-1}f(A)$ of $A \in H(X)$ will also be denoted by $[A]$. Define

$$d(x_1, x_2) = \sup\{|w(A \cup \{x_1\}) - w(A \cup \{x_2\})| A \in H(X)\}$$

for each x_1, $x_2 \in X$. Then d is easily seen to be a pseudo-metric on the set

X with "level" sets equal to the above equivalence classes. Hence, d induces a metric $\tilde{d}$ on the set $\tilde{X}$.

For each $x \in X$ and $r > 0$ the collection

$$B(x, r) = \{ y | d(x, y) < r \}$$

is open: let $y \in B(x, r)$. For each $A \in H(X)$ there exist neighborhoods $O(A)$ of $A \in H(X)$ and $P(A)$ of $y \in X$ such that

$$|w(B \cup \{x\}) - w(B \cup \{z\})| < r$$

for each $B \in O(A)$ and for each $z \in P(A)$. By the compactness of $H(X)$ we can select a finite cover $\{O(A_1), \ldots, O(A_n)\}$ from the covering $\{O(A) | A \in H(X)\}$, and we put $P = \cap_{i=1}^{n} P(A_i)$. Then P is a neighbourhood of $y \in X$, and it is easily seen that $P \subset B(x, r)$. Hence $\tilde{d}$ is compatible with the topology of the compact space $\tilde{X}$.

If $A \in H(X)$, then $w(A) = w([A])$. Indeed, let

$$\mathscr{F} = \{ A \cup F | F \subset [A] \text{ is finite and nonempty} \}.$$

Then $\mathscr{F}$ converges to $[A]$, whence the w-images converge to $w[A]$. Select a sequence $(A \cup F_n)_{n \in \mathbb{N}}$ converging to $w[A]$, and put $D = \cup_n F_n$. Note that $w(A \cup \overline{D}) = w[A]$. Writing

$$D = \{ x_n | n \in \mathbb{N} \}; G_n = \{ x_1, \ldots, x_n \}$$

(the indexation need not be injective; D may be finite), we find that

$$w(A) = w(A \cup G_1) = \ldots = w(A \cup G_n) = w(A \cup G_{n+1}\} = \ldots$$

since each x_n is w-equivalent to some point of A. Now $(A \cup G_n)_{n \in \mathbb{N}}$ converges to $A \cup \overline{D}$ whence $(w(A \cup G_n))_{n \in \mathbb{N}}$ converges to $w(A \cup \overline{D}) = w([A])$. Therefore, $w(A) = (w([A])$.

Define $\tilde{w}$: $H(\tilde{X}) \to [0,1]$ by

$$\tilde{w}(B) = w(f^{-1}(B)), \quad B \in H(\tilde{X}).$$

If follows from the above argument that for each $A \in H(X)$,

$$w(A) = w(f^{-1}f(A)) = \tilde{w}(f(A)).$$

As $H(X)$ and $H(\tilde{X})$ are compact, the mapping $H(f)$: $H(X) \to H(\tilde{X})$ with $H(f)(A) = f(A)$ is an identification map. Hence $\tilde{w}$ is continuous, proving (1).

Let g: $(X, w) \to (Y, \mu)$ be onto, and let $g(x_1) = g(x_2)$. Then for each $A \in H(X)$,

$$g(A \cup \{x_1\}) = g(A \cup \{x_2\}),$$

and consequently

$$w(A \cup \{x_1\}) = \mu g(A \cup \{x\}) = \mu g(A \cup \{x_2\}) = w(A \cup \{x_2\}).$$

This shows that $f(x_1) = f(x_2)$, and we find a unique map h: $Y \to \tilde{X}$ with $hg = f$. Also, if $A \in H(Y)$, then $fg^{-1}(A) = h(A)$, whence

$$\mu(A) = w(g^{-1}(A)) = \tilde{w}(fg^{-1}(A)) = \tilde{w}(h(A)),$$

proving that h is a morphism $(Y, \mu) \to (\tilde{X}, \tilde{w})$. This settles (2).

Assume now that the above μ is a Whitney map. then for each $y_1 \neq y_2$ in Y,

$$\tilde{w}(h(\{y_1\}) = \mu(\{y_1\}) < \mu(\{y_1, y_2\}) = \tilde{w}(h\{y_1, y_2\})),$$

showing that $h(y_1) \neq h(y_2)$, i.e. h is a homeomorphism. Consequently, $\tilde{w}$ is a Whitney mapping too.

2.3. Corollary: *Let X be compact Hausdorff, and let w: $H(X) \to [0,1]$ be continuous, monotonic, and such that $w(\{x\}) = 0$ for each $x \in X$. Then (X, w) can be obtained from a Whitney map iff for each $A \subset B$ in $H(X)$, $w(A) = w(B)$ implies $B \subset [A]$ (notation introduced above).*

Proof: By 2.2(3), (X, w) can be obtained from a Whitney map iff $\tilde{w}$ is a Whitney map. Using the explicit construction of $(\tilde{X}, \tilde{w})$ then yields the desired result.

2.4. Questions

(1) As we already explained in the introduction, we have introduced the present generalizations of Whitney mappings in order to construct "remote" points in superextensions, [5]. To be precise, such a point is obtained in the following way. Let w: $H(X) \to [0,1]$ have property (II). Then the (2-linked) system of closed sets in X,

$$\mathscr{M} = \{ M \in H(X) | w(M) > w(\overline{X \setminus M}) \},$$

has the property that each nowhere dense closed subset of X misses some member of $\mathscr{M}$, and $\mathscr{M}$ extends to a "remote" maximal linked system, that is: one which consists entirely of sets with nonempty interior.

Is there a similar construction possible to obtain "remote" n-linked system on X? If so, then this would provide an extension (on compact spaces) of a result in [1] concerning G-spaces.

(2) A fairly recent development in continual, theory is the study of so-called Whitney properties, and of Whitney reversible properties, [4].

The idea is that if $C(X)$ denotes the subspace of $H(X)$ of all subcontinua of X, and if

$$w: H(X) \to [0,1]$$

is a Whitney map, then certain properties of X may carry over to the spaces

$$w^{-1}(t) \cap C(X), \quad t \in [0,1],$$

or vice versa.

It is clear that by broadening the concept of a Whitney function, the chances for a topological property to be "Whitney (reversible)" get smaller. Are there any interesting topological properties left which are preserved, in one way or the other, by Whitney-like mappings? A similar question could be asked for maps with property (I) or (II), but these conditions are probably too weak to get significant results.

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