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SELF-DUALITY OF KÄHLER SURFACES

Mitsuhiro Itoh

1. Introduction and main theorems

On an oriented Riemannian 4-manifold (M, g) the star operator $*$ is defined on the space of 2-forms Λ^2 by

$$\alpha \Lambda * \beta = (\alpha, \beta)_g \mathrm{d}v_g, \qquad \alpha, \beta \in \Lambda^2. \tag{1.1}$$

The operator $*$ depends essentially only on the conformal structure given by the metric g. Since $* \circ * = id$ on Λ^2, Λ^2 splits into eigenspaces as $\Lambda^2 = \Lambda^2_+ + \Lambda^2_-$, where Λ^2_+ and Λ^2_- are the eigenspaces corresponding to eigenvalues $+1$ and -1, respectively. A 2-form which belongs to Λ^2_+ (respectively, to Λ^2_-) is called self-dual (respectively, anti-self-dual).

Let W be Weyl's conformal curvature tensor of g. Then W is regarded as an End(TM)-valued 2-form.

DEFINITION [1]: An oriented Riemannian 4-manifold is called self-dual (respectively, anti-self-dual) if W is self-dual (respectively, anti-self-dual) as a 2-form.

The first Pontrjagin number $p_1(M)$ is written by

$$p_1(M) = 1/4\pi^2 \int_M \left(|W_+|^2 - |W_-|^2\right) \mathrm{d}v_g \tag{1.2}$$

with respect to the self-dual part W_+ and the anti-self-dual part W_- of the Weyl's tensor W. Since the signature $\tau(M)$ is given by $\tau(M) = 1/3 p_1(M)$, a compact, self-dual (respectively, anti-self-dual) 4-manifold (M, g) has nonnegative (respectively, nonpositive) signature and $\tau(M) = 0$ if and only if (M, g) is conformally flat [1].

It is known [1] that a self-dual Riemannian 4-manifold (M, g) admits a holomorphic Penrose fibering, that is, there exists a $P_1(\mathbb{C})$-bundle over M whose canonical almost complex structure is integrable. Further its Penrose fibering is a Kähler manifold if and only if (M, g) is conformally equivalent to the 4-sphere S^4 or the complex projective plane $P_2(\mathbb{C})$ with standard metrics [8].

Note that a self-dual 4-manifold is anti-self-dual if the orientation is reversed.

On a compact, self-dual Riemannian 4-manifold of positive scalar curvature, moduli space of irreducible self-dual Yang-Mills connections admits a structure of manifold of certain dimension [1]. A similar statement is also obtained in [9] with respect to moduli space of irreducible anti-self-dual Yang-Mills connections on a compact Kähler surface (M, g) of positive scalar curvature, whether (M, g) is self-dual or not. Therefore it seems to be an interesting problem to characterize geometrically self-dual 4-manifolds and anti-self-dual 4-manifolds. In this article we discuss geometrical descriptions of such 4-manifolds whose metrics are Kähler.

Let (M, g) be a Kähler surface. The complex structure J induces canonically an orientation $\{x^1, x^2, x^3, x^4\}$, where $z^1 = x^1 + \sqrt{-1}\,x^2$, $z^2 = x^3 + \sqrt{-1}\,x^4$ are local holomorphic coordinates of M.

It is shown by a straight computation that a Kähler surface of constant holomorphic curvature is self-dual with respect to the canonical orientation. The following shows that the converse is also true if we restrict a Kähler surface to be Einstein.

THEOREM *1: Let (M, g) be a Kähler surface. If it is self-dual with respect to the canonical orientation and it is Einstein, then it is of constant holomorphic curvature.*

This is shown in §2 by the aid of curvature conditions in Lemma 2.2. The following theorem is known further with respect to self-dual Kähler surfaces.

THEOREM A *[5]: Every self-dual Kähler surface with constant scalar curvature is locally symmetric.*

Moreover, each self-dual metric is characterized in terms of Bochner curvature tensor, introduced by Bochner ([14]).

THEOREM B *[13]: Let (M, g) be a Hermitian manifold of complex dimension two. Then the Bochner curvature tensor of g is the anti-self-dual part of the Weyl's conformal curvature tensor W of g.*

The complete characterization of compact self-dual Kähler surfaces is obtained in the following

THEOREM C *[2,3,5]: Let (M, g) be a compact, self-dual Kähler surface. Then (M, g) is either a space of constant holomorphic curvature ($P_2(\mathbb{C})$, a compact quotient of unit disk D^2 or a Kählerian flat torus T^2) or a compact quotient of a product space of $P_1(\mathbb{C})$ and the Poincaré disk D^1 with metrics of opposite curvature.*

REMARKS: (i) Theorem C was obtained in [3] by B.Y. Chen in terms of Bochner-Kähler metrics (i.e., Kähler metrics whose Bochner curvature tensor vanishes) and independently by Derdzinski in [5] using Theorem A. Also Bourguignon verified this theorem by the aid of theorems with respect to harmonic curvature tensor [2].

(ii) Derdzinski obtained an example of non-compact, self-dual Kähler surface which is not locally symmetric [4].

Now we consider anti-self-dual Kähler surfaces. The following theorem characterizes these surfaces in terms of scalar curvature (refer to Problem 41, Problem section in [16]).

THEOREM 2: *Let (M, g) be a Kähler surface. Then it is anti-self-dual if and only if its scalar curvature vanishes everywhere.*

REMARK: From this theorem we claim that the total scalar curvature of a compact, anti-self-dual Kähler surface is necessarily zero and its Ricci form is anti-self-dual and is harmonic as a 2-form. Further we have another topological restriction $c_1(M)^2[M] \leqslant 0$.

The following gives a complete classification of compact, conformally flat Kähler surfaces, that is, compact, anti-self-dual Kähler surfaces whose signature is zero.

THEOREM 3: *Let (M, g) be a compact, conformally flat Kähler surface. Then (M, g) is either a Kählerian flat torus or a Kählerian ruled surface of genus $k(\geqslant 2)$.*

Since each compact, anti-self-dual Kähler surface (M, g) satisfies that $\tau(M) \leqslant 0$ and $c_1(M)^2[M] \leqslant 0$, we have the following by the aid of Theorem 2 together with the classification of complex surfaces.

THEOREM 4: *Let (M, g) be a compact, anti-self-dual Kähler surface. Then (M, g) is necessarily one of the following*

(i) a Kählerian flat torus,

(ii) a Kähler surface covered by a K 3 surface with a Ricci flat metric,

(iii) a K̈hlerian ruled surface of genus $k(\geqslant 2)$ and

(iv) a Kähler surface which is obtained by blowing up either $P_2(\mathbb{C})$ at least 10 times, a ruled surface of genus 0 at least 9 times or a ruled surface of genus $k(\geqslant 1)$ at least once.

In §2 we state local properties of self-dual Kähler surfaces and of anti-self-dual Kähler surfaces. We discuss in §3 global aspects of the anti-self-duality, from which Theorem 4 is deduced.

2. Local properties of (anti-)self-dual Kähler surfaces

We recall at first the definition of Weyl's conformal curvature tensor.

The Weyl's conformal curvature tensor W of a Riemannian 4-manifold (M, g) is written as [6]

$$\begin{aligned} &g(W(X_1, X_2)X_3, X_4) \\ &\quad = g(R(X_1, X_2)X_3, X_4) \\ &\qquad -\tfrac{1}{2}\{g(X_1, X_4)R^1(X_2, X_3) - g(X_2, X_4)R^1(X_1, X_3) \\ &\qquad +R^1(X_1, X_4)g(X_2, X_3) - R^1(X_2, X_4)g(X_1, X_3)\} \\ &\qquad +\tfrac{1}{6}\rho\{g(X_1, X_4)g(X_2, X_3) - g(X_2, X_4)g(X_1, X_3)\} \end{aligned} \tag{2.1}$$

where R, R^1 and ρ are the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of g respectively, that is, R is defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ with respect to the Levi-Civita connection ∇, and R^1 and ρ are defined by $R^1(X, Y) = \sum_{i=1}^4 g(R(e_i, X)Y, e_i)$ and $\rho = \sum_{i=1}^4 R^1(e_i, e_i)$ respectively, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

Suppose that (M, g) is a Kähler surface. We shall begin with (anti-)self-duality condition of W of the Kähler metric g in terms of complex 2-forms. Before stating the condition we give a characterization of (anti-)self-dual 2-forms by the aid of complex 2-forms.

PROPOSITION 2.1 [9]: *A 2-form α is self-dual if and only if (1,1)-part of α is proportional to the Kähler form Ω, and a 2-form β is anti-self-dual if and only if β is of type (1,1) which satisfies that $(\beta, \Omega)_g = 0$.*

REMARK: The Kähler form Ω is a self-dual form and if a real form of type (1,1) $\sigma = \sqrt{-1}\sum \sigma_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta$ satisfies $\sum g^{\alpha\bar\beta}\sigma_{\alpha\bar\beta} = 0$, then σ is anti-self-dual, where $(g^{\alpha\bar\beta})$ is the inverse of the component matrix of g.

Let $\{E_1, E_2\}$ be a local unitary basis. Unless otherwise stated, Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to 2, while Latin capitals $A, B, C, \ldots$ run over $1, 2, \bar 1$ and $\bar 2$. We set $g_{AB} = g(E_A, E_B)$, $R_{ABCD} = g(R(E_A, E_B)E_C, E_D)$, $R^1_{AB} = R^1(E_A, E_B)$ and $W_{ABCD} = g(W(E_A, E_B)E_C, E_D)$.

Since $\Omega(E_1, E_{\bar 1}) = \Omega(E_2, E_{\bar 2})$ and $\Omega(E_1, E_{\bar 2}) = 0$, we have from Proposition 2.1 that W is self-dual if and only if

$$W_{1\bar 1AB} = W_{2\bar 2AB} \quad \text{and} \quad W_{1\bar 2AB} = 0 \tag{2.2}$$

and W is anti-self-dual if and only if

$$W_{12AB} = 0 \quad \text{and} \quad W_{1\bar 1AB} + W_{2\bar 2AB} = 0, \tag{2.3}$$

for any A and B.

Since g is a Kähler metric, the components of R satisfy that $R_{\alpha\beta CD} = R_{CD\alpha\beta} = 0$, $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}} = R_{\gamma\bar{\delta}\alpha\bar{\beta}}$ and $\overline{R_{\alpha\bar{\beta}\gamma\bar{\delta}}} = R_{\beta\bar{\alpha}\delta\bar{\gamma}}$, and the components of R^1 and ρ are given by $R^1_{\alpha\bar{\beta}} = \Sigma_\gamma R_{\alpha\bar{\beta}\gamma\bar{\gamma}}$ and $\rho = 2\Sigma_\alpha R^1_{\alpha\bar{\alpha}} = 2\Sigma_{\alpha,\beta} R_{\alpha\bar{\alpha}\beta\bar{\beta}}$.

With respect to a self-dual Kähler surface, we obtain the following lemma.

LEMMA 2.2: *Let (M, g) be a Kähler surface. Then, it is self-dual if and only if the components of the Riemannian curvature tensor R satisfy that*

$$R_{1\bar{1}1\bar{1}} + R_{2\bar{2}2\bar{2}} - 4R_{1\bar{1}2\bar{2}} = 0 \text{ (that is, } \rho = 12R_{1\bar{1}2\bar{2}})$$

$$R_{1\bar{1}1\bar{2}} = R_{2\bar{2}1\bar{2}} \quad \text{and} \quad R_{1\bar{2}1\bar{2}} = R_{2\bar{1}2\bar{1}} = 0 \tag{2.4}$$

for any unitary basis $\{E_1, E_2\}$.

PROOF OF LEMMA 2.2: We have from formula (2.1) that

$$W_{1\bar{1}1\bar{1}} = R_{1\bar{1}1\bar{1}} - R^1_{1\bar{1}} + \tfrac{1}{6}\rho \quad \text{and} \quad W_{2\bar{2}1\bar{1}} = R_{2\bar{2}1\bar{1}}.$$

Suppose that (M, g) is self-dual. Then from (2.2) we have $\frac{1}{6}\rho = 2R_{2\bar{2}1\bar{1}}$, which is the first formula. The second of (2.4) is obtained from $W_{1\bar{1}1\bar{2}} = W_{2\bar{2}1\bar{2}}$. To show the last we define a new unitary basis $\{E_1^\theta, E_2^\theta\}$ with real parameter θ, by $E_1^\theta = \cos\theta E_1 + \sin\theta E_2$ and $E_2^\theta = -\sin\theta E_1 + \cos\theta E_2$. Since $\rho = 12R_{1\bar{1}2\bar{2}}$ holds also for this basis, by differentiating this with respect to θ twice and setting $\theta = 0$ we have that $2(R_{1\bar{1}1\bar{1}} + R_{2\bar{2}2\bar{2}} - 4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}) = 0$. Then we have that $R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}} = 0$, that is, $R_{1\bar{2}1\bar{2}}$ is pure imaginary for $\{E_1, E_2\}$. For a new unitary basis $\{e\sqrt{-1}\,\pi/4.E_1, E_2\}$ $R_{1\bar{2}1\bar{2}}$, which is also pure imaginary, is reduced to a real number. Hence we have that $R_{1\bar{2}1\bar{2}} = 0$ for $\{E_1, E_2\}$.

Conversely suppose that (M, g) satisfies (2.4). Then $W_{1\bar{1}1\bar{1}} = W_{2\bar{2}1\bar{1}}$ and $W_{1\bar{1}1\bar{2}} = W_{2\bar{2}1\bar{2}}$ hold from the first equalities of (2.4). Since other equalities $W_{1\bar{1}AB} = W_{2\bar{2}AB}$ and $W_{1\bar{2}AB} = 0$ are easily obtained, W is self-dual from (2.2).

PROOF OF THEOREM 1: Since g is Einstein, $R^1_{\alpha\bar{\beta}} = R_{1\bar{1}\alpha\bar{\beta}} + R_{2\bar{2}\alpha\bar{\beta}} = \rho/4\delta_{\alpha\beta}$ for any unitary basis $\{E_1, E_2\}$. Then we have from the above lemma that

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0 \text{ except for } R_{1\bar{1}1\bar{1}} = R_{2\bar{2}2\bar{2}} = 2R_{1\bar{1}2\bar{2}}.$$

Therefore R has the form $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = c(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta})$ where $c = g(R(E_1, \overline{E_1})E_1, \overline{E_1})$, which may be a local function. With respect to local holomorphic coordinates z^1 and z^2 the components of R can be written as

$$g\big(R(\partial/\partial z\alpha, \partial/\partial\bar{z}\beta, \partial/\partial z\gamma, \partial/\partial\bar{z}\delta\big) = c\big(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\big)$$

where $g_{\alpha\bar{\beta}} = g(\partial/\partial z\alpha, \partial/\partial\bar{z}\beta)$. Covariant-differentiating this with respect to $\partial/\partial z^{\varepsilon}$ and applying the second Bianchi's identity $\nabla_{\varepsilon} R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nabla_{\alpha} R_{\varepsilon\bar{\beta}\gamma\bar{\delta}}$ we conclude that c must be constant. Hence from Proposition 7.6 in [10] (M, g) is a Kähler surface of constant holomorphic curvature.

In the last half part of this section we shall show Theorem 2

PROOF OF THEOREM 2: Let (M, g) be an anti-self-dual Kähler surface. Then from (2.3) $W_{12AB} = 0$ for any A and B. If we set $A = \bar{1}$ and $B = \bar{2}$ in W_{12AB}, then $W_{12\bar{1}\bar{2}} = \frac{1}{2}(R^{1}_{\bar{1}\bar{1}} + R^{1}_{\bar{2}\bar{2}}) - \frac{1}{6}\rho = \frac{1}{6}\rho$, hence we have that $\rho = 0$. Suppose conversely that the scalar curvature ρ vanishes identically. Since $R_{12CD} = 0$, $W_{12CD} = -\frac{1}{2}(g_{1D}R^{1}_{C2} - g_{2D}R^{1}_{C1} + R^{1}_{1D}g_{C2} - R^{1}_{2D}g_{C1})$. If we set $C = \bar{1}$ and $D = \bar{2}$ in this representation, then $W_{12\bar{1}\bar{2}} = \frac{1}{2}(R^{1}_{1\bar{1}} + R^{1}_{2\bar{2}}) = \frac{1}{4}\rho = 0$. We have easily that $W_{12CC} = 0$ and $W_{1212} = 0$. That $W_{1\bar{1}CD} + W_{2\bar{2}CD} = 0$ for any C and D is shown as follows. For $C = 1$ and $D = 2$ we have that $W_{1\bar{1}12} + W_{2\bar{2}12} = -\frac{1}{2}(R^{1}_{21} - R^{1}_{12}) = 0$. We also obtain that $W_{1\bar{1}1\bar{1}} + W_{2\bar{2}1\bar{1}} = R_{1\bar{1}1\bar{1}} + R_{2\bar{2}1\bar{1}} - \frac{1}{2}(R^{1}_{1\bar{1}} + R^{1}_{1\bar{1}}) = 0$ and $W_{1\bar{1}1\bar{2}} + W_{2\bar{2}1\bar{2}} = R_{1\bar{1}1\bar{2}} + R_{2\bar{2}1\bar{2}} - \frac{1}{2}(R^{1}_{21} + R^{1}_{1\bar{2}}) = 0$. Similarly we have that $W_{1\bar{1}CD} + W_{2\bar{2}CD} = 0$ for $(C, D) = (\bar{1}, \bar{2})$, $(\bar{1}, 1)$ and $(\bar{1}, 2)$.
Hence W is anti-self-dual.

3. Global aspects of anti-self-dual Kähler surfaces

Let (M, g) be a compact oriented Riemannian 4-manifold. Then its signature $\tau(M)$ is written as $\tau(M) = b^{+} - b^{-}$ where b^{+} and b^{-} are given by the dimension of the space of real, self-dual harmonic 2-forms and the dimension of the space of real, anti-self-dual harmonic 2-forms, respectively.

Now let (M, g) be a compact Kähler surface. Let Δ and $\Box$ be the real Laplace-Beltrami operator and the complex Laplace-Beltrami operator defined on the space $\Gamma(\Lambda^{k})$ of smooth k-forms, respectively. We notice that $\Delta = 2\Box$ and $\Box$ preserves type of k-forms.

LEMMA 3.1: *Let (M, g) be a compact Kähler surface of nonnegative scalar curvature. Then each holomorphic form of type (2,0) is parallel. Moreover if the scalar curvature is positive at some point, then the geometric genus is equal to zero and $b^{+} = 1$.*

PROOF: Let σ be a holomorphic form of type (2,0). Note that σ is a global section of the canonical line bundle K. From Proposition 5.5 in [12] and $\Box = \bar{\Box}$, $\sigma = 1/2\Sigma\sigma_{\mu\nu}dz^{\mu}\Lambda dz^{\nu}$ is $\bar{\Box}$-harmonic. From the following formula, similar to Theorem 6.1 in [12]

$$(\bar{\Box}\sigma)_{\mu\nu} = -\sum g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\nabla_{\alpha}\sigma_{\mu\nu} + \rho/2\sigma_{\mu\nu}, \tag{3.1}$$

we have that

$$0=\int_M\Big(\sum g^{\mu\bar{\varepsilon}}g^{\nu\bar{\lambda}}g^{\alpha\bar{\beta}}\nabla_\alpha\sigma_{\mu\nu}\overline{\nabla_\beta\sigma_{\varepsilon\lambda}}\Big)\mathrm{d}v_g+\int_M\frac{\rho}{2}\Big(\sum g^{\mu\bar{\varepsilon}}g^{\nu\bar{\lambda}}\sigma_{\mu\nu}\overline{\sigma_{\varepsilon\lambda}}\Big)\mathrm{d}v_g. \tag{3.2}$$

Here ρ denotes the scalar curvature of g. Since ρ is nonnegative, $\nabla_\alpha\sigma_{\mu\nu}=0$. Therefore σ is indeed parallel because $\nabla_{\bar{\alpha}}\sigma_{\mu\nu}=0$. If ρ is positive at some point, then $\sigma=0$, that is, the geometric genus $p_g=\dim H^0(M,K)$ is zero. To verify $b^+=1$ it suffices to show that each real, self-dual harmonic 2-form is proportional to the Kähler form Ω. Let ϕ be a self-dual, real harmonic 2-form. Then from Proposition 2.1 ϕ is written by $\phi=\phi^{2,0}+(\phi^{2,0})^-+a\Omega$ where $\phi^{2,0}$ is a form of type (2,0) and a is a real smooth function. Since $\square\phi=0$, $\phi^{2,0}$ and $a\Omega$ are $\square$-harmonic. From the above we have that $\phi^{2,0}=0$. Since $\square(a\Omega)=(\square a)\Omega$, a must be constant. Thus the lemma is verified.

REMARK: From this lemma, every compact, anti-self-dual Kähler surface has trivial canonical line bundle if $p_g>0$.

The following is given as a remark in §1. We give a proof here.

LEMMA 3.2: *Every compact, anti-self-dual Kähler surface satisfies that $c_1(M)^2[M]\leqslant 0$ ($=0$ if and only if g is Ricci flat) and its Ricci form γ is anti-self-dual and harmonic.*

PROOF: Since the scalar curvature vanishes, γ is anti-self-dual from Remark of Proposition 2.1. Then from (1.1) we have $\gamma\wedge\gamma=-|\gamma|_g^2\mathrm{d}v_g$, hence $c_1(M)^2[M]=-1/(4\pi^2)\int_M|\gamma|_g^2\mathrm{d}v_g$. With respect to the formal adjoint ϑ of ∂ we have that $(\vartheta\gamma)_\alpha=\sum g^{\sigma\bar{\tau}}\nabla_\sigma R^1_{\alpha\bar{\tau}}=\nabla_\alpha(\rho/2)=0$. Hence we have the lemma.

REMARK: By a slight consideration we have the following. Let (M,g) be a compact Kähler surface whose total scalar curvature is zero. Then K_M admits a hermitian metric whose Ricci form is anti-self-dual and also $c_1(M)^2[M]\leqslant 0$.

PROOF OF THEOREM 3: Since a conformally flat, compact Kähler surface (M,g) is also self-dual and has zero signature, Theorem 3 is a corollary of Theorem C. In fact, if (M,g) is not a flat torus, then it is a compact quotient of $P_1(\mathbf{C})\times D^1$ with the metric. By an easy argument (M,g) is a holomorphic bundle over a complex curve (C_1,g_1), $C_1=D^1/\Gamma$ of genus k ($\geqslant 2$) with fibre $(P_1(\mathbf{C}),g_2)$ whose projection is a Kählerian submersion, that is, (M,g) is a Kählerian ruled surface of genus k.

REMARKS: (i) From this theorem and Theorem C we obtain that on each conformally flat, compact Kähler surface which is not flat there is one parameter family $\{g_t\}$ of Kähler metrics of constant scalar curvature c_t, where c_t takes any real value.

(ii) We can exhibit an example of Kählerian ruled surface of genus $\geqslant 2$ which is a nontrivial holomorphic $P_1(\mathbf{C})$-bundle.

PROOF OF THEOREM 4: As was shown above, a compact, anti-self-dual Kähler surface (M, g) has topological restrictions $3\tau(M) = c_1(M)^2[M] - 2c_2(M)[M] \leqslant 0$ and $c_1(M)^2[M] \leqslant 0$. Then compact, anti-self-dual Kähler surfaces are divided into four classes. If $\tau(M) = 0$ and $c_1(M)^2[M] = 0$, then $W = 0$ and $R^1 = 0$, hence g is flat, that is (M, g) is a Kählerian flat torus. In the similar manner we obtain that (M, g) with $c_1(M)^2[M] = 0$ and $\tau(M) < 0$ is covered by a $K3$ surface of a Ricci flat metric. If $\tau(M) = 0$ and $c_1(M)^2[M] < 0$, then (M, g) is conformally flat, but not flat. Then from Theorem 3 (M, g) is a Kählerian ruled surface of genus k $(\geqslant 2)$. Now let (M, g) be a compact, anti-self-dual Kähler surface with $\tau(M) < 0$ and $C_1(M)^2[M] < 0$. If there is a non-trivial holomorphic 2-form on M, then from Lemma 3.1 it is parallel, hence it never vanishes, so the canonical line bundle K is trivial. As a consequence we have $c_1(M) = 0$. But this contradicts to $c_1(M)^2[M] < 0$. Therefore the geometric genus p_g is zero. Let M_0 be a relatively minimal complex surface such that M is obtained by blowing up M_0. Then p_g of M_0 is also zero. By Kodaira's classification theorem [11] M_0 is either $P_2(\mathbf{C})$ or a ruled surface of genus k (i.e., a holomorphic $P_1(\mathbf{C})$-bundle over a complex curve of genus k). Since $c_2 = 3$ and $c_1^2 = 9$ for $P_2(\mathbf{C})$, and $c_2 = 4(1-k)$ and $c_1^2 = 8(1-k)$ for a ruled surface of genus k, and blowing up one point increases c_2 one and decreases τ and c_1^2 one, M is obtained by blowing up $P_2(\mathbf{C})$ at least 10 times, a ruled surface of genus 0 at least 9 times or a ruled surface of genus $k(\geqslant 1)$ at least once.

REMARKS: (i) On each ruled surface there exists a Kähler metric of positive scalar curvature [15]. Every compact complex surface, obtained by blowing up a ruled surface several times admits a Hodge metric whose total scalar curvature is positive [15].

(ii) On the other hand, each ruled surface M of genus $k \geqslant 2$ over a complex curve can be endowed with a Kähler metric of negative scalar curvature under a certain condition. By its definition, M is a holomorphic $P_1(\mathbf{C})$-bundle over D^1/Γ of genus k, where $\Gamma \subset \mathrm{Hol}(D^1)$. Let π; $D^1 \to D^1/\Gamma$ be the covering map. Since π^*M over D^1 is trivial as a smooth bundle and D^1 is Stein, π^*M is also trivial as a holomorphic bundle. Then there is a homomorphism ρ; $\Gamma \to \mathrm{Hol}(P_1(\mathbf{C}))$ such that $M = D^1 \times_\rho P_1(\mathbf{C})$. Since $\mathrm{Hol}(D^1) \subset \mathrm{Aut}(D^1, \bar{g}_1)$ with respect to the Poincare metric $\bar{g}_1$, D^1/Γ admits a metric g_1, locally isomorphic to $\bar{g}_1$. Assume that $\Gamma/\mathrm{Ker}\,\rho$, isomorphic to $\mathrm{Im}\,\rho$, is a finite subgroup. Then $g_2 = \sum_{\rho(\alpha) \in \mathrm{Im}\,\rho} \rho(\alpha)^* \bar{g}_2$ defines a Γ-invariant Kähler metric on $P_1(\mathbf{C})$, where $\bar{g}_2$ is a standard metric. Then M admits a one parameter family of Kähler metrics $\{g_t\}$ $(t > 0)$, locally isometric to $\bar{g}_1 \oplus t\bar{g}_2$. If t tends to the infinity, then the scalar curvature becomes negative everywhere. Note that for sufficiently large t g_t is considered as a Hodge metric. Therefore by the same argument in the proof of Proposition 3 in [16], on any compact complex surface $\tilde{M}$, obtained by blowing up M there is a Hodge metric of negative total scalar curvature.

(iii) Let M be a ruled surface of genus $k \geqslant 2$ with $\#\ \mathrm{Im}\,\rho < +\infty$. Then any complex surface $\tilde{M}$, obtained by blowing up M, admits a Kähler metric whose total scalar curvature is zero by the aid of the above remarks.

(iv) Let M be a complex surface, obtained by blowing up either $P_2(\mathbb{C})$ k times $(k \leqslant 9)$ or $P_1(\mathbb{C}) \times P_1(\mathbb{C})$ j t imes $(j \leqslant 8)$. Then $c_1(M)^2[M] \geqslant 0$. Therefore by Remark of Lemma 3.1 M can not admit any Kähler metric whose total scalar curvature is zero.

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