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EXAMPLES OF NON-AMPLE NORMAL BUNDLES

Norman Goldstein

§1. Introduction

Let \( Z = \text{Gr}(1, \mathbb{P}^3) \) be the Grassmannian of 1 planes in \( \mathbb{P}^3 \), embedded as a quadric hypersurface in \( \mathbb{P}^5 \). In this note, I construct in \( Z \) two smooth surfaces, \( X_3 = \mathbb{P}^2 \) blown up at one point and \( X_4 = \mathbb{P}^1 \times \mathbb{P}^1 \), of degrees 3 and 4 respectively, such that for each, the normal bundle in \( Z \) is not ample. This is in apparent contradiction to a result of Papantonopoulou [4], stating that any smooth surface in \( Z \), having a non-ample normal bundle, must be a linear \( \mathbb{P}^2 \).

I do know of other examples, other than linear \( \mathbb{P}^2 \)’s, although, I will not discuss these, here. The reader is referred to Hartshorne [3] for the definition of ampleness.

In a future paper, I will discuss the motivation for these constructions. It is described, briefly, below, and involves a geometric interpretation of the tangent and cotangent bundles of \( Z \); this generalizes to the \( r \)-dimensional quadric \( Z' \subset \mathbb{P}^{r+1} \). Also, the problem of which \( m \)-dimensional submanifolds \( X'' \subset Z' \) (\( m \geq 2 \)) have non-ample normal bundles reduces, largely, via hyperplane sections, to the study of smooth surfaces in \( Z^{r-m+2} \), whose normal bundles are not ample.

We consider the 4-dimensional quadric, \( Z \). Let \( \mathbb{P}(T^*Z) = (T^*Z \setminus Z)/\mathbb{C}^* \). The sections of \( TZ \) induce a map

\[ \phi : \mathbb{P}(T^*Z) \to \mathbb{P}^n. \]

Let \( X \subset Z \) be a surface, and \( NX \) the normal bundle of \( X \). Let \( \phi' \) denote the restriction of \( \phi \) to \( NX \). The normal bundle, \( NX \), is ample when \( \phi' \) is finite to 1. So, \( NX \) can fail to be ample in 3 ways:

(a) \( \dim \phi'(\mathbb{P}(N^*X)) = 2 \)
(b) \( \dim \phi'(\mathbb{P}(N^*X)) = 3 \) and \( \phi' \) has an infinity of positive dimensional fibres, or
(c) \( \dim \phi'(\mathbb{P}(N^*X)) = 3 \) and \( \phi' \) has only a finite number of positive dimensional fibres.

All these possibilities can occur. The possibility (a) happens when \( X \) is a linear \( \mathbb{P}^2 \) (cf [4]), and (b) when \( X = X_4 \), as in §3; in a future paper, we will see that for (a) and (b), that these are the only possibilities. Finally,
the example $X = X_3$ in §4 shows that (c) can occur with a single positive dimensional fibre.

I would like to thank Andrew Sommese for suggesting that I look at this topic, and for ways of viewing the problem. Towards completing the characterization of $X_4$, I had helpful conversations with Gary Kennedy, Daniel Phillips and Avinash Sathaye. Also, the referee’s suggestions have improved the presentation of this paper.

2. Notation and background material

(2.1) Let $A$ be a submanifold of the manifold $B$. I denote the normal bundle of $A$ in $B$ as $N(B/A) := TB/TA$. Its dual, the conormal bundle, is

$$N^*(B/A) = \{ \alpha \in T^*B : \alpha = 0 \text{ on } TA \}.$$ 

(2.2) Let $[z_0, \ldots, z_5]$ be homogeneous coordinates in $\mathbb{P}^5$, and let $\mathcal{O}(-1)$ be the tautological line bundle; it is the one for which the transition function from the patch $z_i \neq 0$ to the path $z_j \neq 0$ is given by multiplication by $z_jz_i^{-1}$. We consider the well-known Euler sequence (see e.g. [2] p. 409).

$$0 \to T^*(\mathbb{P}^5) \to \mathcal{O}(-1)^{\otimes 6} \to \mathcal{O} \to 0.$$ 

If $\alpha = (\alpha_0, \ldots, \alpha_5) \in T^*(\mathbb{P}^5)$ is represented in $z_i \neq 0$ by $a = (a_0, \ldots, a_5) \in \mathbb{C}^6$, then in $z_j \neq 0$ $\alpha$ is represented by $z_jz_i^{-1}a$.

(2.3) Let $Y$ be a projective subvariety of $\mathbb{P}^5$ and $E \to Y$ a holomorphic vector bundle spanned by global sections. Then, according to Gieseker ([1] Proposition 2.1), $E$ is not ample precisely when there exists a curve $C \subset Y$ and a trivial bundle $\mathcal{O}_C \subset E|_C$.

A “line” in $\mathbb{P}^5$ denotes a linear $\mathbb{P}^1$.

§3. The surface $X_4$

Let $\gamma_1, \gamma_2 : \mathbb{P}^1 \to \mathbb{P}^5$

$$\gamma_1(s, t) = (s^2, 2st, 2t^2, 0, 0, 0)$$

$$\gamma_2(s, t) = (0, 0, 0, s^2, 2st, 2t^2).$$

Let $X = X_4$ be the scroll surface in $\mathbb{P}^5$ determined by $\gamma_1$ and $\gamma_2$, i.e. $X$ is the union of the lines obtained by joining corresponding points of $\gamma_1$ and $\gamma_2$. Since $\gamma_1(\mathbb{P}^1)$ and $\gamma_2(\mathbb{P}^1)$ are contained in disjoint linear spaces, $X$ is smooth. Also, the equation for $Z \subset \mathbb{P}^5$ is $z_0z_5 - z_1z_4 + z_2z_3 = 0$, so that $X \subset Z$, as is easily checked.
Consider the exact sequence of conormal spaces (see (2.1) for notation)

\[ 0 \to N^*(\mathbb{P}^5/Z) \to N^*(\mathbb{P}^5/X) \to N^*(Z/X) \to 0. \]

Let \( \mathcal{L} \) be any line of the ruling of \( X \). We construct a "section"

\[ \sigma: \mathcal{L} \to N^*(\mathbb{P}^5/X) \]

which is both well defined and nowhere zero, modulo \( N^*(\mathbb{P}^5/Z) \). This, then, determines a nowhere zero section

\[ \sigma: \mathcal{L} \to N^*(Z/X) \]

i.e. a trivial bundle \( \mathcal{O}_\mathcal{L} \subset N^*(Z/X) \) so that, by (2.3), \( N(Z/X) \) is not ample. We remark that \( N(Z/X) \) is spanned by global sections since \( T_Z \) is spanned by global sections (\( Z \) is a homogeneous space), and \( N(Z/X) \) is a quotient bundle of \( T_Z \).

Construction of \( \sigma \): Let \( C \times \mathbb{P}^1 \to X \) be the patch \( (t, [\lambda, \mu]) \to (\lambda, 2t\lambda, 2t^2\lambda, \mu, 2t\mu, 2t^2\mu) \). The tangent \( \mathbb{P}^2 \) to \( X \) at \( (t, [\lambda, \mu]) \) is spanned by \((1, 2t, 2t^2, 0, 0, 0), (0, 0, 0, 1, 2t, 2t^2) \) and \((0, \lambda, 2t\lambda, 0, \mu, 2t\mu) \).

Consider the line \( \mathcal{L} \) corresponding to some fixed \( t \). On the patch \( \lambda \neq 0 \), let \( \sigma = (2t^2, -2t, 1, 0, 0, 0) \). Then, \( \sigma \in N^*_t(\mathbb{P}^5/X) \) since \( \sigma \) vanishes on the tangent \( \mathbb{P}^2 \) to \( X \). On \( \mu \neq 0 \), by (2.2), \( \sigma = \mu \lambda^{-1}(2t^2, -2t, 1, 0, 0, 0) \). But, \( N^*(\mathbb{P}^5/Z) \) is spanned by \((z_5, -z_4, z_3, z_2, -z_1, z_0) = (2t^2\mu, -2t\mu, \mu, 2t^2\lambda, -2t\lambda, \lambda) \). Thus \( \sigma = -(0, 0, 0, 2t^2, -2t, 1) \) modulo \( N^*(\mathbb{P}^5/Z) \) and is, therefore, a global nowhere zero section of \( N^*(Z/X) \) over \( \mathcal{L} \). Q.E.D.

Except for linear \( \mathbb{P}^2 \)'s and up to an automorphism of \( Z \), \( X_4 \) is the unique surface in \( Z \) with a non-ample normal bundle, and which contains a positive dimensional family of curves, along each of which \( N(Z/X) \) is not ample. The proof of this will appear elsewhere.

§4. The surface \( X_3 \)

Let \( \delta_1, \delta_2: \mathbb{P}^1 \to \mathbb{P}^5 \),

\[ \delta_1(s, t) = (s, t, 0, 0, 0, 0) \]

\[ \delta_2(s, t) = (0, 0, 0, s^2, st, t^2) \]

and \( X = X_3 \subset Z \) the scroll surface determined by \( \delta_1 \) and \( \delta_2 \). The line \( t = 0 \) is the only curve \( l \subset X \) for which there is a trivial bundle \( \mathcal{O}_l \subset N^*(Z/X)_l \).
I defer the proof of this to a future paper. The construction of $\sigma$ is similar to that in §3 for $X_4$:

$$\text{on } \lambda \neq 0, \quad \sigma = (0, 0, 0, 0, 0, 1), \quad \text{and}$$

$$\text{on } \mu \neq 0, \quad \sigma = -(0, 0, 1, 0, 0, 0).$$

References


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