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## NON-ARCHIMEDEAN (UNIFORMLY) CONTINUOUS MEASURES ON HOMOGENEOUS SPACES

Luc Duponcheel

### I. Introduction

Let  $\mathcal{K}$  be a complete non-archimedean valued field. Let  $G$  be a locally compact zero-dimensional group. If  $G$  has an open compact  $q$ -free subgroup, where  $q$  is the characteristic of the residue class field of  $\mathcal{K}$ , then for every closed subgroup  $H$  of  $G$  the homogeneous space of all left cosets of  $H$  in  $G$  has a  $\mathcal{K}$ -valued quasi-invariant measure  $\mu$  (see [4]). Let  $M^\infty(G/H)$  be the space of all measures on  $G/H$ . In this paper we will prove the following result: an element  $\lambda$  of  $M^\infty(G/H)$  translates continuously for the strong (resp. weak) topology on  $M^\infty(G/H)$  if and only if  $\lambda = f\mu$  where  $f$  is a bounded uniformly continuous (resp. continuous) function on  $G/H$ .

### II. Functions and measures

Let  $\mathcal{K}$  be a complete non-archimedean valued field. Let  $X$  be a locally compact zero-dimensional space. Let  $B_c(X)$  be the ring of all open compact subsets of  $X$ . If  $A$  is an element of  $B_c(X)$  then  $\xi(A)$  will be the  $\mathcal{K}$ -valued characteristic function of  $A$ . Let  $BC(X)$  be the space of all bounded continuous  $\mathcal{K}$ -valued functions on  $X$ . With the norm  $\|f\| = \sup_{x \in X} |f(x)|$ , ( $f \in BC(X)$ ),  $BC(X)$  becomes a non-archimedean Banach space over  $\mathcal{K}$ . With the seminorms  $p_x(f) = |f(x)|$ , ( $f \in BC(X)$ ;  $x \in X$ ),  $BC(X)$  becomes a non-archimedean locally convex space over  $\mathcal{K}$ . The topology (resp. uniformity) on  $BC(X)$  induced by the norm  $\|\cdot\|$  will be called the uniform topology (resp. uniformity). The topology (resp. uniformity) on  $BC(X)$  induced by the seminorms  $p_x$  ( $x \in X$ ) will be called the pointwise topology (resp. uniformity). Let  $BUC(X)$  be the space of all bounded  $\mathcal{U}$ -uniformly continuous  $\mathcal{K}$ -valued functions on  $X$  where  $\mathcal{U}$  is a non-archimedean uniformity on  $X$  compatible with the topology on  $X$ . Let  $C_\infty(X)$  be the space of all continuous  $\mathcal{K}$ -valued functions on  $X$  which vanish at infinity. Both  $BUC(X)$  and  $C_\infty(X)$  are subspaces of  $BC(X)$ . In the same way as  $BC(X)$ ,  $BUC(X)$  and  $C_\infty(X)$  can be made into a non-archimedean Banach space and a locally convex space over  $\mathcal{K}$ . Let  $M^\infty(X)$  be

the space of all *bounded additive*  $\mathcal{X}$ -valued functions on  $B_c(X)$ . The elements of  $M^\infty(X)$  will be called *measures* on  $X$ . With the norm  $\|\lambda\| = \sup_{A \in B_c(X)} |\lambda(A)|$  ( $\lambda \in M^\infty(X)$ ),  $M^\infty(X)$  becomes a non-archimedean Banach space over  $\mathcal{X}$ . ( $M^\infty(X)$ ,  $\|\cdot\|$ ) is the *dual space* of  $(C_\infty(X), \|\cdot\|)$ . The duality is given by

$$(f, \lambda) \rightarrow \int_X f d\lambda = \int_X f(x) d\lambda(x) \quad (f \in C_\infty(X), \lambda \in M^\infty(X)).$$

Let  $\lambda$  be an element of  $M^\infty(X)$ . If for every  $x$  in  $X$  we define  $N_\lambda(x) = \inf_{A \ni x} \sup_{B \subset A} |\lambda(B)|$  then for every  $A$  in  $B_c(X)$  we have  $\sup_{B \subset A} |\lambda(B)| = \sup_{x \in A} N_\lambda(x)$ . In particular we have  $\|\lambda\| = \sup_{x \in X} N_\lambda(x)$ . With the seminorms  $q_x(\lambda) = N_\lambda(x)$  ( $x \in X, \lambda \in M^\infty(X)$ ),  $M^\infty(X)$  becomes a non-archimedean locally convex space over  $\mathcal{X}$ . The topology (resp. uniformity) on  $M^\infty(X)$  induced by the norm  $\|\cdot\|$  will be called the *strong* topology (resp. uniformity). The topology (resp. uniformity) on  $M^\infty(X)$  induced by the seminorms  $q_x$  ( $x \in X$ ) will be called the *weak* topology (resp. uniformity).

If  $\lambda$  is an element of  $M^\infty(X)$  and  $g$  is an element of  $BC(X)$  we can define an element  $g\lambda$  of  $M^\infty(X)$  by setting:

$$\int_X f d g\lambda = \int_X f g d\lambda \quad (f \in C_\infty(X)).$$

It is not difficult to see that  $N_{g\lambda}(x) = |g(x)|N_\lambda(x)$ .

**III. PROPOSITION (1):** *Let  $\lambda$  be an element of  $M^\infty(X)$  with  $\inf_{x \in X} N_\lambda(x) \geq m > 0$ . The map  $g \rightarrow g\lambda$  is a linear homeomorphism from  $BC(X)$  with the uniform (resp. pointwise) topology onto a closed subspace of  $M^\infty(X)$  with the strong (resp. weak) topology.*

**PROOF:** It is clear that the map  $g \rightarrow g\lambda$  from  $BC(X)$  with the uniform (resp. pointwise) topology to  $M^\infty(X)$  with the strong (resp. weak) topology is a linear homeomorphism. We only need to prove that its image is closed.

For the the uniform topology on  $BC(X)$  and the strong topology on  $M^\infty(X)$  this is trivial.

For the pointwise topology on  $BC(X)$  and the weak topology on  $M^\infty(X)$  this runs as follows: let  $(f_\alpha \lambda)_{\alpha \in I}$  be a net in  $M^\infty(X)$  converging weakly to  $\mu$ . It is clear that the net  $(f_\alpha)_{\alpha \in I}$  in  $BC(X)$  converges pointwise to a function  $f$ . We have

$$N_{f(x)\lambda - \mu}(x) \leq \max [N_{f(x)\lambda - f_\alpha \lambda}(x), N_{f_\alpha \lambda - \mu}(x)] \quad (\alpha \in I, x \in X).$$

Therefore

$$\begin{aligned}
 N_{f(x)\lambda-\mu}(x) &= 0 \quad (x \in X). \\
 |f(x)| &\leq \frac{1}{m} |f(x)| N_\lambda(x) = \frac{1}{m} N_{f(x)\lambda}(x) \\
 &\leq \frac{1}{m} \max[N_{f(x)\lambda-\mu}(x), N_\mu(x)] \leq \frac{N_\mu(x)}{m} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 |f(x) - f(y)| &\leq \frac{1}{m} |f(x) - f(y)| N_\lambda(y) = \frac{1}{m} N_{f(x)\lambda-f(y)\lambda}(y) \\
 &\leq \frac{1}{m} \max[N_{f(x)\lambda-\mu}(y), N_{f(y)\lambda-\mu}(y)] \\
 &= \frac{1}{m} N_{f(x)\lambda-\mu}(y). \quad (2)
 \end{aligned}$$

From the inequalities (1) and (2) it is easy to conclude that  $f \in BC(X)$ . (Notice that the function  $N_{f(x)\lambda-\mu}$  is uppersemicontinuous and  $N_{f(x)\lambda-\mu}(x) = 0$ ). Now

$$N_{f\lambda-\mu}(x) \leq \max[N_{f\lambda-f(x)\lambda}(x), N_{f(x)\lambda-\mu}(x)] = 0$$

and we may conclude that  $\mu = f\lambda$  and we are done.

#### IV. Groups and homogeneous spaces

Let  $G$  be a *locally compact zerodimensional group*. For every *open compact subgroup*  $K$  of  $G$  we can define an equivalence relation on  $G$  by setting:

$$s \sim t \Leftrightarrow sK = tK \quad (\text{resp. } s \sim t \Leftrightarrow Ks = Kt) \quad (s, t \in G).$$

Using those equivalence relations we can define the so called *left (resp. right) group uniformity* on  $G$  which is compatible with the topology on  $G$ .

Let  $BLUC(G)$  (resp.  $BRUC(G)$ ) be the space of all *bounded left (resp. right) uniformly continuous*  $\mathcal{X}$ -valued functions on  $G$ . If  $f$  is an element of  $BC(G)$  and  $s$  is an element of  $G$ , we can define an element  $R_s f$  (resp.  $L_s f$ ) of  $BC(G)$  by setting:

$$R_s f(t) = f(ts) \quad (\text{resp. } L_s f(t) = f(st)) \quad (t \in G).$$

If  $f$  is an element of  $BC(G)$  then the functions  $s \rightarrow L_s f$  and  $s \rightarrow R_s f$  from  $G$  to  $BC(G)$  are continuous for the pointwise topology on  $BC(G)$ .

An element  $f$  of  $BC(G)$  is an element of  $BLUC(G)$  (resp.  $BRUC(G)$ )

if and only if the function  $s \rightarrow R_s f$  (resp.  $s \rightarrow L_s f$ ) from  $G$  to  $BC(G)$  is continuous for the uniform topology on  $BC(G)$ . It is important to notice that  $C_\infty(G)$  is a subspace of  $BLUC(G)$  (resp.  $BRUC(G)$ ).

Let  $H$  be a closed subgroup of  $G$ . Let  $G/H$  be the set of all *left cosets* of  $H$  in  $G$ . Let  $\pi: G \rightarrow G/H$  be the *natural quotient map* from  $G$  to  $G/H$ . With the quotient topology  $G/H$  also becomes a locally compact zero-dimensional space.  $G$  has an *action* on  $G/H$  by setting:

$$s(\pi(t)) = \pi(st) \quad (s \in G, \pi(t) \in G/H).$$

For every open compact subgroup  $K$  of  $G$  we can define an equivalence relation on  $G/H$  by setting:

$$\pi(s) \sim \pi(t) \Leftrightarrow \pi(Ks) = \pi(Kt) \quad (\pi(s), \pi(t) \in G/H).$$

Using those equivalence relations we can define the so called *homogeneous uniformity* on  $G/H$  which is compatible with the topology on  $G/H$ . Let  $BUC(G/H)$  be the space of all *bounded uniformly continuous*  $\mathcal{X}$ -valued functions on  $G/H$ . If  $f$  is an element of  $BC(G/H)$  and  $s$  is an element of  $G$  we can define an element  $L_s f$  of  $BC(G/H)$  by setting:

$$L_s f(\pi(t)) = f(\pi(st)) \quad (\pi(t) \in G/H).$$

If  $f$  is an element of  $BC(G/H)$  then the function  $s \rightarrow L_s f$  from  $G$  to  $BC(G/H)$  is continuous for the pointwise topology on  $BC(G/H)$ . An element  $f$  of  $BC(G/H)$  is an element of  $BUC(G/H)$  if and only if the function  $s \rightarrow L_s f$  from  $G$  to  $BC(G/H)$  is continuous for the uniform topology on  $BC(G/H)$ . It is important to notice that  $C_\infty(G/H)$  is a subspace of  $BUC(G/H)$ .

## V. Quasi-invariant measures on homogeneous spaces

Let  $G$  be a *locally compact zero-dimensional group*. Let  $H$  be a *closed subgroup* of  $G$ . Let  $G/H$  be the *homogeneous space* of all left cosets of  $H$  in  $G$ . We suppose that  $G$  has an open compact  $q$ -free subgroup where  $q$  is the characteristic of the *residue class field* of  $\mathcal{X}$ . In that case  $G$  has an *invariant measure*  $m$ .  $H$  also has an *invariant measure*  $n$ . Let  $\Delta$  be the *modular function* on  $G$ . Let  $\delta$  be the *modular function* on  $H$ . Let  $f$  be an element of  $C_\infty(G)$ . We can define an element  $f^b$  of  $C_\infty(G/H)$  by setting:

$$f^b(\pi(s)) = \int_H f(st) dt \quad (\pi(s) \in G/H).$$

The map  $f \rightarrow f^b$  from  $C_\infty(G)$  to  $C_\infty(G/H)$  is linear and continuous and using duality we can define a linear and continuous map  $\lambda \rightarrow \lambda^\#$  from

$M^\infty(G/H)$  to  $M^\infty(G)$ . We have:

$$\int_G f d\lambda^\# = \int_{G/H} f^b d\lambda.$$

For every  $s$  in  $G$  we have  $N_\lambda(\pi(s)) = N_{\lambda^\#}(s)$ .

A quasi-invariant measure on  $G/H$  is a measure  $\mu$  such that  $\mu^\# = \rho m$ , where  $\rho$  is an invertible element of  $BRUC(G)$ . Such a measure does always exist and it is unique up to an invertible element of  $BUC(G/H)$ . We can even say more: for every open compact subgroup  $K$  of  $G$  there exists a quasi-invariant measure  $\mu$  on  $G/H$  with  $\mu^\# = \rho m$  where  $|\rho| \equiv 1$  and  $\rho$  is constant on the right cosets of  $K$  (see [4]).

### VI. (Uniformly) continuous measures

Let  $\lambda$  be an element of  $M^\infty(G/H)$  and  $s$  an element of  $G$ . We can define an element  $\lambda_s$  of  $M^\infty(G/H)$  by setting:

$$\int_{G/H} f d\lambda_s = \int_{G/H} L_s f d\lambda \quad (f \in C_\infty(G/H)).$$

An element  $\lambda$  of  $M^\infty(G/H)$  is called a *continuous* (resp. *uniformly continuous*) *measure* if the function  $s \rightarrow \lambda_s$  from  $G$  to  $M^\infty(G/H)$  is weakly (resp. strongly) continuous.

It is clear that the function  $s \rightarrow \lambda_s$  from  $G$  to  $M^\infty(G/H)$  is weakly (resp. strongly) continuous if and only if the function  $s \rightarrow (\lambda^\#)_s$  from  $G$  to  $M^\infty(G)$  is weakly (resp. strongly) continuous. In order to find the continuous (resp. uniformly continuous) measures on  $G/H$  it suffices to find the continuous (resp. uniformly continuous) measures on  $G$ .

Let  $(K_\alpha)_{\alpha \in I}$  be a fundamental system of open compact  $q$ -free subgroups of  $G$ . We can define functions  $(u(K_\alpha))_{\alpha \in I}$  from  $G$  to  $\mathcal{X}$  by setting

$$u(K_\alpha) = \frac{1}{m(K_\alpha)} \xi(K_\alpha).$$

VII. PROPOSITION (2): *Let  $\lambda$  be an element of  $M^\infty(G)$ . Let  $m$  be an invariant measure on  $G$ .  $\lambda$  is a uniformly continuous (resp. continuous) measure if and only if  $\lambda = fm$  for some element  $f$  of  $BRUC(G)$  (resp.  $BC(G)$ ).*

PROOF: (1) If  $\lambda = fm$  for some element  $f$  of  $BRUC(G)$  (resp.  $BC(G)$ ) then the function  $s \rightarrow \lambda_s$  from  $G$  to  $M^\infty(G)$  is strongly (resp. weakly) continuous as can easily be verified.

(2) Suppose now that the function  $s \rightarrow \lambda_s$  from  $G$  to  $M^\infty(G)$  is strongly (resp. weakly) continuous. For every  $f$  in  $C_\infty(G)$  we can define an element

$f * \lambda$  of  $M^\infty(G)$  by setting:

$$f * \lambda = \int_G f(s) \lambda_s ds$$

$$\left( \text{resp. } \int_G g df * \lambda = \int_G f(s) \left[ \int_G g d\lambda_s \right] ds \quad g \in C_\infty(G) \right).$$

We will prove that  $u(K_\alpha) * \lambda$  converges strongly (resp. weakly) to  $\lambda$ . (\*)

From this it follows (using Proposition 1) that it suffices to prove that for every  $f$  in  $C_\infty(G)$  there exists an element  $F$  of  $BRUC(G)$  (resp.  $BC(G)$ ) with  $f * \lambda = Fm$ . Define

$$F(s) = \int_G \Delta(t)^{-1} f(st^{-1}) d\lambda(t)$$

$$\|F\| = \sup_{s \in G} |F(s)| = \sup_{s \in G} \left| \int_G \Delta(t)^{-1} f(st^{-1}) d\lambda(t) \right|$$

$$\leq \|\lambda\| \sup_{s \in G} \sup_{t \in G} |\Delta(t)^{-1} f(st^{-1})| \leq \|\lambda\| \|f\|.$$

$$\|L_t F - F\| = \sup_{s \in G} |F(ts) - F(s)|$$

$$= \sup_{s \in G} \left| \int_G \Delta(q)^{-1} [f(ts q^{-1}) - f(s q^{-1})] d\lambda(q) \right|$$

$$\leq \|\lambda\| \sup_{s \in G} \sup_{q \in G} |\Delta(q)^{-1} [f(ts q^{-1}) - f(s q^{-1})]|$$

$$\leq \|\lambda\| \|L_t f - f\|.$$

It is clear that  $F$  is an element of  $BRUC(G)$  (in particular it is an element of  $BC(G)$ ). Let  $g$  be an element of  $C_\infty(G)$ .

$$\int_G g df * \lambda = \int_G \int_G f(s) g(t) d\lambda_s(t) ds$$

$$= \int_G \int_G f(s) g(st) d\lambda(t) ds$$

$$= \int_G \int_G f(s) g(st) ds d\lambda(t)$$

$$= \int_G \int_G \Delta(t)^{-1} f(st^{-1}) g(s) ds d\lambda(t)$$

$$\begin{aligned}
&= \int_G g(s) \left[ \int_G \Delta(t)^{-1} f(st^{-1}) d\lambda(t) \right] ds \\
&= \int_G g(s) F(s) ds \\
&= \int_G g dFm
\end{aligned}$$

and we may conclude that  $f * \lambda = Fm$ . We still need to prove (\*):

– For the strong topology this runs as follows:

$$\forall \epsilon > 0 \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha: \sup_{s \in K_\beta} \|\lambda_s - \lambda\| \leq \epsilon.$$

Using the inequality

$$\|u(K_\beta) * \lambda - \lambda\| = \left\| \frac{1}{m(K_\beta)} \int_{K_\beta} (\lambda_s - \lambda) ds \right\| \leq \sup_{s \in K_\beta} \|\lambda_s - \lambda\|$$

we see that

$$\forall \epsilon > 0 \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha: \|u(K_\beta) * \lambda - \lambda\| \leq \epsilon \text{ and we are done.}$$

– For the weak topology this runs as follows:

$$\forall t \in G \forall \epsilon > 0 \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha: \sup_{s \in K_\beta} N_{\lambda_s, -\lambda}(t) \leq \epsilon.$$

If  $p \in K_\beta t$  (say  $p = q^{-1}t$ ,  $q \in K_\beta$ ) then

$$\begin{aligned}
N_{\lambda_s, -\lambda}(p) &= N_{\lambda_s, -\lambda}(q^{-1}t) \\
&= N_{\lambda_{sq}, -\lambda_q}(t) \leq \max(N_{\lambda_{sq}, -\lambda}(t), N_{\lambda_q, -\lambda}(t))
\end{aligned}$$

and we see that

$$\forall t \in G \forall \epsilon > 0 \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha: \inf_{A \ni t} \sup_{s \in K_\beta} \sup_{p \in A} N_{\lambda_s, -\lambda}(p) \leq \epsilon$$

thus

$$\forall t \in G \forall \epsilon > 0 \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha: \inf_{A \ni t} \sup_{s \in K_\beta} \sup_{B \subset A} |\lambda_s(B) - \lambda(B)| \leq \epsilon.$$

Using the inequality

$$N_{u(K_\beta) * \lambda, -\lambda}(t) = \inf_{A \ni t} \sup_{B \subset A} |(u(K_\beta) * \lambda - \lambda)(B)|$$

$$\leq \inf_{A \ni t} \sup_{B \subset A} \sup_{s \in K_\beta} |\lambda_s(B) - \lambda(B)|$$

we may conclude that

$$\forall t \in G \forall \epsilon > 0 \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha: N_{u(K_\beta) * \lambda - \lambda}(t) \leq \epsilon$$

and we are done.

VIII. REMARK: *The function  $f$  of the foregoing proposition is the uniform (resp. pointwise) limit of the functions  $(F_\alpha)_{\alpha \in I}$  defined by:*

$$F_\alpha(s) = \frac{\lambda(K_\alpha s)}{m(K_\alpha s)} \quad (s \in G, \alpha \in I).$$

PROOF:  $\lambda = \lim_{\alpha \in I} u(K_\alpha) * \lambda = \lim_{\alpha \in I} F_\alpha m = (\lim_{\alpha \in I} F_\alpha) m$  with

$$\begin{aligned} F_\alpha(s) &= \int_G \Delta(t)^{-1} u(K_\alpha)(st^{-1}) d\lambda(t) \\ &= \frac{1}{m(K_\alpha)} \int_G \Delta(t)^{-1} \xi(K_\alpha s)(t) d\lambda(t) \\ &= \frac{1}{m(K_\alpha)} \int_G \Delta(s)^{-1} \xi(K_\alpha s)(t) d\lambda(t) \\ &= \frac{\lambda(K_\alpha s)}{m(K_\alpha) \Delta(s)} = \frac{\lambda(K_\alpha s)}{m(K_\alpha s)}. \end{aligned}$$

IX. THEOREM: *Let  $\lambda$  be an element of  $M^\infty(G/H)$ . Let  $\mu$  be a quasi-invariant measure on  $G/H$ .  $\lambda$  is a uniformly continuous (resp. continuous) measure if and only if  $\lambda = f\mu$  for some element  $f$  of  $BUC(G/H)$  (resp.  $BC(G/H)$ ).*

PROOF: Let  $\rho$  be the invertible function of  $BRUC(G)$  with  $\mu^\# = \rho m$ . The function  $s \rightarrow \lambda_s$  is strongly (resp. weakly) continuous if and only if the function  $s \rightarrow (\lambda^\#)_s$  is strongly (resp. weakly) continuous.

(a) If the function  $s \rightarrow (\lambda^\#)_s$  from  $G$  to  $M^\infty(G)$  is strongly (resp. weakly) continuous there exists an element  $g$  of  $BRUC(G)$  (resp.  $BC(G)$ ) with  $\lambda^\# = gm$ . It is easy to see that  $\lambda = f\mu$  where  $f(\pi(s)) = [g(s)/\rho(s)]$  is an element of  $BUC(G/H)$  (resp.  $BC(G/H)$ ).

(b) If there exists an element  $f$  of  $BUC(G/H)$  (resp.  $BC(G/H)$ ) with  $\lambda = f\mu$  then it is easy to see that  $\lambda^\# = gm$  where  $g(s) = \rho(s)f(\pi(s))$  is an element of  $BRUC(G)$  (resp.  $BC(G)$ ) and therefore we see that the function  $s \rightarrow (\lambda^\#)_s$  from  $G$  to  $M^\infty(G)$  is strongly (resp. weakly) continuous.

X. REMARK: *The function  $f$  of the foregoing theorem is the uniform (resp. pointwise) limit of the functions  $(F_\alpha)_{\alpha \in I}$  defined by:*

$$F_\alpha(\pi(s)) = \frac{\lambda(\pi(K_\alpha s))}{\mu(\pi(K_\alpha s))} \quad (\pi(s) \in G/H, \alpha \in I).$$

PROOF: Let  $\nu$  be a quasi-invariant measure on  $G/H$  with  $\nu^\# = \rho m$  where  $|\rho| \equiv 1$  and  $\rho$  is constant on every right coset of every  $K_\alpha$  ( $\alpha \in I$ ). If  $\lambda = g\nu$  then  $g$  is the uniform (resp. pointwise) limit of the functions  $(G_\alpha)_{\alpha \in I}$  where

$$G_\alpha(\pi(s)) = \frac{\lambda^\#(K_\alpha s)}{m(K_\alpha s)\rho(s)}.$$

Now

$$\begin{aligned} \lambda^\#(K_\alpha s) &= \int_{G/H} \xi(K_\alpha s)^b d\lambda = \int_{G/H} \int_H \xi(K_\alpha s)(tr) dr d\lambda(\pi(t)) \\ &= n(K_\alpha s \cap H) \int_{G/H} \xi(\pi(K_\alpha s))(\pi(t)) d\lambda(\pi(t)) \\ &= n(K_\alpha s \cap H) \lambda(\pi(K_\alpha s)) \end{aligned}$$

and

$$\begin{aligned} m(K_\alpha s)\rho(s) &= \int_G \xi(K_\alpha s)(t)\rho(t) dt \\ &= \int_{G/H} \int_H \xi(K_\alpha s)(tr) dr d\pi(t) \\ &= n(K_\alpha s \cap H) \int_{G/H} \xi(\pi(K_\alpha s))(\pi(t)) d\pi(t) \\ &= n(K_\alpha s \cap H) \nu(\pi(K_\alpha s)) \end{aligned}$$

so we may conclude that

$$G_\alpha(\pi(s)) = \frac{\lambda(\pi(K_\alpha s))}{\nu(\pi(K_\alpha s))}.$$

If  $\mu = h\nu$  for some invertible element  $h$  of  $BUC(G/H)$  then, in the same way,  $h$  is the uniform limit of the functions  $(H_\alpha)_{\alpha \in I}$  where

$$H_\alpha(\pi(s)) = \frac{\mu(\pi(K_\alpha s))}{\nu(\pi(K_\alpha s))}.$$

If  $\lambda = f\mu = (g/h)\mu$  then we see that  $f$  is the uniform (resp. pointwise) limit of the functions  $(F_\alpha)_{\alpha \in I}$  where

$$F_\alpha(\pi(s)) = \frac{G_\alpha(\pi(s))}{H_\alpha(\pi(s))} = \frac{\lambda(\pi(K_\alpha s))}{\mu(\pi(K_\alpha s))}$$

and were are finished.

### XI. Final remark

The most important results of this paper can be reformulated as follows:

*Let  $\mu$  be a quasi-invariant measure on  $G/H$ . The map  $f \rightarrow f\mu$  from  $BUC(G/H)$  with the uniform topology (resp.  $BC(G/H)$  with the pointwise topology) to  $M^\infty(G/H)$  with the strong (resp. weak) topology is a linear homeomorphism onto a closed subspace of  $M^\infty(G/H)$ . This subspace consists exactly of those measures wick translate continuously for the strong (resp. weak) toplogy on  $M^\infty(G/H)$ .*

We can always find a quasi-invariant measure  $\mu$  on  $G/H$  with  $N_\mu \equiv 1$ . In that case the map  $f \rightarrow f_\mu$  from  $BUC(G/H)$  to  $M^\infty(G/H)$  is a linear isometry.

### References

- [1] A.C.M. VAN ROOIJ: Non-archimedean Functional Analysis, Marcel Dekker, Inc., New York and Basel (1978).
- [2] N. BOURBAKI: XXIX, Elem. de Math., Livre IV, Intégration chap. 7 et 8, Hermann, Paris (1954).
- [3] L. DUPONCHEEL: Non-archimedean Induced Representations and Related Topics, Thesis (1979).
- [4] L. DUPONCHEEL: Non-archimedean quasi-invariant measures on homogeneous spaces. *Indag. Math.*, Volumen 45, Fasciculus 1 (1983).

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