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INFINITESIMAL VARIATIONS OF HODGE STRUCTURE (I)

James Carlson ★, Mark Green ★★, Phillip Griffiths ★★★ and Joe Harris ☆

0. Introduction

(a) General remarks

The Hodge structure of a smooth algebraic curve C consists of its Jacobian variety $J(C)$ together with the principal polarization determined by the intersection form Q on $H_1(C, \mathbb{Z})$. It is well known that this is equivalent to giving the pair $(J(C), \Theta)$, where $\Theta \subset J(C)$ is a divisor uniquely determined up to translation by the property that its fundamental class be Q under the identification $H^2(J(C), \mathbb{Z}) \cong \mathrm{Hom}(\Lambda^2 H_1(C, \mathbb{Z}), \mathbb{Z})$. Beginning with the inversion of the elliptic integral and continuing through current research, the polarized Hodge structure $(J(C), \Theta)$ has played an essential role in the theory of algebraic curves. As signposts we mention Abel's theorem, the Jacobi inversion theorem, Riemann's theorem, the Riemann singularity theorem, the Andreotti-Mayer theorem, and the use of the Jacobian variety in the study of special divisors (cf. [2] and [20] for precise statements of these results). In sum, one might say that in addition to the direct geometric arguments that one expects to use in studying algebraic curves, Hodge theory provides an additional unexpected and penetrating technique.

The theory of abelian integrals on curves was partially extended to higher dimensions by Picard, Poincaré, and Lefschetz, among others (cf. [41], [32]). This development culminated in the work of Hodge in the 1930's (cf. [27]), and constitutes what is now called classical Hodge theory for a smooth projective variety.

In recent years classical Hodge theory has been extended to general algebraic varieties (mixed Hodge theory; cf. [10], [17]) and to families of algebraic varieties (variations of Hodge structure, cf. [16], [9]). These two extensions interact in the precise description of the limiting behaviour of the Hodge structure of a variety as it acquires singularities (cf. [44], [46]).

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At present one may feel that Hodge theory and its extensions constitutes a subject of formal symmetry and some depth (cf. the recent papers [6], [8], and [50]).

Given this it is reasonable to expect that Hodge theory should have applications to algebraic geometry at least somewhat comparable to what happens for curves. But unfortunately this is not yet the case. Certainly classical Hodge theory has its well known applications (Lefschetz (1,1) theorem, Hodge index theorem, etc.), and there are more recent isolated successes such as the global Torelli theorem for K3 surfaces (cf. [40], [39]) *. The two extensions of Hodge theory have applications to local and global monodromy questions, which in turn has applications to degeneration problems (cf. [13] and [48] for further applications to Torelli questions). Also, variation of Hodge structures has been useful in classification questions (cf. [15], [28]), and mixed Hodge theory has proved fruitful in the study of singularities (cf. [43] for just one nice application). Nevertheless, we feel that some of the expected deep interaction between Hodge theory and geometry is not yet present in higher dimensions, as evidenced by the lack of progress on the fundamental problem of higher codimensional algebraic cycles.

If one accepts this premise then there naturally arises the question as to *why*? Partly the reason may be historical: It is possible to argue that the extent to which the theory of curves was developed by transcendental methods is simply a reflection of the training of the 19th century mathematicians. Subsequently, the theory of algebraic surfaces was already largely developed by the Italian school some thirty years before Hodge's work, essentially by extending their projective methods for studying curves. Finally, the theory of higher dimensional varieties is still in its infancy. But this ignores the lack of progress on cycles, as well as other matters such as the fact that the Torelli theorem seems to be frequently true (in some form – cf. [5], [7], [47] and [48]), and to this extent the Hodge structure provides good moduli.

We suggest that there is a more precise technical reason for this lack of interaction between Hodge theory and geometry. Namely, it is the fact that

in higher dimensions a generic Hodge structure
does not come from geometry

(this is formulated more precisely in Section 1(a)). (1)** In particular, there appears to be no natural way of attaching to a polarized Hodge structure of weight $n \geqslant 2$ a geometric object such as Θ. (2) Put differently,

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* Since this paper was written, Ron Donagi has proved that the period map has degree one in the case of almost all hypersurfaces in $\mathbb{P}_n$. His paper will appear in this journal.

**These numbers refer to notes at the end of the paper.

theory has yet to take into account the special features of those Hodge structures arising from an algebraic variety.

Now even though a general Hodge structure of weight $n \geqslant 2$ does not come from geometry, there are indications that a non-trivial global variation of Hodge structure does arise geometrically. Moreover, Hodge theory is frequently most useful in investigating problems in which there are parameters (here we mention the theory of moduli, and the Picard-Lefschetz method of studying a given variety by fibering it by a general pencil of hypersurface sections). Motivated by these observations, in this series of papers we shall introduce and study a refinement of a Hodge structure called an infinitesimal variation of Hodge structure. (3) As with a usual Hodge structure this is given by linear algebra data. However, whereas a Hodge structure has no algebraic invariants, an infinitesimal variation has too many invariants. Consequently, our main task has been to isolate a few of those (five, to be precise) that have geometric interpretations. In this paper we shall give these five constructions, and shall then study the first of these in detail.

Before turning to more specific remarks, we should like to emphasize that this work is experimental and raises more questions than it answers. Moreover, two of the most important integredients in the definition of a polarized Hodge structure, the integral lattice and the second Hodge-Riemann bilinear relation, thus far play only a minor role in the theory of infinitesimal variations of Hodge structure. However, each of our five constructions does provide direct interaction between formal Hodge theory on the one hand and the geometry of projective varieties on the other.

Finally, we should like to acknowledge valuable conversations with Chris Peters about infinitesimal variations of Hodge structure. His criticism and suggestions have been extremely useful, and a set of unpublished notes by Peters-Steenbrink was quite helpful in preparing this manuscript. We are indebted to the referee, who did an exceptionally careful job and forced us to insert occasional readable passages.

Also, we would like to thank Joanne S. Kirk, our typist for her skill and patience.

(b) Specific remarks

This paper is organized as follows:

In Section I we recall the requisite background material, and from among the plethora of invariants of an infinitesimal variation of Hodge structure list five that have thus far proved useful in geometry. (4)

In Section II we give some results, under the title of infinitesimal Schottky relations, (5) concerning the first of these constructions.

Finally, in Section III we study high degree hypersurface sections of a fixed variety, and among other things give some computations of infinitesimal Schottky relations in this case.

In more detail, associated to an infinitesimal variation of Hodge structure $V=\{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ there is a linear system of quadrics

$$\mathcal{G}(V) \subset \operatorname{Sym}^2 H^{n,0}$$

that we call the *infinitesimal Schottky relations* of V. In case V arises from a 1*st*-order deformation of a smooth, projective variety X there is an exact sequence

$$0 \to I_{\varphi_K(K)}(2) \to \mathcal{G}(V) \to \text{image}\, \nu \cap \ker \lambda \to 0,$$

where $I_{\varphi_K(X)}(2)$ is the linear system of quadrics through the canonical model $\varphi_K(X)$ and

$$\text{image}\, \nu \cap \ker \lambda \subset H^0(X, K^2)$$

with

$$\nu : \operatorname{Sym}^2 H^0(X, K) \to H^0(X, K^2)$$

being the obvious map. The mysterious ingredient here is ker λ, $^{(6)}$ and in Section II(b) we consider the case where the 1*st*-order deformation occurs in a fixed $\mathbb{P}^N$. We define the *Gauss linear system*

$$\Gamma_{2K} \subset H^0(X, K^2)$$

to be the subspace generated by quadratic differentials vanishing on ramification loci of some projection $X \to \mathbb{P}^n$ ($n = \dim X$), and prove that

$$\Gamma_{2K} \subset \ker \lambda .$$

In Section 2(c) we refine this result and, as an illustration of one of our main heuristic principles (2.c.1), show that

$$\text{image}\, \mu_1 \subset \ker \lambda$$

where

$$\mu_0 : \Lambda^n H^0(X, L) \otimes H^0(X, KL^{-n}) \to H^0(X, \Omega^{n-1} \otimes \Omega^n)$$

$$\mu_1 : \ker \mu_1 \to H^0(X, K^2)$$

are the generalizations of the two main maps in Brill-Noether theory [2].

In Section 3(a) we introduce some commutative algebra formalism into the theory of infinitesimal variation of Hodge structure and use this to prove a strenthened infinitesimal version of a classical result of M.

Noether. Finally, in Section 3(b) we discuss the infinitesimal Torelli conjecture and use again the commutative algebra formalism to verify some special cases (one of which is known) of this conjecture.

Scattered througout are a number of examples and little results. Also, we rederive and put in a general setting the main theorem of [4]. Finally, from time to time we pose specific problems and conjectures, such as the global Torelli for extremal varieties (cf. (3.b.27)).

1. Preliminaries

(a) Review of definitions from Hodge theory *

Definition: A *Hodge structure of weight n*, denoted by $\{H_{\mathbf{Z}}, H^{p,q}\}$, is given by a finitely generated free abelian group $H_{\mathbf{Z}}$ together with a *Hodge decomposition*

$$\begin{cases} H = \bigoplus\limits_{p+q=n} H^{p,q} \\ H^{p,q} = \overline{H^{p,q}} \end{cases} \tag{1.a.0}$$

on its complexification $H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes \mathbf{C}$.

Associated to $\{H_{\mathbf{Z}}, H^{p,q}\}$ is a *Hodge filtration*

$$F^p = H^{n,0} \oplus \ldots \oplus H^{p,n-p},$$

a decreasing filtration on H satisfying the condition that

$$F^p \oplus \bar{F}^{n-p+1} \to H \tag{1.a.1}$$

be an isomorphism for $p = 0, \ldots, n$. Conversely, a decreasing filtration

$$(0) = F^{n+1} \subset F^n \subset \ldots \subset F^1 \subset F^0 = H$$

that satisfies (1.a.1) gives a Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}\}$ of weight n where

$$H^{p,q} = F^p \cap \bar{F}^q.$$

The *Hodge numbers* are defined by

$$\begin{cases} h^{p,q} = \dim H^{p,q} \\ f^p = \dim F^p = h^{n,0} + \ldots + h^{p,n-p}. \end{cases}$$

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* A good general reference and source of specific references for this section is [17].

EXAMPLE: The n^{th} cohomology of a compact Kähler manifold X gives a Hodge structure of weight n where

$$H_{\mathbf{Z}} = H^n(X, Z)/(\text{torsion})$$

$$H^{p,q} = H^{p,q}(X)$$

(the notations are standard, and are given in Chapter 1 of [20]). This is our main example; others may be obtained by applying linear algebra constructions to Hodge structures coming from compact Kähler manifolds.

DEFINITION: A *polarized Hodge structure of weight n*, denoted by $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$, is a Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}\}$ of weight n together with a bilinear form

$$Q: H_{\mathbf{Z}} \times H_{\mathbf{Z}} \to \mathbb{Z}$$

that satisfies

$$\begin{cases} Q(\psi, \varphi) = (-1)^n Q(\varphi, \psi) & \\ Q(\psi, \varphi) = 0 & \psi \in H^{p,q},\ \varphi \in H^{p',q'} \text{ and } p' \neq q \\ (\sqrt{-1})^{p-q} Q(\psi, \bar{\psi}) > 0 & 0 \neq \psi \in H^{p,q} \end{cases}$$

We shall refer to (I) and (II) as the *Hodge-Riemann bilinear relations*. In terms of the Hodge filtration (1.a.0) they are

$$Q(F^p, F^{n-p+1}) = 0$$

$$Q(C\psi, \psi) > 0 \tag{1.a.2}$$

where the *Weil operator C* is defined by

$$\begin{cases} C_p \psi = (\sqrt{-1})^{p-q} \psi,\ \psi \in H^{p,q} \\ C = \bigoplus\limits_{p=n}^{0} C_p. \end{cases}$$

EXAMPLE: A polarized algebraic variety (X, ω) is given by a compact, complex manifold X together with the Chern class $\omega = c_1(L)$ of an ample line bundle $L \to X$.

We will identify (X, ω) with (X, ω') when $\mathbf{Q} \cdot \omega = \mathbf{Q} \cdot \omega'$.

Suppose that $\dim X = n + k$ and recall that the *primitive cohomology* is

$$H^n_{\text{prim}}(X, \mathbb{Q}) = \ker\{\omega^{k+1} : H^n(X, \mathbb{Q}) \to H^{n+2k+2}(X, \mathbb{Q})\}.$$

Because of the *hard Lefschetz theorem*

$$\omega^k : H^n(X, \mathbb{Q}) \xrightarrow{\sim} H^{n+2k}(X, \mathbb{Q})$$

and *Lefschetz decomposition*

$$H^n(X, \mathbb{Q}) = \bigoplus_{m \geqslant 0} \omega^m \quad H^{n-2m}_{\text{prim}}(X, \mathbb{Q})$$

we may think of the primitive cohomology as providing the basic building blocks for the cohomology of X. Setting

$$H_{\mathbb{Z}} = H^n(X, \mathbb{Z}) \cap H^n_{\text{prim}}(x, \mathbb{Q})$$

$$H^{p,q} = H^{p,q}(X) \in H^n_{\text{prim}}(X, \mathbb{C})$$

$$Q(\varphi, \psi) = (-1)^{n(n-1)/2} \int_X \varphi \wedge \psi \wedge \omega^k,$$

we obtain a polarized Hodge structure of weight n.

A *sub-Hodge structure* of a Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}\}$ is given by $\{H'_{\mathbb{Z}}, \mathrm{H}'^{p,q}\}$ where $H'_{\mathbb{Z}} \subset H_{\mathbb{Z}}$ is a subgroup, $H'^{p,q} = H' \cap H^{p,q}$, and where

$$H' = \bigoplus_{p+q=n} H'^{p,q}.$$

In this case there is a natural quotient *Hodge structure* $\{H''_{\mathbb{Z}}, H''^{p,q}\}$ where

$$H''_{\mathbb{Z}} = H_{\mathbb{Z}}/H'_{\mathbb{Z}} \qquad \text{(modulo torsion)}$$

$$H''^{p,q} = H^{p,q}/H'^{p,q}$$

More generally, a *morphism* λ *between Hodge structures* $\{H_{\mathbb{Z}}, H^{p,q}\}$, $\{H'_{\mathbb{Z}}, H'^{p,q}\}$ of respective weights n, $n + 2m$ is given by a linear map

$$\lambda : H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$$

satisfying

$$\lambda(H^{p,q}) \subseteq H'^{p+m,q,m}.$$

In this case the kernel, image, and cokernel of λ are all Hodge structures.

Given Hodge structures $\{H_{\mathbb{Z}}, H^{p,q}\}$, $\{H'_{\mathbb{Z}}, H'^{p',q'}\}$ of respective weights n, n', the complexification of $H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}}$, and $\operatorname{Hom}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$ have natural Hodge decompositions of respective weights $n + n'$ and $n - n'$.

A sub-Hodge structure $\{H'_{\mathbb{Z}}, H'^{p,q}\}$ of a polarized Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ inherits a natural polarization form $Q = Q \mid H'_{\mathbb{Z}}$. In particular, Q' is non-degenerate and the complement

$$H''_{\mathbb{Z}} = (H')^{\perp} \cap H_{\mathbb{Z}}$$

has an induced polarized Hodge structure where, e.g., $H''^{p,q} = H^{p,q} \cap H''$.

The standard constructions of linear algebra also leave invariant the set of polarized Hodge structures.

DEFINITION: A Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}\}$ is said to *arise from geometry* in case it may be obtained from the Hodge structures of polarized algebraic varieties by standard linear algebra constructions. (7)

EXAMPLE: If $f\colon X \to Y$ is a morphism of smooth projective varieties, then setting

$$H'_{\mathbb{Z}} = \ker\{f^* : H^n(Y, \mathbb{Z}) \to H^n(X, \mathbb{Z})\}/\text{torsion}$$

$$H^{p,q} = \ker\{f^* : H^{p,q}(X) \to H^{p,q}(X)\}$$

gives a Hodge structure arising from geometry.

By our remarks, any Hodge structure arising from geometry will have a polarization induced from the standard polarizations on the cohomology of polarized algebaric varieties, and we shall generally consider it with such a polarization.

To define the classifying spaces for polarized Hodge structures, we assume given a finitely generated free abelian group $H_{\mathbb{Z}}$, a nondegenerate bilinear form

$$Q\colon H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$$

satisfying the (skew) symmetry relation preceding the Hodge-Riemann bilinear relation, and Hodge numbers $h^{p,q}$ satisfying

$$\begin{cases} h^{p,q} = h^{q,p} \\ \displaystyle\sum_{p+q=n} h^{p,q} = \operatorname{rank} H_{\mathbb{Z}} \\ \displaystyle\sum_{p+q=n} (-1)^q h^{p,q} = \operatorname{sgn} Q_{\mathbb{R}}, \end{cases} \tag{1.a.3}$$

where $\operatorname{sgn} Q_{\mathbb{R}}$ in the signature of Q on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$.

DEFINITION: The *classifying space* D is the set of all polarized Hodge structures $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ where $H_{\mathbf{Z}}$, Q are given above and $h^{p,q} = \dim H^{p,q}$.

If we let

$$G = \operatorname{Aut}(H_{\mathbb{R}}, Q)$$

be the automorphisms of $H_{\mathbb{R}}$ that preserve Q, the G operates on D in the obvious way. It is well known that, under this action, D is a homogeneous complex manifold of the form

$$D = G/H \tag{1.a.4}$$

where $H \subset G$ is a compact subgroup. (cf. [19], [18], and [44]). To describe the complex structure on D is is convenient to give the following

DEFINITION: The *dual classifying space* $\check{D}$ is the set of all filtrations $\{F^p\}$ that satisfy the first Hodge-Riemann bilinear relation (I).

If $G(f^p, H)$ is the Grassmann manifold of all linear subspaces $F^p \subset H$ where $\dim F^p = f^p$, then there is an obvious inclusion

$$\check{D} \subset \prod_{p \geqslant 1} G(f^p, H), \tag{1.a.5}$$

and it may be proved that $\check{D}$ is a smooth algebraic subvariety. In fact, if

$$G_C = \operatorname{Aut}(H, Q)$$

is the complexification of G, then it may be shown that $G_{\mathbf{C}}$ acts transitively on $\check{D}$. Thus $\check{D}$ is a homogeneous algebraic variety of the form

$$\check{D} = G_{\mathbf{C}}/B$$

where $B \subset G_{\mathbf{C}}$ is a parabolic subgroup. It may further be shown that

$$D \subset \check{D}$$

is the open G-orbit of a point; consequently D is a homogeneous complex manifold with $H = G \cap B$.

Before defining a variation of Hodge structure we need to describe a distinguished sub-bundle $T_h(D) \subset T(D)$. For this we recall that if $W \in G(k, H)$ is a k-plane in the vector space H, then there is a canonical identification

$$T_W(G(k, H)) \cong \operatorname{Hom}(W, H/W)$$

described as follows: if $w \in W$ and $\xi \in T_W(G(k, H))$, then we choose an arc $\{W_t\}$ in $G(k, H)$ with $W_0 = W$ and tangent ξ, and vectors $w(t) \in W_t$ with $w(0) = w$. Then the homomorphism $\xi \in \operatorname{Hom}(W, H/W)$ is given by

$$\xi(w) = \text{projection of } \left.\frac{\mathrm{d}w_t}{\mathrm{d}t}\right|_{t=0} \text{ into } H/W. \tag{1.a.6}$$

Combining this with (1.a.5) gives an inclusion

$$T_F(\check{D}) \subset \bigoplus_{p \geqslant 1} \operatorname{Hom}(F^p, H/F^p) \tag{1.a.7}$$

where $F = \{F^p\} \in \check{D}$. If we write elements on the right hand side of (1.a.7) as $\xi = \oplus_{p>1}\xi_p$ where $\xi_p \in \operatorname{Hom}(F^p, H/F^p)$, then it is easy to see that $T_F(\check{D})$ is the set of $\xi = \oplus\, \xi_p$ satisfying the conditions: the diagram

$$\begin{tikzcd} F^p \arrow[r, "\xi_p"] \arrow[d] & H/F^p \arrow[d] \\ F^{p-1} \arrow[r, "\xi_{p-1}"] & H/F^{p-1} \end{tikzcd} \tag{1.a.8}$$

is commutative, and

$$Q(\xi_p\varphi, \psi) + Q(\varphi, \xi_{n-p+1}\psi) = 0 \qquad \varphi \in F^{n-p+1}.$$

Because of (1.a.8) we may unambiguously write the last equation as

$$Q(\xi\varphi, \psi) + Q(\varphi, \xi\psi) = 0 \qquad \varphi \in F^p, \quad \psi \in F^{n-p+1} \tag{1.a.9}$$

We then define the *horizontal space*

$$T_{F,h}(\check{D}) \subset T_F(\check{D})$$

by the additional condition

$$\xi(F^p) \subset F^{p-1}. \tag{1.a.10}$$

It is clear that these horizontal subspaces give a holomorphic subbundle

$$T_h(\check{D}) \subset T(\check{D})$$

invariant under the action of $G_{\mathbf{C}}$, and we set

$$T_h(D) = T_h(\check{D})|_D.$$

DEFINITION: A *variation of Hodge structure* is given by a mapping

$$\varphi : S \to \Gamma \backslash D \tag{1.a.11}$$

where S is a complex-analytic variety, Γ is a subgroup of $G_{\mathbf{Z}} = \mathrm{Aut}(H_{\mathbf{Z}}, Q)$, and φ satisfies the following conditions:

(i) φ is *holomorphic* (this makes sense since Γ acts properly discontinuously on D and therefore $\Gamma \backslash D$ is a complex-analytic variety);

(ii) φ is *locally liftable*; i.e., each point $s \in S$ has neighborhood $\mathcal{U}$ in this $\varphi | \mathcal{U}$ lifts to a mapping

$$\tilde{\varphi} : \mathcal{U} \to D; \tag{1.a.12}$$

and

(iii) φ is *horizontal* in the sense that the differential of one (and hence any) local lifting (1.a.12) satisfies

$$\tilde{\varphi}_* : T_s(S) \to T_{\tilde{\varphi}(s),h}(D).$$

We shall sometimes abuse notation and write this as

$$\varphi_* : T(S) \to T_h(D). \tag{1.a.13}$$

The horizontality condition (1.a.13) is sometimes referred to as the *infinitesimal period relation.*

If $\pi : \tilde{S} \to S$ denotes the universal covering of S with the fundamental group $\pi_1(S, s_0) = \pi_1$ being viewed as a group of deck transformations of $\tilde{S}$, then the local liftability property implies that there is a diagram of holomorphic mappings

$$\begin{tikzcd} \tilde{S} \arrow[r, "\tilde{\varphi}"] \arrow[d] & C \arrow[d] \\ S \arrow[r, "\varphi"] & \Gamma \backslash D \end{tikzcd} \tag{1.a.14}$$

Moreover, there is a *monodromy representation*

$$\rho : \pi_1 \to \Gamma$$

with the property that

$$\tilde{\varphi}(\gamma \tilde{s}) = \rho(\gamma) \tilde{\varphi}(\tilde{s}) \qquad \tilde{s} \in \tilde{S}, \quad \gamma \in \pi_1.$$

If we agree to identify holomorphic vector bundles over S with locally free sheaves of $\mathcal{O}_S$-modules, then the trivial bundle $\tilde{S} \times H$ induces on S a locally free sheaf $\mathcal{H}$ having an integrable connection

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_S,$$

the *Gauss-Manin connection*. Since $H = H_{\mathbf{Z}} \otimes \mathbf{C}$ there is a locally constant subsheaf $\mathcal{H}_{\mathbf{Z}} \subset \mathcal{H}$. Moreover, the Hodge structure at $\tilde{\varphi}(\tilde{s}) \in D$ induces a filtration

$$\{0\} \subset \mathcal{F}^n_s \subset \ldots \subset \mathcal{F}^1_s \subset \mathcal{F}^0_s = \mathcal{H}_s$$

at $s = \pi(\tilde{s})$, where the $\mathcal{F}^p$ are the holomorphic vector bundles over S obtained by pulling back the universal sub-bundle over $G(f^p, H)$ via the map $D \to G(f^p, H)$. If we define the *Hodge bundles* by

$$\mathcal{H}^{p,q} = \mathcal{F}^p/\mathcal{F}^{p+1}$$

then there is a C^∞ (*not* holomorphic) Hodge decomposition

$$\begin{cases} \mathcal{H} = \bigoplus\limits_{p+q=n} \mathcal{H}^{p,q} \\ \mathcal{H}^{p,q} = \overline{\mathcal{H}^{q,p}}. \end{cases}$$

Finally, the quadratic form on $H_{\mathbf{Z}}$ induces a locally constant quadratic form Q on $\mathcal{H}$, and the infinitesimal period relation is

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_S. \tag{1.a.15}$$

Summarizing, the variation of Hodge structure gives the data

$$\mathcal{U} = \{\mathcal{H}_{\mathbf{Z}}, \mathcal{H}^{p,q}, \nabla, Q, S\} \tag{1.a.16}$$

subject to the conditions explained above.

Conversely, given the data (1.a.16) subject to the above conditions we may construct a variation of Hodge structure

$$\varphi : S \to \Gamma \backslash D. \tag{1.a.17}$$

In the sequel we shall interchangeably think of a variation of Hodge structure as given by the data (1.a.16) or by a holomorphic mapping (1.a.17).

EXAMPLE: Let $\mathcal{X}$, S be connected complex manifolds and let

$$f : \mathcal{X} \to S$$

be a smooth proper morphism with connected fibres $X_s = f^{-1}(s)$. Suppose moreover that there is a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, hook] \arrow[dr, "f"'] & & S \times \mathbb{P}^r \arrow[dl, "\pi_S"] \\ & S & \end{tikzcd}$$

where π_S is the projection onto S. We shall refer to this situation as a *projective family* $\{X_s\}_{s \in S}$. We fix a base point $s_0 \in S$ and set $H_{\mathbf{Z}} = H^n(X_{s_0}, \mathbf{Z}) \cap H^n_{\mathrm{prim}}(X_{s_0}, \mathbf{Q})$. Since the hyperplane class $\omega \in H^2(X_{s_0}, \mathbf{Z})$ and cup-product are both invariant under the action of the fundamental group $\pi_1(S, s_0)$ on $H^n(X_{s_0}, \mathbf{Z})$, there is induced on $H_{\mathbf{Z}}$ a bilinear form Q and linear representation

$$\rho : \pi_1(S, s_0) \to \mathrm{Aut}(H_{\mathbf{Z}}, Q).$$

Setting $\Gamma = \rho(\pi_1(S, s_0))$ we may define a mapping

$$\varphi : S \to \Gamma \backslash D$$

by

$$\varphi(s) = \{\text{polarized Hodge structure on } H^n(X_s, \mathbf{Z}) \cap H^n_{\mathrm{prim}}(X_s, \mathbf{Q})\}.$$

It is well known that the three conditions for a variation of Hodge structure are satisfied (cf. [9], [19]).

We shall say that such a variation of Hodge structure *arises from a geometric situation*. (Actually, this concept should be extended in an analogous manner to saying that a fixed Hodge structure arises from geometry in case it is constructed by linear algebra from Hodge structures of polarized varities, but we shall not discuss this extension here.) We shall have occasion to use the following

DEFINITION: An *extended variation of Hodge structure* is given by a complex-analytic variety $\mathfrak{M}$ and a holomorphic mapping

$$\varphi : \mathfrak{M} \to \Gamma \backslash D$$

such that the restriction of φ to a dense Zariski open subset $S \subset \mathfrak{M}$ is a variation of Hodge structure in the usual sense.

What this means is that there is a proper analytic subvariety $Z \subset \mathfrak{M}$ such that on $S = \mathfrak{M} - Z$ we have a variation of Hodge structure as previously defined, but φ may fail to be locally liftable in the neighborhood of points $s \in Z$.

EXAMPLE: Let $\mathcal{T}_g$ be the Teichmüller space [3] for compact Riemann surface of genus $g \geqslant 1$ and Γ_g the Teichmüller modular group. Then (cf. [11])

$$\mathfrak{M}_g = \mathcal{T}_g/\Gamma_g$$

is a quasi-projective variety whose points are in a one-to-one correspondence with the smooth algebraic curves of genus g. The classifying space for the polarized Hodge structures of curves is the Siegel generalized upper-half-plane $\mathcal{H}_g$, and the usual period mapping of curves gives an extended variation of Hodge structure

$$\varphi : \mathfrak{M}_g \to Sp(2g, \mathbb{Z})\backslash\mathcal{H}_g.$$

The mapping φ fails to be locally liftable around points of $\mathfrak{M}_g$ corresponding to curves having non-trivial automorphisms (except that the hyperelliptic involution doesn't count when $g = 2$). The aforementioned notes of Peters-Steenbrink contain an excellent discussion of fine vs. coarse moduli schemes.

REMARK: Given any variation of Hodge structure

$$\varphi : S \to \Gamma\backslash D \tag{1.a.18}$$

where S is a Zariski open set in a smooth algebraic variety $\bar{S}$, there is a maximal Zariski open set $S' \subset S$ to which φ extends to define an extended variation of Hodge structure

$$\varphi' : S' \to \Gamma\backslash D$$

where φ' is *proper*. To obtain S' we look at the divisor components D_i of $\bar{S} - S$. If the local monodromy transformation around a simple point of some D_i is of finite order, then by [21] we may holomorphically extend φ across $D_i - (\cup_{j \neq 1} D_i \cap D_j)$. Call this new mapping

$$\varphi_1 : S_1 \to \Gamma\backslash D.$$

If W is a codimension $\geqslant 2$ component of $\bar{S} - S_1$, then φ_1 is locally liftable in $\mathfrak{U} \cap S_1$ where μ is a neighborhood of a general point of W. By [18] we may then extend φ_1 to all of $\mathfrak{U}$. Continuing in this way we obtain our maximal extension of (1.a.18).

Needless to say this process is easier to carry out in theory than in practice (take $\bar{S} \cong \mathbb{P}^{d(d+3)}/2$ to be the parameter space for plane curves of degree d and $S \subset \bar{S}$ the open set corresponding to smooth curves). In fact, one of the central problems in application of Hodge theory is to determine which degenerate varieties must be added in order to make the period mapping proper (cf. [13], [14]).

Finally, we shall use the following:

DEFINITION: Let $\mathfrak{M}$ be a moduli scheme for some class of polarized algebraic varieties, and suppose that the Hodge structure of a general member of $\mathfrak{M}$ gives an extended variation of Hodge structure

$$\varphi : \mathfrak{M} \to \Gamma \backslash D. \tag{1.a.19}$$

We shall say that the weak *global Torelli theorem* holds in case φ has degree one (as a mapping onto its image).

We note that this depends on the particular subgroup $\Gamma \subset \operatorname{Aut}(H_{\mathbf{Z}}, Q)$. In practice period mappings (1.a.19) frequently are finite-to-one, but there seem to be no criteria enabling us to say that φ is a Galois covering, so that the weak global Torelli theorem holds for a suitable Γ.

(b) Review of intermediate Jacobians and normal functions

To a Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}\}$ of weight $n = 2m - 1$ we associate the following complex torus J. As a real torus

$$J = H_{\mathbf{R}}/H_{\mathbf{Z}}. \tag{1.b.1}$$

To define the complex structure on J we set

$$\begin{cases} H' = F^m H = H^{2m-1,0} \oplus \ldots \oplus H^{m,m-1} \\ H'' = \overline{F^m H} = H^{m-1,m} \oplus \ldots \oplus H^{0,2m-1} \end{cases}$$

so that

$$\begin{cases} H = H' \oplus H'' \\ H'' = \overline{H}'. \end{cases}$$

If we make the identification

$$H/H' \cong H''$$

and denote by Λ the lattice obtained by projecting $H_{\mathbf{Z}}$ to H'', then J is the complex torus given by

$$J = H''/\Lambda.$$

We note that the Lie algebra of J is

$$\mathfrak{L}(J) = H''. \tag{1.b.3}$$

EXAMPLE: For the Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}\}$ associated to $H^{2m-1}(X, \mathbf{Z})$

for X a compact Kähler manifold, the resulting complex torus is the m^{th} intermediate Jacobian $J^m(X)$. (9)

For later purposes it will be convenient to have an alternate description of $J^m(X)$. Set

$$H^{2k-1'}(X)=F^kH^{2k-1}(X)=H^{2k-1,0}(X)\oplus\ldots\oplus H^{k,k-1}(X)$$

$$H^{2k-1''}(X)=\overline{F^kH^{2k-1}(X)}.$$

If $\dim_{\mathbf{C}} X=n$, then the cup-product pairing

$$H^{2n-2m+1'}(X)\otimes H^{2m-1''}(X)\to\mathbf{C}$$

is non-degenerate, and hence the dual of the Lie algebra of $J^m(X)$ is canonically given by (10)

$$H^{1,0}(J^m(X))=H^{2n-2m+1'}(X), \tag{1.b.4}$$

and therefore

$$J^m(X)\cong\left(H^{2n-2m+1'}(X)\right)^*/\Lambda^* \tag{1.b.5}$$

where Λ^* is the image of the map

$$\alpha: H_{2n-2m+1}(X,\mathbb{Z})\to\left(H^{2n-2m+1'}(X)\right)^*$$

defined by

$$\alpha(\gamma)(\omega)=\int_\gamma\omega.$$

In the sequel we will drop the "α".

Intermediate Jacobians are useful in the study of cycles on a smooth algebraic variety X, as will now be briefly described. We denote by $\mathfrak{Z}^m(X)$ the group of codimension $-$ m algebraic cycles on X and by

$$\mathfrak{Z}_h^m(X)\subset\mathfrak{Z}^m(X)$$

the subgroup of cycles homologous to zero. Using the description (1.b.5) of $J^m(X)$ we define the *Abel-Jacobi mapping*

$$u:\mathfrak{Z}_h^m(X)\to J^m(X)$$

by assigning to each $Z\in\mathfrak{Z}_h^m(X)$ the linear function on $H^{2n-2m+1}{}'(X)$ given by

$$\omega\to\int_C\omega$$

where C is a chain with $\partial C = Z$. It is known (cf. [19], [21], and [33]) that u varies holomorphically with Z, and therefore maps to zero in $J^m(X)$ the subgroup

$$\mathfrak{Z}_r^m(X) \subset \mathfrak{Z}_h^m(X)$$

of cycles rationally equivalent to zero. Moreover, u satisfies the following differential condition: Denote by

$$\mathfrak{L}_h(J^m(X)) \subset \mathfrak{L}(J^m(X))$$

the subspace of the Lie algebra of $J^m(X)$ given by

$$\begin{aligned}\mathfrak{L}_h(J^m(X)) &= H^{m-1,m}(X)\\ &= F^{n-m+2}H^{2n-2m+1}(X)^{\perp} \cap H''(X),\end{aligned}$$

and suppose that $\{Z_b\}_{b \in B}$ is a complex-analytic family of codimension-m algebraic cycles on X. Choosing a base point $b_0 \in B$ there is a holomorphic mapping

$$u\colon B \to J^m(X)$$

defined by

$$u(b) = u(Z_b - Z_{b_0}).$$

The differential restriction is

$$u_*\colon T_b(B) \to \mathfrak{L}_h(J^m(X)),$$

or equivalently by what was said above

$$u^* F^{n-m+2}H^{2n-2m+1}(X) = 0 \tag{1.b.6}$$

Returning to the general discussion, suppose that $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ is a polarized Hodge structure of weight $2m-1$. Using the bilinear form there is a natural identification

$$H'' \cong (H')^*.$$

Thus

$$\begin{cases} H^{1,0}(J) = H' \\ J = (H')^*/\Lambda^* \end{cases}$$

where Λ^* is the image of the map

$$H_{\mathbb{Z}} \to H/F^m \xrightarrow{\sim} F^{m*}.$$

We remark that there are natural identifications

$$\begin{cases} H_{\mathbb{Z}} \cong H_1(J, \mathbb{Z}) \\ \operatorname{Hom}(\Lambda^2 H_{\mathbb{Z}}, \mathbb{Z}) \cong H^2(J, \mathbb{Z}). \end{cases}$$

Using the second of these, the polarizing form Q may be viewed as a class

$$q \in H^2(J, \mathbb{Z}).$$

By the first Hodge-Riemann bilinear relation, $q \in H^{1,1}(J) \cap H^2(J, \mathbb{Z})$ and is therefore the Chern class of a holomorphic line bundle $L \to J$. The curvature Θ of this line bundle is given by the second Hodge-Riemann bilinear relation, from which it follows that (see [44])

$$\begin{cases} \Theta > 0 & \text{on} \quad H^{m-1,m} \oplus H^{m-3,m+2} \oplus \dots \subset \mathcal{L}(J) \\ \Theta < 0 & \text{on} \quad H^{m-2,m+1} \oplus H^{m-4,m+3} \oplus \dots \subset \mathcal{L}(J). \end{cases}$$

In particular, if $m = 1$ then (J, Θ) is a *polarized abelian variety*. In any case, if B is a complex-analytic variety and

$$u : B \to J$$

is a holomorphic mapping satisfying

$$u_* : T(B) \to \mathcal{L}_h(J),$$

then $\Theta > 0$ on the image variety $u(B)$.

According to (1.b.6), this is the situation for Abel-Jacobi mappings. (11)

Referring to the discussion in Section 1(a), it is essentially clear how to define a variation of Hodge structure

$$\mathcal{V} = \{\mathcal{H}_{\mathbb{Z}}, \mathcal{H}^{p,q}, \nabla, S\}$$

in the absence of a polarizing form: We should be given an analytic variety S, a locally free sheaf $\mathcal{H} \to S$ having an integrable connection ∇, a subsheaf $\mathcal{H}_{\mathbb{Z}} \subset \mathcal{H}$ of locally constant sections, and holomorphic subbundles $\mathcal{F}^p \subset \mathcal{H}$ such that all axioms for a variation of Hodge structure, other than those involving the polarizing form, are satisfied.

Suppose now that $\{\mathcal{H}_{\mathbb{Z}}, \mathcal{H}^{p,q}, \nabla, S\}$ is a variation of Hodge structure of weight $2m-1$. In the obvious way we may construct a complex analytic fibre space

$$\pi : \mathcal{J} \to S$$

of complex tori $J_s = \pi^{-1}(s)$. We shall also denote by $\mathcal{J}$ the corresponding sheaf of holomorphic sections of this fibre space, and by

$$\mathcal{L}(\mathcal{J}) = \mathcal{H}/\mathcal{F}^m$$

the sheaf of Lie algebras. Thus there is an exact sheaf sequence

$$0 \to \mathcal{H}_{\mathbf{Z}} \to \mathcal{L}(\mathcal{J}) \to \mathcal{J} \to 0. \tag{1.b.7}$$

Observing that the Gauss-Manin connection

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_S$$

satisfies

$$\nabla(\mathcal{H}_{\mathbf{Z}}) = (0)$$

$$\nabla(\mathcal{F}^m) \subset \mathcal{F}^{m-1} \otimes \Omega^1_S,$$

there is an induced map

$$\nabla : \mathcal{J} \to \left(\mathcal{H}/\mathcal{F}^{m-1}\right) \otimes \Omega^1_S. \tag{1.b.8}$$

DEFINITION: (i) *The sheaf of normal functions* is the subsheaf

$$\mathcal{J}_h \subset \mathcal{J}$$

defined by the kernel of the mapping (1.b.8)

(ii) *A normal function* is a global cross-section

$$\nu \in H^0(S, \mathcal{J}_h).$$

REMARKS: (i) When S is a quasi-projective variety, a normal function will be required to satisfy an additional "growth condition" at infinity (cf. [22], [12]).

(ii) In general, for any variation of Hodge structure the sections ν of the subsheaf

$$\left(\mathcal{H}/\mathcal{F}^k\right)_h \subset \left(\mathcal{H}/\mathcal{F}^k\right)$$

defined by the condition

$$\nabla\tilde{\nu} \in \mathcal{F}^{k-1} \otimes \Omega^1_S,$$

where $\tilde{\nu} \in \mathcal{H}$ is any lifting of ν, will be said to be *quasi-horizontal*; these are the sections that have geometric meaning.

EXAMPLE: Suppose that

$$f: \mathcal{X} \to S$$

is a proper and smooth holomorphic mapping between complex manifolds where $\mathcal{X}$ is Kähler (but may not be compact). Let $\mathcal{U} = \{\mathcal{H}_{\mathbf{Z}}, \mathcal{H}^{p,q}, \nabla, S\}$ be the variation of Hodge structure with

$$\begin{cases} (\mathcal{H}_{\mathbf{Z}})_s = H^{2m-1}(X_s, \mathbf{Z})/\text{torsion} \\ \mathcal{H}_s^{p,q} = H^{p,q}(X_s) \end{cases}$$

where $X_s = f^{-1}(s)$. Let $\mathcal{Z} \subset \mathcal{X}$ be a codimension-m analytic cycle such that each intersection

$$Z_s = \mathcal{Z} \cdot X_s \in \mathcal{Z}_h^m(X_s) \tag{1.b.9}$$

is homologous to zero (note: in practice, $\mathcal{X}$ will be algebraic and we may vary $\mathcal{Z}$ by a rational equivalence, so that the intersections (1.b.9) are defined). We set

$$\nu_{\mathcal{Z}}(s) = u_s(Z_s) \in J^m(X_s)$$

where

$$u_s: \mathcal{Z}_h^m(X_s) \to J^m(X_s)$$

is the Abel-Jacobi mapping. It is known ([21]) that $\nu_{\mathcal{Z}}$ is a holomorphic section of $\mathcal{J} \to S$ and that is satisfies

$$\nabla \nu_{\mathcal{Z}} = 0$$

where ∇ is defined by (1.b.8). Moreover, in case $\mathcal{X}$ and $\mathcal{Z}$ are algebraic, $\nu_{\mathcal{Z}}$ satisfies the required growth conditions (cf. [12]), and therefore defines a normal function.

EXAMPLE: In moduli problems one is frequently given not only the variation of Hodge Structure but also a (perhaps multi-valued) normal function.

For instance, let $\mathcal{M}_4^*$ be the moduli space of non-hyperelliptic smooth curves of genus 4. We will identify these curves with their canonical models $C \subset \mathbb{P}^3$, and recall that

$$C = Q \cap V$$

where Q, V are respectively a quadric surface, cubic surface. In general, Q is smooth and the difference of the two rulings on Q cuts out a pair of distinct g_3^1's on C. Labelling these g_3^1's by $|D|$ and $|D'|$, there is given over $\mathfrak{M}_4^*$ a normal function ν, defined up to ± 1, by

$$\nu(C) = u_C(D - D') \in J(C).$$

This normal function vanishes over the locus of curves having an effective theta characteristic, and from the infinitesimal invariant $\delta\nu$ defined in Section 1(c) below (which also equals $\delta(-\nu)$) we may reconstruct a general curve $C \in \mathfrak{M}_4^*$.

For another example we consider the family $\{X_s\}_{s\in S}$ of smooth quintic threefolds $X_s \subset \mathbb{P}^4$, where $S \subset \mathbb{P}$ $(\mathrm{Sym}^5\,\mathbb{C}^5)$ is the natural parameter space. On a general X_s there are 1002 distinct lines L_j, and considering differences we obtain (very) multivalued normal functions by setting

$$\nu_{ij}(s) = u_{X_s}(L_i - L_j).$$

As one motivation for studying normal functions we suggest [23] and [50]. In particular we would like informally to recall one result from these papers. To state this we assume given a variation of Hodge structure $\{H_{\mathbf{Z}}, \mathcal{H}^{p,q}, \nabla, S\}$ of weight $2m-1$ and a normal function $\nu \in H^0(S, \mathcal{J}_h)$. Using the exact cohomology sequence

$$\begin{tikzcd} H^0(S, \mathcal{L}(\mathcal{J})) \arrow[r] & H^0(S, \mathcal{J}) \arrow[r, "\delta"] & H^1(S, \mathcal{H}_{\mathbf{Z}}) \\ & H^0(S, \mathcal{J}_h) \arrow[u, phantom, "\cup"] & \end{tikzcd} \tag{1.b.10}$$

of (1.b.7) we define the fundamental class $\eta(\nu)$ by

$$\eta(\nu) = \delta\nu \in H^1(S, \mathcal{H}_{\mathbf{Z}}).$$

Example: Let $f\colon \mathcal{X} \to S$ be as in the example above, and let $\mathcal{Z} \subset \mathcal{X}$ be a codimension-m cycle satisfying (1.b.9). Then, on the one hand $\mathcal{Z}$ has a fundamental class ${}_{\mathfrak{z}} \in H^{2m}(\mathcal{X}, \mathbb{Z})$. On the other hand the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q_{f_*}\mathbb{Q}) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{Q})$$

degenerates at E_2 (cf. [10]), and hence modulo torsion there is a natural inclusion

$$H^1(S, \mathcal{H}) \hookrightarrow H^{2m}(\mathcal{X}, \mathbb{Z}). \tag{1.b.11}$$

Under this inclusion the fundamental class of $\nu_{\mathfrak{z}}$ may be shown to be equal to ${}_{\mathfrak{z}}$ (see [50]).

Now suppose that $X \subset \mathbb{P}^r$ is a smooth projective variety of dimension $2m$. Denote by $S \subset \mathbb{P}^{r^*}$ the Zariski open set $\{H_s\}_{s \in S}$ of hyperplanes such that

$$X_s = H_s \circ X$$

is smooth, and define

$$\mathcal{X} \subset S \times X$$

by $\mathcal{X} = \{(s, x) : x \in X_s\}$. Then the projection

$$f : \mathcal{X} \to S$$

gives a family of the type we have been considering. Suppose that

$$\eta_0 \in H^{m,m}_{\text{prim}}(X) \cap H^{2m}(X, \mathbb{Z}) \tag{1.b.12}$$

is a primitive integral class of type (m, m), and denote by $\eta \in H^{2m}(\mathcal{X}, \mathbb{Z})$ the pullback of η_0 to $\mathcal{X}$. The result we are referring to is

Given η_0 *satisfying* (1.b.12), *there exists a normal function* ν *with fundamental class* $\eta(\nu) = \eta$. (12) (1.b.13)

(c) Infinitesimal variations of Hodge structure and some invariants

DEFINITION: An *infinitesimal variation of Hodge structure* $V = \{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ is given by a polarized Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ together with a vector space T and linear mapping

$$\delta : T \to \bigoplus_{1 \leqslant p \leqslant n} \operatorname{Hom}(H^{p,q}, H^{p-1,q+1})$$

that satisfies the two conditions:

$$\delta(\xi_1)\delta(\xi_2) = \delta(\xi_2)\delta(\xi_1) \qquad \xi_1, \xi_2 \in T \tag{1.c.1}$$

$$Q(\delta(\xi)\varphi, \psi) + Q(\varphi, \delta(\xi)\psi) = 0$$

$$\xi \in T \text{ and } \varphi \in F^p, \psi \in F^{n-p+1}. \tag{1.c.2}$$

REMARKS: Condition (1.c.2) is just (1.a.9), while (1.c.1) may be explained as follows:

Given a variation of Hodge structure

$$\varphi : S \to D,$$

which for present purpose we write as a variable Hodge filtration

$$\varphi(s)=\{F_s^p\}, \qquad F_s^p \subset H,$$

for each point $s_0 \in S$ there is an associated infinitesimal variation of Hodge structure given by

$$\varphi_*: T_{s_0}(S) \to T_{\varphi(s_0)}(D). \tag{1.c.3}$$

Given $\xi_1, \xi_2 \in T_{s_0}(S)$ we choose local coordinates $(s^1,\dots,s^m)$ around s_0 so that

$$\xi_i = \partial/\partial s^i|_{T_s0(S)}.$$

Then for a holomorphic section $\psi(s)$ $F_s^p \subset H$ we have

$$\frac{\partial^2\psi(s)}{\partial s^1\partial s^2}=\frac{\partial^2\psi(s)}{\partial s^2\partial s^1},$$

which in particular implies that,

$$\frac{\partial}{\partial s^1}\left(\frac{\partial\psi(s)}{\partial s^2}\right)\equiv\frac{\partial}{\partial s^2}\left(\frac{\partial\psi(s)}{\partial s^1}\right) \bmod F_s^{p-2}$$

i.e.,

$$\varphi_*\left(\frac{\partial}{\partial s_1}\right)\varphi_*\left(\frac{\partial}{\partial s_2}\right)\psi=\varphi_*\left(\frac{\partial}{\partial s_2}\right)\varphi_*\left(\frac{\partial}{\partial s_1}\right)\psi$$

so this equation is just (1.c.1) for the infinitesimal variation of Hodge structure (1.c.3). The slightly subtle point is that, even though the infinitesimal variation of Hodge structure (1.c.3) depends only on the first order behaviour of the mapping φ and s_0, that the condition (1.c.2) be satisfied depends on second order considerations.

Put somewhat differently, every linear subspace $E \subset T_{\{F^p\}}(D)$ of the tangent space to D at a point $\{F^p\} \in D$ is the tangent space to many local submanifolds $N \subset D$ passing through $\{F^p\}$, but there are non-trivial conditions on E in order that we may choose N to be an integral manifold of the horizontal differential system I on D. (13)

DEFINITION: The infinitesimal variation of Hodge structure $V=\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ is said to *arise from geometry* in case there exists a projective family $\mathfrak{X} \to S$, where

$$S=\operatorname{Spec}(\mathbf{C}[s^1,\dots,s^m]/m^2), \qquad m=\text{maximal ideal}, \tag{1.c.4}$$

whose associated variation of Hodge structure is V.

REMARKS: An exposition, from the point of view of this paper, of what it means to have a projective family with base space (1.c.4) is given in Chapter VII of [2]. Again, the slightly subtle point is that not every family with base (1.c.4) gives rise to an infinitesimal variation of Hodge structure. It turns out that for this a sufficient condition is that $\mathcal{X} \to S$ be the restriction of a projective family over $\mathrm{Spec}(\mathbf{C}[s^1, \ldots, s^m]/m^3)$ (i.e., we should have a 2nd order variation of the central fibre). (14)

The matters are reconsidered somewhat more systematically below, cf. the discussion following (1.c.11).

Because of the homogeneity of D there are no linear algebra invariants of a polarized Hodge structure (although there may be other invariants that are transcendental in the coordinates (in $\check{D}$) of the Hodge structure, such as the theta divisor of a principally polarized abelian variety). On the other hand, there are a plethora of linear algebra invariants of an infinitesimal variation of Hodge structure, (15) and the problem then becomes one of sifting out from among this multitude those that have geometric meaning in case the infinitesimal variation of Hodge structure arises from geometry. We shall now give five invariants that have geometric significance and that have proven useful in applications.

In this discussion, $V = \{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ will be a fixed infinitesimal variation of Hodge structure of weight n.

Construction #1. Given a vector space $\mathcal{U}$ we identify $\mathrm{Hom}(\mathcal{U}, \mathcal{U}^*)$ with $\mathcal{U}^* \otimes \mathcal{U}^*$, and will denote by $\mathrm{Hom}^{(s)}(\mathcal{U}, \mathcal{U}^*) = \mathrm{Sym}^2 \mathcal{U}^*$ the subspace of symmetric transformations in $\mathrm{Hom}(\mathcal{U}, \mathcal{U}^*)$ (thus, $\varphi \in \mathrm{Hom}^{(s)}(\mathcal{U}, \mathcal{U}^*)$ means that $\langle \varphi(u_1), u_2 \rangle = \langle \varphi(u_2), u_1 \rangle$ for all $u_1, u_2 \in \mathcal{U}$). For a polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ there are natural isomorphisms

$$H^{p,n-p*} \cong H^{n-p,p}$$

induced by the non-degenerate pairing

$$Q: H^{p,n-p} \otimes H^{n-p,p} \to \mathbf{C}.$$

Given $\xi_1, \ldots, \xi_n \in T$, the linear mapping

$$\delta(\xi_1) \ldots \delta(\xi_n): H^{n,0} \to H^{0,n} \tag{1.c.5}$$

is, using respectively (1.c.1) and (1.c.2), readily seen to be symmetric in $\xi_1, \ldots, \xi_n$ and symmetric as an element in $\mathrm{Hom}(H^{n,0}, H^{n,0*})$. Thus we have a linear mapping

$$\delta^n: \mathrm{Sym}^n T \to \mathrm{Hom}^{(s)}(H^{n,0}, H^{n,0*}) \tag{1.c.6}$$

that will be referred to as the n^{th} *iterate of the differential* δ. The dual of (1.c.6) is a linear mapping

$$\delta^{*n}: \mathrm{Sym}^2 H^{n,0} \to \mathrm{Sym}^n T^* \tag{1.c.7}$$

that will be referred to as the n^{th} *interate of the codifferential* δ^*.

DEFINITION: $\mathcal{I}(V)$ will denote the linear system of quadrics on $\mathbb{P}H^{0,n}$ given by the kernel of δ^{*n}.

In this way, to an infinitesimal variation of Hodge structure we have intrinsically associated a linear systems of quadrics that will itself have invariants such as the base locus, locus of singular quadrics, etc.

Construction #2. This is a variant of the first construction. We consider the symmetric linear transformation

$$\delta^{n-2p}: \mathrm{Sym}^{n-2p} T \to \mathrm{Hom}^{(s)}(H^{n-p,p}, H^{n-p,p^*}). \tag{1.c.8}$$

DEFINITION: We denote by $\Sigma_{p,k} \subset \mathbb{P}T$ the determinantal variety defined by

$$\Sigma_{p,k} = \{\xi : \mathrm{rank}\, \delta^{n-2p}(\xi) \leqslant k\}.$$

As special cases we set

$$\begin{cases} \Sigma_k = \Sigma_{0,k} \\ \Sigma = \Sigma_{h^{n,0}-1}; \end{cases}$$

the latter is given by

$$\sum = \{\xi \in \mathbb{P}T : \det \delta^n(\xi) = 0\}.$$

In this paper we shall discuss the projective interpretation of the first invariant (and of a generalization of it to be given below), and in the third paper of this series we will discuss the geometric meaning of the second invariant. In each case there are specific open questions, and a deeper understanding of infinitesimal variations of Hodge structure depends on their resolution together with further computation of example.

Construction #3: This invariant will be defined in case of even weight $n = 2m$.

DEFINITION: Given a Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}\}$ of weight $n = 2m$, the space of *Hodge classes* is defined by

$$H_{\mathbf{Z}}^{m,m} = H_{\mathbf{Z}} \cap H^{m,m}.$$

DEFINITION: Given an infinitesimal variation of Hodge structure $V = \{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ of weight $2m$ and Hodge class $\gamma \in H_{\mathbf{Z}}^{m,m}$, we set

$$H^{m+k,m-k}(-\gamma)$$
$$= \{\psi \in H^{m+k,m-k} : Q(\delta^k(\xi)\psi, \gamma) = 0 \qquad \text{for all} \quad \xi \in T\}.$$

This invariant will be discussed in the second paper of this series. If γ is the fundamental class of a primitive algebraic m-cycle Γ on a smooth variety X of dimension $2m$, then it will be easy to see that (taking $k = m$)

$$H^0(X, K(-\operatorname{supp}\Gamma)) \subseteq H^{2m,0}(-\gamma), \tag{1.c.9}$$

where the left hand side denotes the holomorphic $2m$-forms vanishing on the support of the cycle Γ. There is a generalization of (1.c.9) to the intermediate Hodge groups $H^{m+k,m-k}$, and in some cases we will be able to prove that (1.c.9) is an equality. In this way we will be able to show, e.g., that a smooth surface $X \subset \mathbb{P}^3$ with the same infinitesimal variation of Hodge structure as the Fermat surface $F_d = \{x_0^d + x_1^d + x_2^d + x_3^d = 0\}$ must be projectively equivalent to F_d, for $d \geqslant 5$.

These first three constructions give invariants of an infinitesimal variation of Hodge structure, which is linear algebra data abstracting the description of the differential of a variation of Hodge structure. Our next construction will be based on the 1st order behaviour of a normal function, and for this some preliminary discussion is necessary.

To begin we consider a classifying space D for polarized Hodge structures of odd weight $2m - 1$. Over D we may, in the obvious way, construct a fibre space

$$\tilde{\omega} : \mathbb{J} \to D \tag{1.c.10}$$

of complex tori whose fibre over a point $\{F^p\} \in D$ is the corresponding intermediate Jacobian

$$F^m \backslash H / H_{\mathbf{Z}} = J(\{F^p\}).$$

The action of $G_{\mathbf{Z}} = \operatorname{Aut}(H_{\mathbf{Z}}, Q)$ on D lifts to an action on $\mathbb{J}$. Given any variation of Hodge structure

$$\varphi : S \to \Gamma \backslash D,$$

the corresponding family of complex tori $\mathcal{J} \to S$ introduced in Section 1 (b) is obtained by pulling back the universal family (1.c.10) to the universal covering $\tilde{S}$ of S, and then passing to the quotient by the action of $\pi_1(S)$. Thus we may think of $\mathbb{J} \to D$ is being a classifying space for families of polarized intermediate Jacobians.

There is, however, one important difference. Whereas D is a homogeneous space for the group $G = \operatorname{Aut}(H_{\mathbb{R}}, \mathbb{Q})$, so that any two tangent spaces to D "look alike", the automorphism group of (1.c.10) is the discrete group $G_{\mathbb{Z}}$, whose action on $\mathbb{J}$ is very far from being transitive. For example, in the classical case where $D = \mathcal{H}_g$ is the Siegel generalized upper-half-space and $\mathbb{J} \to \mathcal{H}_g$ is the versal family of principally polarized abelian varieties, this *non*-homogeneity is reflected by the very fortunate circumstance that the theta divisor is different for different abelian varieties. It is for this reason that there is no existing theory of "the differential of a normal function". In fact, it seems difficult to give good geometric meaning to all of the first order behaviour of a normal function.

On the other hand, we can give meaning to at least part of this infinitesimal behaviour as follows: First, we observe that it makes perfectly good sense to speak of a variation of Hodge structure

$$\varphi : S \to D \tag{1.c.11}$$

when S is a non-reduced analytic space. * For example, when

$$S = S_k = \operatorname{Spec} \mathbb{C}\left[s^1, \dots, s^m\right]/m^{k+1}, \tag{1.c.12}$$

where $m = \{s^1, \dots, s^m\}$ is the maximal ideal in $\mathbb{C}[s^1, \dots, s^m]$, a variation of Hodge structure

$$\varphi : S_k \to D$$

may be thought of as a k^{th} order variation of a given Hodge structure (or, equivalently, as a *k-jet* of variation of Hodge structure). With this terminology, an infinitesimal variation of Hodge structure $V = \{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ gives

$$\varphi : S_1 \to D \tag{1.c.13}$$

where

$$T^* = m/m^2. \tag{1.c.14}$$

* The referee remarks that here the Gauss-Manin connection is not determined by (1.c.11).

Secondly, let (1.c.11) be a variation of Hodge structure of weight $2m-1$ with

$$\mathcal{J} \rightarrow S$$

the corresponding family of complex tori, and $\nu \in H^0(S, \mathcal{J}_h)$ a normal function; observe that this also makes sense when S is non-reduced.

DEFINITION: An *infinitesimal normal function* (V, ν) is given by an infinitesimal variation of Hodge structure V together with $\nu \in H^0(S_1, \mathcal{J}_h)$ where S_1 is the analytic space (1.c.12) (when $k=1$) and $\mathcal{J} \rightarrow S_1$ is given by (1.c.13). (16)

We are now ready to define the invariant $\delta\nu$ associated to an infinitesimal normal function. The construction proceeds in two steps.

Step one. Given an infinitesimal variation of Hodge structure $V = \{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ of weight $2m-1$, we define

$$\Xi \subset \mathbb{P}T \times \mathbb{P}H^{m,m-1}$$

by the condition

$$\Xi = \{(\xi, \omega) : \delta(\xi)\omega = 0\}$$

(what we mean here is that $\delta(\tilde{\xi})\tilde{\omega} = 0$ for any liftings of ξ, ω to T, $H^{m,m-1}$ respectively).

Using the notation of construction #2 and setting $h = h^{m,m-1}$, the projection of Ξ on the first factor induces a fibering

$$\Xi \rightarrow \Sigma_{m-1,h-1} \tag{1.c.16}$$

whose fibre over $\xi \in \Sigma_{m-1,h-1} \subset \mathbb{P}T$ is the projective space $\mathbb{P}(\ker \delta(\xi))$. Consequently, from the theory of determinantal varieties (cf. Chapter II of [2] for a discussion from this point of view) it follows that (1.c.16) is a natural candidate for a desingularization of $\Sigma_{m-1,h-1}$, by analogy with the standard desingularization of the $m \times m$ matrix of rank $\leqslant r$.

Step two. Let (V, ν) be an infinitesimal normal function and

$$v : S_1 \rightarrow H \tag{1.c.17}$$

any lifting that induces ν. Such liftings clearly exist, and any other lifting is of the form

$$\tilde{v} = v + \lambda + f$$

where

$$\begin{cases} \lambda : S_1 \to H_{\mathbf{Z}} \\ f: S_1 \to \mathcal{F}^m \end{cases}$$

For a tangent vector $\xi \in T$, it follows from these equations that

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\xi} = \frac{\mathrm{d}v}{\mathrm{d}\xi} + \frac{\mathrm{d}f}{\mathrm{d}\xi};$$

in particular, denoting by $F^m + \delta(\xi)F^m \subset F^{m-1}$ the span of F^m and $\delta(\xi)F^m$ in F^{m-1},

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\xi} \equiv \frac{\mathrm{d}v}{\mathrm{d}\xi} \text{modulo } F^m + \delta(\xi)F^m.$$

On the other hand, by the differential condition that defines normal functions,

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} \in F^{m-1}.$$

If we now observe that

$$F^{m-1}/(F^m + \delta(\xi)F^m) \cong H^{m-1,m}/\delta(\xi)H^{m,m-1},$$

and

$$Q(\delta(\xi)H^{m,m-1}, \omega) = 0 \qquad (\text{from } \delta(\xi)\omega = 0 \text{ and 1.c.2.}),$$

then we may give the

DEFINITION: The *infinitesimal invariant* $\delta\nu$ is given by

$$\delta\nu = Q\left(\frac{\mathrm{d}v}{\mathrm{d}\xi}, \omega\right).$$

REMARKS: (i) If we denote by $\mathcal{O}(k, l)$ the restriction to Ξ of the line bundle $\mathcal{O}_{\mathbb{P}T}(k) \otimes \mathcal{O}_{\mathbb{P}H^{m,m-1}}(l)$ on $\mathbb{P}T \times \mathbb{P}H^{m,m-1}$, then

$$\delta\nu \in H^0(\Xi, \mathcal{O}(1, 1)). \tag{1.c.18}$$

(ii) The motivation for this construction stems from the following differential geometric consideration: Let H be a vector space of even dimension $2p$ and $G = G(p, H)$ the Grassmann manifold of p-planes $F \subset H$. Over G we have the universal bundle sequence

$$0 \to S \to H \to Q \to 0,$$

where the fibres are respectively

$$S_F = F, \qquad H_F = H, \qquad Q_F = H/F.$$

Given on open set $\mathcal{U} \subset G$ and holomorphic section $\nu \in H^0(\mathcal{U}, Q)$ we ask the question:

Is ν the projection of a constant section $v \in H$?

Since there is no $GL(H)$-invariant connection on Q, this question does not appear to have an easy natural answer.

However, if we make the standard identification

$$T(G) \cong \operatorname{Hom}(S, Q)$$

and define

$$\Sigma \subset \mathbb{P}T(G)$$

by

$$\Sigma = \bigcup_{F \in G} \Sigma_F,$$

where

$$\Sigma_F = \{\xi \in \mathbb{P}T_F(G) : \xi(F) \text{ is a proper subspace of } Q_F\},$$

then the pullback of Q to Σ has a natural *partial connection*. That is, given any lifting v of ν the expressions

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} \in H/(F + \xi \cdot F) \tag{1.c.19}$$

are well-defined (up to scalars) on Σ. The vanishing of (1.c.19) is clearly a *necessary* condition that ν be induced from a constant section of $H \to G$, and provides the motivation for our construction of $\delta\nu$.$^{(17)}$

Our last invariant is also motivated by local differential geometric considerations of the Grassmannian, together with the following analogy: In Euclidean differential geometry (i.e., in the study of submanifolds of $\mathbb{R}^N$), it is the 2nd order invariants interpreted as 2nd fundamental form that play the dominant role. Therefore it makes sense to look also for 2nd order invariants of a variation of Hodge structure. Following some preliminary remarks on the Grassmannian, we will define one of these.

Let H be an n-dimensional vector space and denote by $\mathcal{F}(H)$ the set of all frames $\{e_1, \ldots, e_n\}$ in H. On $\mathcal{F}(H)$ we have the structure equations of a moving frame (cf. the exposition in [24])

$$\begin{cases} \mathrm{d}e_i = \omega_i^j e_j \\ \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j \end{cases} \tag{1.c.20}$$

where we use summation convention and the index range $1 \leqslant ij,\ k \leqslant n$. Now let $G = G(p, H)$ be the Grassmann manifold of p-planes $F \subset H$. There is a fibering

$$\pi : \mathcal{H}(H) \to G(p, H) \tag{1.c.21}$$

defined by

$$\pi\{e_1, \ldots, e_n\} = \text{span of } \{e_1, \ldots, e_p\}.$$

If we use the additional index range

$$\begin{cases} 1 \leqslant \alpha, \beta \leqslant p \\ p + 1 \leqslant \mu, \nu \leqslant n, \end{cases} \tag{1.c.22}$$

then from

$$\mathrm{d}e_\alpha \equiv \omega_\alpha^\mu e_\mu \bmod F,$$

we infer that the forms $\{\omega_\alpha^\mu\}$ are horizontal for the fibering (1.c.21), and in fact give a basis for

$$\pi^* T^*(G(p, H)) \subset T^*(\mathcal{F}(H)).$$

Now let $M \subset G$ be a submanifold of codimension r and set $\mathcal{F}(M) = \pi^{-1}(M)$. Then on $\mathcal{F}(M)$ there is, at each point, an r-dimensional space of matrices $b = (b_\mu^\alpha)$ giving the relations

$$b_\mu^\alpha \omega_\alpha^\mu = 0 \tag{1.c.23}$$

that define the conormal spaces $N^*(M)$ to $T(M)$ in $T(G)$. Setting

$$Db_\mu^\alpha = \mathrm{d}b_\mu^\alpha + b_\mu^\beta \omega_\beta^\alpha - b_\nu^\alpha \omega_\mu^\nu$$

the exterior derivative of (1.c.23) gives, using (1.c.20),

$$Db_\mu^\alpha \wedge \omega_\alpha^\mu = 0. \tag{1.c.24}$$

Let $\omega^1,\dots,\omega^m$ be a local coframe on M, where $m=p(n-p)-r$ is the dimension of M, and use the additional index range

$$1 \leqslant a, b \leqslant m.$$

Then on $\mathcal{F}(M)$ we have

$$\omega^\mu_\alpha = h^\mu_{\alpha a}\omega^a$$

$$\left(Db^\alpha_\mu h^\mu_{\alpha a}\right) \wedge \omega^a = 0.$$

This relation plus the Cartan lemma imply that

$$Db^\alpha_\mu h^\mu_{\alpha a} = h_{ab}\omega^b, \qquad h_{ab} = h_{ba}. \tag{1.c.25}$$

The quadratic differential form

$$h_{ab}\omega^a\omega^b \tag{1.c.26}$$

is well defined in $\mathcal{F}(M)$ and is a section of

$$\pi^*\left(N(M) \otimes \operatorname{Sym}^2 T^*(M)\right).$$

It will be convenient to write (1.c.26) as

$$Db^\alpha_\mu \omega^\mu_\alpha, \tag{1.c.27}$$

where the multiplication is symmetric multiplication of l-forms.

The fibres of (1.c.21) are given by linear substitutions

$$\begin{cases} \tilde{e}_\alpha = A^\beta_\alpha e_\beta \\ \tilde{e}_\mu = B^\nu_\mu e_\nu + B^\alpha_\mu e_\alpha \end{cases} \tag{1.c.28}$$

If we set

$$\omega^{\lambda\mu}_{\beta\gamma} = \omega^\lambda_\beta \omega^\mu_\gamma - \omega^\lambda_\gamma \omega^\mu_\alpha \qquad \text{(symmetric multiplication)}, \tag{1.c.29}$$

then it is straightforward to verify that, under a substitution (1.c.28),

$$D\tilde{b}^\alpha_\mu \omega^\mu_\alpha \equiv Db^\alpha_\mu \omega^\mu_\alpha \bmod\left\{\omega^{\lambda\mu}_{\beta\gamma}\text{'}s\right\}.$$

On the other hand, the quadratic differential forms (1.c.29) are just the 2×2 minors of the linear transformation in the subspace

$$T_F(M) \subset \operatorname{Hom}(F, H/F)$$

(these linear transformations may be viewed as matrices whose entries are elements of $T_F^*(M)$).

DEFINITION: The *2nd fundamental form* of M in $G(p, H)$ is the space of quadratic differential forms

$$Db_\mu^\alpha \omega_\alpha^\mu \bmod\{\omega_{\alpha\gamma}^{\lambda\mu}\text{'s}\}.$$

REMARKS: (i) If $T = T_F(M)$ is a typical tangent space to M, then the base *base locus*

$$\Sigma^1 \subset \mathbb{P}T$$

of the quadrics $\omega_{\alpha\gamma}^{\lambda\mu} \in \mathrm{Sym}^2 T^*$ is well-defined; in fact, it is clear that

$$\Sigma^1 = \{\xi \in \mathbb{P}T : \mathrm{rank}\,\xi = 1\}.$$

Then the 2nd fundamental form of M in $G(p, H)$ cuts out a well-defined linear system of quadrics on Σ_1. However, it contains information even when Σ_1 is empty.

(ii) If we view the tangent spaces to M as linear subspaces

$$T_F(M) \subset \mathrm{Hom}(F, H/F),$$

then we have canonical inclusions

$$N_F^*(M) \subset \mathrm{Hom}(H/F, F) \subset \mathrm{Hom}(H, H).$$

Thus, associated to $M \subset G(p, H)$ there is a *Gauss map*

$$\gamma : M \to G(p, H \otimes H^*)$$

defined by

$$\gamma(F) = N_F^*(M).$$

As in ordinary Euclidean differential geometry, the differential of γ contains the information in the 2nd fundamental form.

Finally we can define our last Hodge-theoretic invariant. With S_2 given by (1.c.12) we consider a 2nd order variation of Hodge structure

$$\varphi : S_2 \to D \tag{1.c.31}$$

of odd weight $2m-1$. If $\dim H = 2p$ there is an associated map

$$\psi : S_2 \to G(p, H)$$

induced by the map

$$\begin{cases} D \to G(p, H) \\ \{F^p\} \mapsto F^m \subset H \end{cases}$$

DEFINITION: The *2nd fundamental form* of the 2nd order infinitesimal variation of Hodge structure is given by the 2nd fundamental form of $\psi(S_2)$ in $G(p, H)$.

In Part IV of this series of papers we will geometrically interpret the second fundamental form for families of algebraic curves, and also for some special higher dimensional examples.

2. Infinitesimal Schottky relations

(a) The basic diagram

With our previous notation (cf. (1.c.121)

$$S_1 = \operatorname{Spec} \mathbb{C}[s^1, \dots, s^m]/m^2,$$

we recall that an infinitesimal variation of Hodge structure $V = \{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ is said to come from geometry (cf. the discussion above (1.c.4)) in case there exists a projective family

$$\mathcal{X} \to S_1 \tag{2.a.1}$$

whose associated infinitesimal variation of Hodge structure is V. Thus, in particular

$$T^* = m/m^2. \tag{2.a.2}$$

In practice this means the following: First the central or reduced fibre of (2.a.1) should be a smooth polarized variety (X, ω) whose n^{th} primitive cohomology is the Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$. Next, with the notation (2.a.2), we denote by

$$\rho : T \to H^1(X, \Theta) \tag{2.a.3}$$

the *Kodaira-Spencer mapping* [29]. A basic fact is that the differential δ of the variation of Hodge structure associated to (2.a.1) may be expressed in terms of ρ, and when this is done we may sometimes "compute" the infinitesimal variation of Hodge structure V. $^{(18)}$

More precisely, we first ignore the polarization and ask how the subspace

$$H^{p,q}(X) \subset H^n(X, \mathbb{C}) \qquad (p + q = n)$$

moves when X is deformed in the direction of a tangent vector $\xi = \sum_i \xi^i \partial/\partial s^i \in T$. Recalling that

$$H^{p,q}(X) \cong H^q(X, \Omega^p),$$

the answer is that the differential

$$\delta_p(\xi) \in \mathrm{Hom}\big(H^q(X, \Omega^p), H^{q+1}(X, \Omega^{p-1})\big)$$

is given by cup-product with $\rho(\xi)$. (19) Equivalently, the diagram

$$\begin{tikzcd} T \arrow[r, "\delta_p"] \arrow[d, "\rho"'] & \mathrm{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)) \\ H^1(X, \Theta) \arrow[ur, dashed, "\kappa"'] & \end{tikzcd} \tag{2.a.4}$$

is commutative, where κ is the mapping given by cup-product (see [19]).

One consequence is that, since (2.a.1) is a projective family,

$$\rho(\xi) \cdot \omega = 0 \text{ in } H^{0,2}(X) \tag{2.a.5}$$

for all $\xi \in T$. Recalling the definition of the primitive cohomology, it follows that the cup-product with $\rho(\xi)$ maps primitive spaces to primitive spaces. In other words, setting

$$H^{p,q} = H^{p,q}_{\mathrm{prim}}(X)$$

$$H^1(X, \Theta)_\omega = \{\theta \in H^1(X, \Theta) : \theta \cdot \omega = 0 \text{ in } H^{0,2}(X)\}, \tag{2.a.6}$$

the diagram (2.a.4) has the following commutative sub-diagram

$$\begin{tikzcd} \mathrm{T} \arrow[r, "\delta_p"] \arrow[d, "\rho"'] & \mathrm{Hom}(H^{p,q}, H^{p-1,q+1}) \\ H^1(X, \Theta)_\omega \arrow[ur, "\kappa"'] & \end{tikzcd}$$

Summarizing, if we set $\delta = \oplus\, \delta_p$ then *the differential of the variation of*

Hodge structure associated to (2.a.1) *is expressed by the commutativity of the diagram*

$$\begin{tikzcd}
T \arrow[rr, "\delta"] \arrow[d, "\rho"'] & & \oplus\, \mathrm{Hom}(H^{p,q}, H^{p-1,q+1}) \\
H^1(X, \Theta)_\omega \arrow[urr, "\kappa"'] & &
\end{tikzcd} \tag{2.a.7}$$

where ρ *is the Kodaira-Spencer mapping.*

In our work we shall use the following:

DEFINITION: The *infinitesimal Torelli theorem* holds for the polarized variety (X, ω) in case the mapping

$$\kappa : H^1(X, \Theta)_\omega \to \oplus\, \mathrm{Hom}(H^{p,q}, H^{p-1,q+1})$$

is injective.

REMARKS: There are several ways in which this definition is a misnomer. The most serious is the phenomena pointed out in Oort and Steenbrink [38] that, due to the presence of automorphisms, the period map with source the coarse moduli scheme may be injective although the tangent mapping κ for the period map with source the fine moduli space may fail to be injective.

Our goal is to interpret cohomologically, and eventually geometrically, the 1st construction in Section 1 (d). For this some preliminary remarks are necessary. Namely, we shall define natural mappings

$$\mathrm{Sym}^q H^1(X, \Theta) \to H^q(X, \Theta), \tag{2.a.8}$$

denoted by

$$(\theta_1, \dots, \theta_q) \to \theta_1 \dots \theta_q (\theta_i \in H^1(X, \Theta)).$$

For this we use the Dolbeault isomorphism

$$H^q(X, \Theta) \cong H^{0,q}_{\bar\partial}(X, \Theta).$$

For a vector-valued (0, 1) form given locally by

$$\theta = \sum \theta^i_j \partial/\partial z^i \otimes d\bar{z}^j,$$

we set

$$\theta^q = \sum \theta^{i_1}_{j_1} \dots \theta^{i_q}_{j_q} \partial/\partial z^{i_1} \wedge \dots \wedge \partial/\partial z^{\iota_q} \otimes \mathrm{d}\bar{z}^{J_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{J_q}. \qquad (2.a.9)$$

In other words, thinking of θ as a section of Hom($\bar{T}$, T) where $T \to X$ is the holomorphic tangent bundle, θ^q is the induced section of Hom($\Lambda^q \bar{T}$, $\Lambda^q T$). It is easy to verify that the polarization of the map (2.a.9) induces on Dolbeault cohomology a natural map (2.a.8). In particular, if

$$\dim X = n,$$

then (2.a.9) is given for $q = n$ by

$$\theta^n = \det \| \theta^i_j \| \partial/\partial z^1 \wedge \dots \wedge \partial/\partial z^n \otimes \mathrm{d}\tilde{z}^1 \wedge \dots \wedge \mathrm{d}\tilde{z}^n. \qquad (2.a.10)$$

We remark that the natural sheaf map

$$\otimes^q \Theta \to \Lambda^q \Theta \qquad (2.a.11)$$

together the the ordinary cup-product induce

$$\otimes^q H^1(X, \Theta) \to H^q(X, \otimes^q \Theta) \to H^q(X, \Lambda^q \Theta),$$

and (2.a.8) is the composite. The reason it is symmetric is that both the cup-product and (2.a.11) are alternating.

We also remark that the composition of the Kodaira-Spencer mapping (2.a.3) with (2.a.8) induces

$$\rho^q : \mathrm{Sym}^q T \to H^q(X, \Lambda^q \Theta). \qquad (2.a.12)$$

We now consider the iterated differential

$$\delta^q : \mathrm{Sym}^q T \to \mathrm{Hom}(H^{n-k,k}, H^{n-k-q,k+q})$$

of the infinitesimal variation of Hodge structure associated to (2.a.1) (here we drop the subscript on δ). Although completely straightforward to verify, (20) a basic fact is:

The following diagram is commutative

$$\begin{tikzcd}
\mathrm{Sym}^q T \arrow[rr, "\delta^q"] \arrow[dr, "\rho^q"'] & & \mathrm{Hom}(H^{n-k,k}, H^{n-k-q,k+q}) \\
& H^q(X, \Lambda^q\Theta), \arrow[ur, "\kappa"'] &
\end{tikzcd} \tag{2.a.13}$$

where ρ^q is given by (2.*a*.12) *and κ by cup-product.*

In particular when $q = n - 2k$ we have

$$\begin{tikzcd}
\mathrm{Sym}^{n-2k} T \arrow[rr] \arrow[dr, "\rho^{n-2k}"'] & & \mathrm{Hom}^{(s)}(H^{n-k,k}, H^{k,n-k}) \\
& H^{n-2k}(X, \Lambda^{n-2k}\Theta) \arrow[ur, "\kappa"'] &
\end{tikzcd} \tag{2.a.14}$$

When $k = 0$ this diagram reduces to

$$\begin{tikzcd}
\mathrm{Sym}^n T \arrow[rr] \arrow[dr, "\rho^n"'] & & \mathrm{Hom}^{(s)}(H^{n,0}, H^{0,n}) \\
& H^n(X, \Lambda^n\Theta) \arrow[ur, "\kappa"'] &
\end{tikzcd} \tag{2.a.15}$$

The dual of (2.a.15) gives what we shall call our *basic diagram* $^{(21)}$

$$\begin{tikzcd}
\mathrm{Sym}^2 H^0(X, K) \arrow[rr, "\delta^{*n}"] \arrow[dr, "\nu"'] & & \mathrm{Sym}^n T^* \\
& H^0(X, K^2) \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

Here, ν is the usual multiplication of sections (it is easily verified that this is the dual of κ), and $\lambda = (\rho^n)^*$. As will be seen below, the basic diagram provides one link between the infinitesimal variation of Hodge structure and the projective geometry of X.

For example, recalling the definition

$$\mathcal{J}(V) = \ker \delta^{*n}$$

of the linear system of quadrics associated to the infinitesimal variation of Hodge structure V arising from (2.a.1), we have the exact sequence

$$0 \to \ker \nu \to \mathcal{J}(V) \to (\ker \lambda) \cap (\text{image } \nu) \to 0 \qquad (2.a.16)$$

Definition: In case the infinitesimal variation of Hodge structure V arises from geometry, we shall refer to $\mathcal{J}(V)$ as the *infinitesimal Schottky relations* for the projective family $\mathcal{X} \to S_1$ that defines V.

The motivation for this terminology comes from the case of algebraic curves discussed below.

To interpret the infinitesimal Schottky relations we consider the canonical mapping

$$\varphi_K \colon X \to \mathbb{P}H^n(X, \mathcal{O}) \qquad (2.a.17)$$

Choosing a basis $\omega_0, \omega_1, \ldots, \omega_r$ for $H^0(X, K)$ gives a set of homogeneous coordinates in

$$\mathbb{P}H^n(X, \mathcal{O}) = \mathbb{P}H^0(X, K)^* = \mathbb{P}^r,$$

and we shall refer to $\mathbb{P}^r$ as the space of the canonical image of X. It is clear that:

> $\ker \nu = I_{\varphi_K(X)}(2)$ *is the space of quadrics in* $\mathbb{P}^r$ *that pass through the canonical image* $\varphi_K(X)$.

Moreover, in case the quadrics in $\mathbb{P}^r$ cut out a complete linear system (i.e. ν is onto), then (2.a.16) reduces to

$$0 \to I_{\varphi_K(X)}(2) \to \mathcal{J}(V) \to \ker \lambda \to 0. \qquad (2.a.18)$$

Example: Suppose that $X = C$ is a smooth curve of genus $g \geqslant 2$, and let

$$\mathcal{X} \to S_1$$

be the 1st order part of the local moduli space (*Kuranishi space*, cf. [31]) of C. Then

$$T = H^1(X, \Theta)$$

and λ is an isomorphism. It is well known that if $g = 2$ or $g \geqslant 3$ and C is non-hyperelliptic, then

$$\nu : \mathrm{Sym}^2 H^0(C, K) \rightarrow H^0(C, K^2)$$

is surjective (this is the theorem of *Max Noether*, cf. [20] and [47]). In this case the infinitesimal Torelli theorem holds (in fact, when suitably interpreted it always holds – cf. [38]).

Example: (Continuation of preceding example): Now suppose that C is non-hyperelliptic, non-trigonal, and not a smooth plane quintic. A general curve of genus $g \geqslant 5$ has this property. By the theorem of *Babbage-Enriques-Petri* (cf. [20] and [42])

$$\varphi_K(C) = \bigcap_{Q \in I_{\varphi_K(C)}(2)} Q; \tag{2.a.20}$$

i.e., the canonical curve is the intersection of the quadrics in $\mathcal{G}(V)$. Form this we conclude that:

The weak global Torelli theorem holds
for smooth curves of genus $g \geqslant 5$. (2.a.21)

Proof: Suppose the extended period mapping

$$\varphi : \mathfrak{M}_g \rightarrow \Gamma \backslash \mathcal{H}_g$$

has degree $d \geqslant 1$, and choose a Z which is a non-singular point of the variety $\varphi(\mathfrak{M}_g)$ and also Z is a regular value of the map $\mathfrak{M}_g \overset{\varphi_g}{\rightarrow} \varphi(\mathfrak{M}_g)$, and such that

$$\varphi^{-1}(Z) = \{C_1, \ldots, C_d\}$$

consists of d distinct curves of genus $g \geqslant 5$, each of which satisfies (2.a.20). Then the C_i all have the same 1st order infinitesimal variation of Hodge structure, and hence all the $\varphi_K(C_i)$ must coincide. This can only happen if $d = 1$.

Briefly, *whenever we have a moduli space whose general member can be reconstructed from its infinitesimal variation of Hodge structure* (of *any* order), *then the weak global Torelli theorem holds.* *

Remark: Later we shall extend this result to the case $g = 4$, using the infinitesimal invariant $\delta\nu$ associated to the naturally defined normal function.

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* This has been carried through by Ron Donagi for hypersurfaces in $\mathbb{P}_n$.

Of course, (2.a.21) follows from the usual Torelli theorem for curves for which there are by now a large number of proofs (cf. [2]). However, it has the advantage of using only ingredients that generalize to higher dimension (in particular, it does *not* use the theta divisor).

(b) Infinitesimal Schottky relations and the Gauss linear system

Suppose that $V=\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ is an infinitesimal variation of Hodge structure of weight n arising from a projective family (2.a.1) whose central fibre is an n-dimensional polarized algebraic variety (X, ω). We consider the canonical mapping

$$\varphi_K : X \to \mathbb{P}^r,$$

and assume for the moment that the quadrics in $\mathbb{P}^r$ cut out the complete linear system $H^0(X, K^2)$; i.e., the mapping

$$\nu : \mathrm{Sym}^2 H^0(X, K) \to H^0(X, K^2) \qquad (2.\mathrm{b}.1)$$

should be surjective. Then (2.a.14) gives the exact sequence

$$0 \to \ker \nu \to \mathcal{G}(V) \to \ker \lambda \to 0 \qquad (2.\mathrm{b}.2)$$

where λ is the mapping

$$\lambda : H^0(X, K^2) \to \mathrm{Sym}^n T^* \qquad (2.\mathrm{b}.3)$$

induced from the dual of the n^{th} iteratie of the Kodaira-Spencer mapping (2.a.3). Since $\ker \nu$ is just the linear system of quadrics passing through $\varphi_K(X)$, we may interpret $\ker \lambda$ as a linear subsystem of the system cut out on $\varphi_K(X)$ by the quadrics in $\mathbb{P}^r$. *From the exact sequence* (2.*b*.2) *we see the geometric interpretation of the infinitesimal Schottky relations* $\mathcal{G}(V)$ *resides in understanding the linear subsystem*

$$\ker \lambda \subset H^0(X, K^2).$$

We shall give a geometric theorem that explains part of $\ker \lambda$. (22)

To explain this we assume given a projective embedding

$$X \subset \mathbb{P}^N. \qquad (2.\mathrm{b}.4)$$

We denote by $L \to X$ the hyperplane line bundle and assume that $c_1(L)$ is a rational multiple of the polarizing class ω. If $N \to X$ is the normal bundle, then it is well-known that $T = H^0(X, N)$ parametrizes the 1st order deformations of X *in* $\mathbb{P}^N$ (cf. [30]). We therefore consider the corresponding projective family

$$\mathcal{X} \to S_1 \qquad (2.\mathrm{b}.5)$$

where $S_1 = \operatorname{Spec} \mathbb{C}[s^1, \ldots, s^m]/m^2$ and

$$m/m^2 = T^*,$$

and assume that (2.b.5) gives an infinitesimal variations of Hodge structure V (this is satisfied, e.g., if the deformations of $X \subset \mathbb{P}^N$ corresponding to $\xi \in H^0(X, N)$ are all unobstructed). We recall that the Kodaira-Spencer mapping

$$\rho : T \to H^1(X, \Theta)$$

is the coboundary map in the exact cohomology sequence of

$$0 \to \Theta \to \Theta_{\mathbb{P}^N} \otimes \mathcal{O}_X \to N \to 0 \tag{2.b.6}$$

where $\Theta_{\mathbb{P}^N}$ is the tangent sheaf of $\mathbb{P}^N$.

We denote by $\mathbb{G}(n, N)$ the Grassmannian of $\mathbb{P}^n$'s in $\mathbb{P}^N$ and consider the Gauss mapping

$$\gamma : X \to \mathbb{G}(n, N)$$

If $u = (u^1, \ldots, u^n)$ are local holomorphic coordinates on an open set $\mathcal{U} \subset X$ and

$$u \to X(u)$$

is a holomorphic mapping from $\mathcal{U}$ to $\mathbb{C}^{N+1} - \{0\}$ that gives the inclusion $\mathcal{U} \subset X \subset \mathbb{P}^N$ via the projection $\mathbb{C}^{N+1} - \{0\} \to \mathbb{P}^N$, the composition of γ with the Plücker embedding

$$\mathbb{G}(n, N) \subset \mathbb{P}(\Lambda^{n+1}\mathbb{C}^{N+1})$$

is given by

$$u \to X(u) \wedge \frac{\partial X(u)}{\partial u^1} \wedge \ldots \wedge \frac{\partial X(u)}{\partial u^n}. {}^{(23)} \tag{2.b.7}$$

Denoting by $H \to \mathbb{G}(n, N)$ the hyperplane line bundle, it follows from (2.b.7) that

$$\gamma^{-1}(H) = KL^{n+1}.$$

Consequently, the Gauss mapping is given by a sub-linear system of $|KL^{n+1}|$ on X, and we denote by

$$\Gamma \subset H^0(X, KL^{n+1}) \tag{2.b.9}$$

the corresponding linear subspace.

DEFINITION: The Gauss linear system, denoted by Γ_{2K}, is the image of

$$\Gamma \otimes H^0(X, KL^{-(n+1)}) \to H^0(X, K^2)^{(24)}$$

EXAMPLE: Suppose that

$$X \subset \mathbb{P}^{n+1}$$

is a smooth hypersurface with defining equation

$$F(x^0, x^1, \ldots, x^{n+1}) = 0.$$

The Gauss map

$$\gamma : X \to \mathbb{P}^{n+1*}$$

is the restriction to X of the map

$$\gamma_X : \mathbb{P}^{n+1} \to \mathbb{P}^{n+1*}$$

given by

$$\gamma_X(x) = \left[\frac{\partial F}{\partial x^0}(x), \frac{\partial F}{\partial x^1}(x), \ldots, \frac{\partial F}{\partial x^{n+1}}(x)\right]$$

If $\deg X = d$, then $K \cong \mathcal{O}_X(d-n-2)$ $(K = K_X)$ and $KL^{n+1} \cong \mathcal{O}_X(d-1)$. It is clear that Γ in (2.b.9) is the subspace of $H^0(X, \mathcal{O}_X(n-1)) \cong$ {homogeneous forms of degree $n-1$} spanned by the $\partial F/\partial x^i(x)$. Consequently the *Gauss linear system*

$$\Gamma_{2K} \subset H^0(X, \mathcal{O}_X(2d-2n-4))$$

is *simply the homogeneous part of degree* $2d - 2n - 4$ *in the Jacobian ideal*

$$J_F = \left\{\frac{\partial F}{\partial x^0}, \frac{\partial F}{\partial x^1}, \ldots, \frac{\partial F}{\partial x^{n+1}}\right\}.$$

The result we wish to prove here is the

THEOREM: *The Gauss linear system is always included the space of infinitesimal Schottky relations* $\mathcal{I}(V)$. *More precisely, in the basic diagram* (2.a.16) *we have*

$$\nu^{-1}(\Gamma_{2K}) \subseteq \ker \delta^{*n}.^{(25)} \tag{2.b.10}$$

We will give two proofs of this result. The first one involves a somewhat novel idea in deformation-theoretic computations, while the second will lay the ground for later proofs of similar results.

FIRST PROOF: This proof shows that the relation

$$\nu^{-1}(\Gamma_{2K}) \subseteq \ker \delta^{*n}$$

has a "universal character"; i.e., is a consequence of pulling back relations on a certain flag manifold under the refined Gauss map defined below.

To set up we denote by $\Phi_P = \Theta_{\mathbb{P}^N} \otimes \mathcal{O}_X$ the restriction to X of the tangent bundle to $\mathbb{P}^N$ and consider the big commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O} \arrow[r, phantom, "="] \arrow[d] & \mathcal{O} \arrow[d] & & \\
0 \arrow[r] & \Phi \arrow[r] \arrow[d] & \Phi_P \arrow[r] \arrow[d, "\pi"] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Theta \arrow[r, "j"] \arrow[d] & \Theta_P \arrow[r] \arrow[d] & N \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{2.b.11}$$

Here, $\Phi_P \cong \oplus^{N+1} L$, the middle column is the restriction to X of the *Euler sequence* on $\mathbb{P}^N$, (26) the bottom row is the standard sequence (2.b.6), and $\Phi = \pi^{-1}(\Theta)$. Recalling that the fibre of $L \to \mathbb{P}^N$ over $p \in \mathbb{P}^N$ is just the line $L_p \subset \mathbb{C}^{N+1}$ corresponding to p, we may interpret fibre of $\Phi \to X$ as being the $(n+1)$-plane

$$\Phi_p \subset \mathbb{C}^{N+1}$$

lying over the usual projective tangent plane $T_p(X) \cong \mathbb{P}^n$ to p. (27)

We denote by $G = \mathbb{G}(0, n, N)$ the manifold of all flags

$$\{p \subset \mathbb{P}^n \subset \mathbb{P}^N\}$$

and consider the *refined Gauss mapping*

$$\hat{\gamma}: X \to G$$

defined for $p \in X$ by

$$\hat{\gamma}(p) = \{p \subset T_p(X) \subset \mathbb{P}^N\}. \tag{2.b.12}$$

In other words, the flag in $\mathbb{C}^{N+1}$ corresponding to the flag (2.b.12) in $\mathbb{P}^N$ is

$$L_p \subset \Phi_p \subset \mathbb{C}^{N+1}.$$

From this follows our first main observation:

(2.b.11) *is the pullback to* X *under the refined Gauss mapping* $\hat{\gamma}$ *of a similar universal diagram over* G. (2.b.13)

We shall denote this similar diagram by the same symbols as in (2.b.11) but with a hat ^ over the entries; thus

$$\hat{\gamma}^*\hat{\mathcal{O}} = \mathcal{O}, \qquad \hat{\gamma}^*\hat{\Theta} = \Theta, \qquad \hat{\gamma}^*\hat{N} = N$$

and so forth.

The exact sequence (2.b.6) is defined by an extension class

$$\psi \in H^1(X, \Theta \otimes N^*), \tag{2.b.14}$$

and clearly

$$\psi = \hat{\gamma}^*(\hat{\psi})$$

where

$$\hat{\psi} \in H^1(G, \hat{\Theta} \otimes \hat{N}^*)$$

defines the extension

$$0 \to \hat{\Theta} \to \hat{\Theta}_P \to \hat{N} \to 0.$$

Next we recall the following linear algebra construction: Given vector spaces A, B, C and $\psi \in A \otimes B \otimes C$, there is induced a vector

$$\psi^n \in \Lambda^n A \otimes \Lambda^n B \otimes \mathrm{Sym}^n C.$$

Indeed, we may think of

$$\psi \in \mathrm{Hom}(A^*, B) \otimes C$$

as a matrix whose entries are linear functions of C^*. Then the $n \times n$ minors of ψ are homogeneous polynomials of degree n on C^* and give the element (2.b.15), viewed as sitting in $\mathrm{Hom}(\Lambda^n A^*, \Lambda^n B) \otimes \mathrm{Sym}^n C$.

We now use the Dolbeault isomorphism

$$H^1(X, \Theta \otimes N^*) \cong H^{0,1}_{\bar{\partial}}(X, \Theta \otimes N^*),$$

and for each point $p \in X$ apply the above linear algebra construction when

$$A = \overline{T}_p^*(X), B = T_p(X), C = N_p(X)$$

and $\psi = \psi(p)$ is the value at p of a Dolbeault representative of the extension class ψ. This gives

$$\psi^n \in H_{\bar{\partial}}^{0,n}(X, \Lambda^n\Theta \otimes \mathrm{Sym}^n N^*), \tag{2.b.16}$$

and our second main observation is:

The n^{th} iterate of the differential
$\mathrm{Sym}^n H^0(X, N) \otimes H^0(X, K) \to H^n(X, \mathcal{O})$ (2.b.17)
of the infinitesimal variation of Hodge structure
corresponding to (2.b.5) *is induced by the cup-product*
with ψ^n in (2.b.16).

This follows from the naturality of cup-products with exact cohomology sequences plus the observation that the coboundary map in the exact cohomology sequence of (2.b.6) is given by cup-product with the extension class ψ.

Now we observe that there are homogeneous line bundles $\hat{L}$ and $\hat{K}$ over G that pull back under $\hat{\gamma}$ to L and K over X. In fact, there is an obvious diagram

$$\begin{tikzcd}
& G \subset P^N \times \mathbb{G}(n, n) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^N & & \mathbb{G}(n, N)
\end{tikzcd}$$

such that $\pi_1 \circ \hat{\gamma}$ is the given inclusion $X \subset \mathbb{P}^N$ and $\pi_2 \circ \hat{\gamma} = \gamma$. In particular

$$\begin{cases} \hat{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^N}(1) \\ \hat{K}\hat{L}^{n+1} = \pi_2^* \mathcal{O}_{\mathbb{G}(n,N)}(1) \end{cases}$$

We consider the cup-product pairing

$$H^0(G, \hat{K}\hat{L}^{n+1}) \otimes H^n(G, \hat{K}^* \otimes \mathrm{Sym}^n \hat{N}^*)$$

$$\to H^n(G, \hat{L}^{n+1} \otimes \mathrm{Sym}^n \hat{N}^*), \tag{2.b.18}$$

and claim: Theorem (2.b.10) *will follow if we can show that*

$$H^0(G, \hat{K}\hat{L}^{n+1}) \otimes \hat{\psi}^n \to 0 \tag{2.b.19}$$

in (2.b.18). This is because:

(i) $\Gamma \subset H^0(X, KL^{n+1})$ is given by

$$\Gamma = \hat{\gamma}^* H^0(G, \hat{K}\hat{L}^{n+1})$$

(it is easy to see that $\pi_2^* H^0(\mathbb{G}(n, N), \mathcal{O}(1)) = H^0(G, \hat{K}\hat{L}^{n+1})$));

(ii) The observation that (2.b.19) implies that on X the mapping

$$\Gamma \otimes \psi^n \otimes H^n\big(X, L^{n+1} \otimes \operatorname{Sym}^n N^*\big)^* \to \mathbb{C}$$

is zero;

and

(iii) Noting that

$$H^n\big(X, L^{n+1} \otimes \operatorname{Sym}^n N^*\big)^* \cong H^0\big(X, KL^{-(n+1)}\big) \otimes \operatorname{Sym}^n N)$$

so that by (ii)

$$\Gamma \otimes H^0\big(X, KL^{-(n+1)}\big) \otimes \psi^n \otimes H^0\big(X, \operatorname{Sym}^n N\big) \to 0,$$

which implies the desired result.

To establish (2.b.19) we will prove the stronger

LEMMA: $H^n(G, \hat{L}^{n+1} \otimes \operatorname{Sym}^n \hat{N}^*) = (0)$.

PROOF: We consider the manifold

$$F = \mathbb{G}(0, n, n+1, N)$$

of all flags in $\mathbb{C}^{N+1}$

$$S_1 \subset S_{n+1} \subset S_{n+2} \subset \mathbb{C}^{N+1}$$

where S_k is a linear subspace of dimension k. Clearly, the natural mapping

$$\pi : F \to G \tag{2.b.21}$$

defined by

$$\pi\{S_1 \subset S_{n+1} \subset S_{n+2} \subset \mathbb{C}^{N+1}\} = \{S_1 \subset S_{n+1} \subset \mathbb{C}^{N+1}\}$$

is a projective bundle with fibres the lines in $\mathbb{C}^{N+1}/S_{n+1}$. Thus

$$F = \mathbb{P}\hat{N}$$

is the projective bundle associated to $\hat{N} \to G$. We denote by E the tautological line bundle on $\mathbb{P}\hat{N}$ whose fibres are

$$(S_{n+2}/S_{n+1})^* \otimes S_1^*$$

(note that the fibre of $\hat{N}$ over a point $\{S_1 \subset S_{n+1} \subset \mathbb{C}^{N+1}\} \in G$ is

$$\left(\mathbb{C}^{N+1} \otimes S_1^*\right)/S_{n+1} \otimes S_1^* \cong \left(\mathbb{C}^{N+1}/S_{n+1}\right) \otimes S_1^*).$$

Using the standard isomorphism

$$R_\pi^0 E^n \cong \mathrm{Sym}^n \hat{N}^*,$$

a spectral sequence argument applied to (2.b.21) gives

$$H^n\left(G, \hat{L}^{n+1} \otimes \mathrm{Sym}^n \hat{N}^*\right) \cong H^n\left(F, \hat{L}^{n+1} \otimes E^n\right) \qquad (2.b.22)$$

Now consider the natural fibering

$$\tilde{\omega} : F \to \mathbb{G}(0, n+1, N) \qquad (2.b.23)$$

defined by

$$\tilde{\omega}\left\{S_1 \subset S_{n+1} \subset S_{n+2} \subset \mathbb{C}^{N+1}\right\} = \left\{S_1 \subset S_{n+2} \subset \mathbb{C}^{N+1}\right\}.$$

The fibres of $\tilde{\omega}$ are $\mathbb{P}^n$'s, and the restrictions of E and $\hat{L}$ to a typical fibre are given by

$$\begin{cases} E|\mathbb{P}^n \cong \mathcal{O}(-1) \\ \hat{L}|\mathbb{P}^n = \mathcal{O} \end{cases}$$

Since

$$H^n(\mathbb{P}^n, \mathcal{O}(-n)) = (0)$$

the Leray spectral sequence of (2.b.23) implies that

$$H^n\left(F, \hat{L}^{n+1} \otimes E^n\right) = (0).$$

When combined with (2.b.22) we obtain the lemma. Q.E.D.

REMARK: Since $H^n(\mathbb{P}^n, \mathcal{O}(-n-1)) \neq (0)$, the lemma is false for $H^n(G, \hat{L}^{n+1} \otimes \mathrm{Sym}^{n+1}\hat{N}^*)$. In this respect, Theorem (2.b.10) appears to be somewhat delicate.

SECOND PROOF: In this argument all cohomology will be computed over X, and so we just write $H^1(X, \Theta) = H^1(\Theta)$, etc. The Kodaira-Spencer mapping

$$\rho: H^0(N) \to H^1(\Theta)$$

may be reinterpreted as follows: Let $s_0, s_1, \dots, s_r$ be a basis for $H^0(L)$ and set

$$s = \begin{pmatrix} s_0 \\ \vdots \\ s_r \end{pmatrix}.$$

Then, for each $p \in X$

$$\begin{cases} s_0(p) = \operatorname{span} s(p) \subset H^0(L)^* \\ S_1(p) = \operatorname{span}\left\{ s(p), \dfrac{\partial s}{\partial z_1}(p), \dots, \dfrac{\partial s}{\partial z_n}(p) \right\} \subset H^0(L)^* \end{cases} \tag{2.b.24}$$

each define bundles of ranks 1 and $n+1$, respectively, on X. (The notation in (2.b.24) means this: In terms of a local trivialization of L and local coordinates $z_1, \dots, z_n$ on X, define the indicated vectors $s(p)$, $\partial s/\partial z_i(p) \in \mathbb{C}^{r+1} \cong H^0(L)^*$. Then the resulting subspaces are independent of choices.)

The inclusion

$$S_0 \to H^0(L)^*$$

is the dual of the evaluation map

$$H^0(L) \to L,$$

and so

$$S_0 \simeq L^* \tag{2.b.25}$$

The map

$$S_0 \otimes \Theta \to S_1/S_0$$

$$S \otimes \frac{\partial}{\partial z_i} \to \frac{\partial s}{\partial z_i}$$

is an isomorphism, so

$$S_1/S_0 \simeq L^* \otimes \Theta \tag{2.b.26}$$

The sequence (2.b.6) can now be rewritten

$$0 \to (S_1/S_0) \otimes L \to \left(H^0(L)^*/S_0\right) \otimes L \to \left(H^0(L)^*/S_1\right) \otimes L \to 0 \tag{2.b.27}$$

If

$$\boldsymbol{g} \in H^0(N) = H^0\left(\left(H^0(L)^*/S_1\right) \otimes L\right)$$

and if we lift $\boldsymbol{g}$ to a C^∞ section of $H^0(L)^* \otimes L$

$$g \in H^0(L)^* \otimes \mathcal{Q}^0(L),$$

then

$$\bar{\partial} g \in \mathcal{Q}^{0,1}(S_1 \otimes L).$$

Thus locally

$$\bar{\partial} g = as + \sum_{i=1}^{n} b_i \frac{\partial s}{\partial \bar{z}_i}$$

where $(b_1, \ldots, b_n)$ transforms as a element η of $\mathcal{Q}^{0,1}(\Theta)$. An especially nice form of this equation is to write

$$\bar{\partial} g = \eta \lrcorner \partial s \bmod s \tag{2.b.28}$$

where ∂s is the image of the canonical element of $\mathcal{O} \cong (S_1/S_0) \otimes L \otimes \Omega_X^1$ in $(H^0(L)^*/S_0) \otimes L \otimes \Omega_X^1$ (cf. (2.b.26)), and $\lrcorner$ represents the contraction

$$\Theta_X \otimes \Omega_X^1 \to \Theta_X.$$

Note that $\bar{\partial} g$ represents $\rho(\boldsymbol{g})$ in $H^1(\Theta)$.

The Gaussian system arises from the map

$$H^0(L) \to S_1^*$$

wedged $n+1$ times to give

$$\Lambda^{n+1} H^0(L) \to \det S_1^*$$

By (2.b.26) we have

$$\det S_1^* \cong S_0^* \otimes \det(S_1/S_0)^* \cong K \otimes L^{n+1}.$$

If $e_0, \dots, e_r$ is the standard basis for $\mathbb{C}^{r+1} = H^0(L)$, the map

$$\Lambda^{n+1} H^0(L) \to K \otimes L^{n+1}$$

is

$$e_{i_1} \wedge \dots \wedge e_{i_{n+1}} \to \left(s \wedge \frac{\partial s}{\partial z_1} \wedge \dots \wedge \frac{\partial s}{\partial z_n} \right)_{i_1 \dots i_{n+1}}$$

The Gaussian system is just the image Γ_{2K} of

$$\Lambda^{n+1} H^0(L) \otimes H^0(K \otimes L^{-(n+1)}) \to H^0(2K).$$

We wish to show that if

$$\eta = \rho(g), \qquad g \in H^0(N)$$

then

$$\eta \wedge \dots \wedge \eta \in H^n(X, \Lambda^n \Theta)$$

annihilates Γ_{2K}. Recalling (2.b.28) we have

$$\eta \wedge \dots \wedge \eta \lrcorner \bar{\partial} s \boldsymbol{\wedge} \dots \boldsymbol{\wedge} \bar{\partial} s = \bar{\partial} g \boldsymbol{\wedge} \dots \boldsymbol{\wedge} \bar{\partial} g \quad \bmod s$$

where the boldface wedges are really "double wedges", being a wedge both as vectors in $H^0(L)$ and as differential forms. This double wedge is symmetric rather than anti-symmetric. The contraction symbol $\lrcorner$ denotes the duality

$$\det \Theta_X \otimes K_X \to \mathcal{O}_X.$$

So

$$\begin{aligned} &\eta \wedge \eta \wedge \dots \wedge \eta \lrcorner (s \boldsymbol{\wedge} \bar{\partial} s \boldsymbol{\wedge} \dots \boldsymbol{\wedge} \bar{\partial} s) \\ &= s \wedge \bar{\partial} g \boldsymbol{\wedge} \dots \boldsymbol{\wedge} \bar{\partial} g \\ &= \bar{\partial}(s \wedge g \wedge \bar{\partial} g \boldsymbol{\wedge} \dots \boldsymbol{\wedge} \bar{\partial} g) \end{aligned}$$

If

$$\sum_{I = (i_1, \dots, i_{n+1})} \beta_I (s \boldsymbol{\wedge} \bar{\partial} s \boldsymbol{\wedge} \dots \boldsymbol{\wedge} \bar{\partial} s)_I \in \Gamma_{2K},$$

$$\beta_I \in H^0(K \otimes L^{-(n+1)})$$

then

$$(\eta \wedge \ldots \wedge \eta) \lrcorner \sum \beta_I (s \wedge \partial s \wedge \ldots \wedge \partial s)_I$$

$$= \bar{\partial}\left(\sum \beta_I (s \wedge g \wedge \partial g \wedge \ldots \wedge \partial g)_I\right.$$

and thus under the Serre duality pairing

$$H^n(X, \det \Theta) \otimes H^0(X, K_X^2) \to H^n(X, K_X)$$

we have shown that $\eta \wedge \eta \wedge \ldots \wedge \eta$ annihilates Γ_{2K}. Q.E.D.

EXAMPLE: Referring to the example preceeding the statement of (2.b.10), it can be shown (cf. [4] and §3.(b) below) that:

> *In case $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree $d \geqslant 2n+4$, equality holds in Theorem* (2.b.10); *i.e.* (28)

$$\nu^{-1}(\Gamma_{2K}) = \ker \delta^{*n} \tag{2.b.29}$$

In down to earth terms, suppose we set

$$S_k = H^0(\mathbb{P}^{n+1}, \mathcal{O}(k))$$

($\cong \mathrm{Sym}^k \mathbb{C}^{n+2*}$), and suppose we are given the following data:

$$\left\{\begin{array}{ll} \text{(i)} & \text{The infinitesimal variation of Hodge structure} \\ & v = \{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\} \text{ corresponding to the 1st} \\ & \text{order variations of } X \subset \mathbb{P}^{n+1}; \\ \text{(ii)} & \text{The Poincaré residue isomorphism} \\ & H^{n,0} \cong S_{d-n-2}{}^{(29)} \end{array}\right. \tag{2.b.30}$$

Then we claim that (loc. cit.):

> *A general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2n+4$ can be reconstructed, up to a projective transformation, from the data* (2.b.30). (2.b.31)

PROOF: By (2.b.29) we know the homogeneous component

$$J_{F,2d-2n-4} \subset S_{2d-2n-4}$$

of the Jacobian ideal in degree $2d-n-4$. Since $2d-2n-4 \geqslant d-1$, we may apply Macaulay's theorem (cf. [4] for a "residue proof") to determine

$$(J_F)_{d-1} \subset S_{d-1}.$$

To establish (2.b.31) it remains to prove the

LEMMA: *If* Aut $X = \{e\}$, *then X is uniquely determined, up to a projective transformation, by the Jacobian ideal J_F.* (2.b.32)

PROOF: Let $\mathcal{U}_d \subset S_d$ parametrize the smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$, and let $G = \subset \operatorname{Aut} \mathcal{U}_d$ be the group of transformations induced by the projectivities on $\mathbb{P}^{n+1}$. By, Mumford's theorem [37] the following quotient exists

$$M_d = \mathcal{U}_d / G,$$

and moreover if $X \in \mathcal{U}_d$ has no automorphisms (which is generically the case if $d \geqslant 3$), $^{(30)}$ then the corresponding point of M_d is smooth with tangent space a subspace of $H^1(X, \Theta)$. Now suppose that X, X' have the same Jacobian ideals

$$J_{F,d-1} = J_{F',d-1}.$$

Set $F_t = (1-t)F + tF'$. Then in particular

$$J_{F_{t,d}} = J_{F,d}$$

for F_t smooth, hence for a general t. This says exactly that F_t projects to an arc in M_d whose tangent vector is identically zero, $^{(31)}$ and thus this arc must be constant. Equivalently, the hypersurfaces $F_t(x) = 0$ are all projectively equivalent to X. Q.E.D.

REMARK: Unfortunately, we cannot use (2.b.31) to prove the weak global theorem for hypersurfaces of large degree. What must be additionally established is that the assumption (ii) is superfluous. Intuitively the reason to this is as follows: If we consider the composite map

$$\operatorname{Sym}^2(S_{d-n-2}) \xrightarrow{\alpha} S_{2d-2n-4} \xrightarrow{\beta_F} S_{2d-2n-4}/J_{F,2d-2n-4} \qquad (2.b.32)$$

then by (2.b.29)

$$\ker \delta^{*n} = \ker(\beta_F \circ \alpha).$$

It follows that the purely Hodge-theoretic object ker δ^{*n} has two pieces, a *fixed part* ker α, and then the *variable part* ker $\beta_F \subset \operatorname{Sym}^2(S_{d-n-2})/\ker \alpha$. To make sense out of this seems to require 2nd order information on the variation of Hodge structure corresponding to $X \subset \mathbb{P}^{n+1}$, and the formalism for this is only partially developed.

It is perhaps worth remarking on the nature of the problem in the case of a plane curve given in affine coordinates by

$$f(x,y)=0.$$

The period matrix has entries

$$\int_\gamma \frac{p(x,y)\mathrm{d}x}{f_y(x,y)}, \qquad \gamma \in H_1(X, \mathbb{Z}), \tag{2.b.33}$$

and what must be done is to tell from the periods (2.b.33), for a fixed $p(x,y)$ of degree $d-3$ and all cycles γ, whether $p(x,y)$ is decomposable; i.e.

$$p(x,y)=(ax+by+c)^{d-3}.$$

As will be seen below, this problem can be resolved in a number of special cases.

(c) Infinitesimal Schottky relations and the generalized Brill-Noether theory

We denote by $\mathcal{W}_d^r$ the set of pairs (C, L) where C is a smooth curve of genus g and $L \to C$ is a line bundle satisfying

$$\begin{cases} \deg L = d \\ h^0(C,L) \geqslant r+1. \end{cases}$$

It is known that $\mathcal{U}_d^r$ has naturally the structure of a (determinantal) algebraic variety (cf. [2]) (32), and we denote by

$$\mathfrak{M}_{g,d}^r \subset \mathfrak{M}_g$$

the image of $\mathcal{W}_d^r$ in the moduli space $\mathfrak{M}_g$ of curves of genus g.

One of our main heuristic principles is the following:

The local moduli theory for a higher dimensional variety is analogous to the local moduli of pairs $(C, L) \in \mathcal{W}_d^r$. (2.c.1)

For example, the local moduli space of a smooth hypersurface of degree d

$$X \subset \mathbb{P}^{n+1}, \qquad n \geqslant 2 \quad \text{and} \quad d \neq 4 \quad \text{if} \quad n=2,$$

is obtained by varying the equation of X. However, only a proper subvariety of the moduli space $\mathcal{M}_{(d-1)(d-2)/2}$ is given by smooth plane curves of degree $d \geqslant 5$. (33)

Although it is possible to make (2.c.1) more precise we shall not do so here. (34) We would like to observe that if one takes $\mathcal{U}_d^r$ rather than $\mathcal{M}_g$ as a model for local moduli spaces in higher dimensions, then all the various phenomena such as singularities, obstructions, nowhere reduced moduli schemes, etc. already occur in the curve level where they are more visible. More importantly, there is the general Brill-Noether theory that may be used as a model for questions such as Torelli and finding the infinitesimal Schottky relations, of which the latter will be the object of this section.

Specifically, we recall the recipe for computing the Zariski tangent space $T \subset H^1(C, \Theta)$ to the image $\mathcal{M}_{g,d}^r$ of $\mathcal{W}_d^r$ near a point $(C, L) \in \mathcal{W}_d^r$. From the natural mapping

$$\mu_0 : H^0(C, L) \otimes H^0(C, KL^{-1}) \to H^0(C, K) \tag{2.c.2}$$

there is constructed a natural "derived" map

$$\mu_1 : \ker \mu_0 \to H^0(C, K^2) \tag{2.c.3}$$

and Brill-Noether theory gives that

$$T = (\text{image}\, \mu_1)^{\perp}.$$

Moreover, under the natural map

$$\begin{tikzcd}
\Lambda^2 H^0(C,L) \otimes H^0(C,KL^{-2}) \arrow[r, phantom, "\subset"] \arrow[dr, "\tau"] & H^0(C,L) \otimes (H^0(C,L) \otimes H^0(C,KL^{-2})) \arrow[d, "id \otimes (\text{multiplication})"] \\
 & H^0(C,L) \otimes H^0(C,KL^{-1}),
\end{tikzcd}$$

$\Lambda^2 H^0(C, L) \otimes H^0(C, KL^{-2})$ maps to $\ker \mu_0$ (this is clear), and in [2] it is shown that

$$\mu_1 \circ \tau : \Lambda^2 H^0(C, L) \otimes H^0(C, KL^{-2}) \to H^0(C, K^2)$$

has image the Gauss linear system corresponding to the Gauss map

$$\gamma_L : C \to \mathbb{P}(\Lambda^2 H^0(C, L)^*)$$

associated to

$$\varphi_L : C \to \mathbb{P}H^0(C, L)^*.$$

In other words we may say that *in the Brill-Noether theory the Gauss* *linear system gives the "easy" part of the equations that define* $T(\mathfrak{M}^r_{g,d})$. (Actually, "easy" has the following precise geometric meaning: Given $(C, L) \in \mathfrak{W}^r_d$, the Gauss linear system describes those directions $\xi \in H^1(C, \Theta)$ such that every *pencil* in $|L|$ deforms to first order in the direction ξ. As will be seen below, there are examples where $\Gamma_{2K} = (0)$ but $\mu_1 \neq 0$.)

In the preceding section we determined that part of the infinitesimal Schottky relations for a general $X \subset \mathbb{P}^N$ corresponding "easy" part of the infinitesimal Brill-Noether theory for curves, and in this section we shall give the analogue of μ_0 and μ_1 in general.

We will use repeatedly that for a vector bundle E and a subbundle F of E, there is a collection of exact sequences

$$\begin{cases} 0 \to A_1 \to \Lambda^k E \to \Lambda^k(E/F) \to 0 \\ 0 \to A_2 \to A_1 \to F \otimes \Lambda^{k-1}(E/F) \to 0 \\ 0 \to \Lambda^k F \to A_{k-1} \to \Lambda^{k-1} F \otimes (E/F) \to 0 \end{cases} \tag{2.c.4}$$

where the bundles A_i are defined inductively as the kernels of each new sequence.

Let $L \to X$ be a holomorphic line bundle over a smooth variety of dimension n. Let S_0, S_1 be the bundles defined in (2.b.24), and let Q_0, Q_1 be the quotient bundles defined by the exact sequences

$$0 \to S_0 \to H^0(L)^* \to Q_0 \to 0 \tag{2.c.5}$$

$$0 \to S_1 \to H^0(L)^* \to Q_1 \to 0 \tag{2.c.6}$$

(Since only the variety X is involved we will write $H^0(L)$ in place or $H^0(X, L)$, etc.)

There is the obvious sequence

$$0 \to S_0 \to S_1 \to S_1/S_0 \to 0.$$

We will define maps for $0 \leqslant k < n/2$

$$\begin{cases} \psi_k : H^{2k-1}(\Omega^{n-2k} \otimes \Omega^{n-1}) \to H^{2k}(\Omega^{n-2k} \otimes K) \\ \pi_k : H^{2k}(\Omega^{n-2k} \otimes K) \to H^{2k}(\Omega^{n-2k} \otimes K)/\operatorname{im} \psi_k \\ \mu_{0,k} : \Lambda^n H^0(L) \otimes H^{2k}(\Omega^{n-2k}(L^{-n})) \to H^{2k}(\Omega^{n-2k} \otimes \Omega^{n-1}) \\ \mu_{1,k} : \ker \mu_{0,k} \to H^{2k}(\Omega^{n-2k} \otimes K)/\operatorname{im} \psi_K \\ \tau_k : \Lambda^{n+1} H^0(L) \otimes H^{2k}(\Omega^{n-2k}(L^{-n-1})) \\ \qquad \to \Lambda^n H^0(L) \otimes H^{2k}(\Omega^{n-2k}(L^{-n})) \\ \gamma_k : \Lambda^{n+1} H^0(L) \otimes H^{2k}(\Omega^{n-2k}(L^{-n-1})) \to H^{2k}(\Omega^{n-2k}(L^{-n})) \end{cases}$$

(2.c.7)

and prove the

Theorem: *For an infinitesimal variation of Hodge structure arising from geometry, and with*

$$\lambda_k : H^{2k}(X, \Omega^{n-2k} \otimes K) \to \mathrm{Sym}^{n-2k} T^* \tag{2.c.8}$$

dual to ρ^{n-2k} *in* (2.a.12), *we have*

(i) $\operatorname{im} \tau_k \subset \ker \mu_{0,k}$

(ii) $\pi_k \circ \gamma_k = \mu_{1,k} \circ \tau_k$

(iii) $\lambda_k \circ \psi_k = 0$

(iv) $\pi_k^{-1}(\operatorname{im} \mu_{1,k}) \subset \ker \lambda_k$.

Proof: From (2.b.26) and (2.c.4)–(2.c.6) we have a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & \Lambda^n(S_1/S_0)^* \simeq K \otimes L^n \arrow[d] \\
\Lambda^n H^0(L) \arrow[r] & \Lambda^n S_1^* \arrow[d] \\
 & S_0^* \otimes \Lambda^{n-1}(S_1/S_0)^* \simeq \Omega^{n-1} \otimes L^n \arrow[d] \\
 & 0
\end{tikzcd} \tag{2.c.9}$$

Tensoring with $\Omega^{n-2k}(L^{-n})$ and taking cohomology gives

$$\begin{tikzcd}
 & H^{2k-1}(\Omega^{n-2k} \otimes \Omega^{n-1}) \arrow[d, "\psi_k"] \\
 & H^{2k}(\Omega^{n-2k} \otimes K) \arrow[d] \\
\Lambda^n H^0(L) \otimes (\Omega^{n-2k}(L^{-n})) \arrow[r, dash] \arrow[rd, dashed, "\mu_{0,k}"'] & H^{2k}(\Lambda^n S_1^* \otimes \Omega^{n-2k}(L^{-n})) \arrow[d] \\
 & H^{2k}_{,}(\Omega^{n-2k} \otimes \Omega^{n-1})
\end{tikzcd}$$

where the vertical sequence is exact. The map ψ_k is defined by the vertical map indicated, it is cup product with the extension class in $H^1(\Omega^1)$ of the vertical sequence of (2.c.9) – this extension class is a multiple of $c_1(L)$.

The composition indicated by a dotted arrow defines $\mu_{0,k}$. By the vertical exactness we obtain

$$\mu_{1,k} : \ker \mu_{0,k} \to H^{2k}(\Omega^{n-2k} \otimes K)/\operatorname{im} \psi_k .$$

The map

$$\pi_k : H^{2k}(\Omega^{n-2k} \otimes K) \to H^{2k}(\Omega^{n-2k} \otimes K)/\operatorname{im} \psi_k$$

is the canonical projection.

From (2.c.4) and (2.c.5), we obtain the sequence

$$0 \to \Lambda^{n+1}Q_0^* \to \Lambda^{n+1}H^0(L) \to \Lambda^n Q_0^* \otimes S_0^* \to 0.$$

As Q_0^* is a sub-bundle of the trivial bundle $H^0(L)$ and $S_0^* \simeq L$, we obtain

$$\begin{tikzcd}
\Lambda^{n+1}H^0(L) \arrow[r] \arrow[dr, dashed] & \Lambda^n Q_0^* \otimes L \arrow[d] \\
 & \Lambda^n H^0(L) \otimes L
\end{tikzcd}$$

Tensoring with $\Omega^{n-2k}(L^{-n-1})$ and taking H^{2k} we obtain a map

$$\tau_k : \Lambda^{n+1}H^0(L) \otimes H^{2k}(\Omega^{n-2k}(L^{-n-1}))$$
$$\to \Lambda^n H^0(L) \otimes H^{2k}(\Omega^{n-2k}(L^{-n})).$$

From the map

$$\Lambda^{n+1}H^0(L) \to \Lambda^{n+1}S_1^* \simeq K \otimes L^{n+1}$$

tensored with $\Omega^{n-2k}(L^{-n-1})$ we obtain

$$\gamma_k : \Lambda^{n+1}H^0(L) \otimes H^{2k}(\Omega^{n-2k}(L^{-n-1})) \to H^{2k}(\Omega^{n-2k} \otimes K).$$

The commutative diagram

$$\begin{tikzcd}
\Lambda^n Q_0^* \otimes L \arrow[r] \arrow[d] & \Lambda^n H^0(L) \otimes L \arrow[d] \\
\Lambda^n (S_1/S_0)^* \otimes L \arrow[r] & \Lambda^n S_1^* \otimes L
\end{tikzcd}$$

shows that $\mu_{0,k} \circ \tau_k = 0$. The commutative diagram

$$\begin{tikzcd}
\Lambda^{n+1}H^0(L) \arrow[r] \arrow[d] & \Lambda^n Q_0^* \otimes L \arrow[d] \\
\Lambda^{n+1}S_1^* \arrow[r, phantom, "\simeq"] & \Lambda^n(S_1/S_0)^* \otimes L
\end{tikzcd}$$

shows that $\pi_k \circ \gamma_k = \mu_{1,k} \circ \tau_k$.

Properties (iii) and (iv) are somewhat deeper. If

$$g \in T = H^0(X, N)$$

then by (2.b.28), and using notations from there,

$$\bar{\partial} g = \eta \lrcorner \partial s \bmod s.$$

Given a class in $H^{2k-1}(\Omega^{n-2k} \otimes \Omega^{n-1})$, we may represent its Dolbeault class by

$$\alpha \in \mathcal{A}^{0,2k-1}(\Omega^{n-2k} \otimes \Omega^{n-1}).$$

Under the isomorphism

$$\Omega^{n-2k} \otimes \Omega^{n-1} \simeq S_0^* \otimes \Lambda^{n-1}(S_1/S_0)^* \otimes \Omega^{n-2k}(L^{-n})$$

we get

$$\alpha^{\#} \in \mathcal{A}^{0,2k-1}\big(S_0^* \otimes \Lambda^{n-1}(S_1/S_0)^* \otimes \Omega^{n-2k}(L^{-n})\big)$$

where α, $\alpha^{\#}$ are related by

$$\alpha^{\#}(s \wedge \partial s \wedge \partial s \wedge \ldots \wedge \partial s) = \alpha.$$

If we lift $\alpha^{\#}$ to

$$\tilde{\alpha}^{\#} \in \mathcal{A}^{0,2k-1}\big(\Lambda^{n-1}S_1^* \otimes \Omega^{n-2k}(L^{-n})\big)$$

using (2.c.9), then

$$\psi_k(\alpha) = \bar{\partial}(\tilde{\alpha}^{\#})|_{\Lambda^n(S_1/S_0^*)}(\partial s \wedge \partial s \wedge \ldots \wedge \partial s) \in \mathcal{A}^{0,\partial k}(\Omega^{n-2k} \otimes K).$$

Let

$$\beta = \bar{\partial}(\tilde{\alpha}^{\#})|_{\Lambda^n(S_1/S_0)^*} \in \mathcal{A}^{0,2k}\big(\Lambda^n(S_1/S_0)^* \otimes \Omega^{n-2k}(L^{-n})\big)$$

and extend β to

$$\tilde{\beta} \in \mathcal{A}^{0,2k}\big(\Lambda^n Q_0^* \otimes \Omega^{n-2k}(L^{-n})\big)$$

So

$$\psi_k(\alpha) = \tilde{\beta}(\partial s \wedge \ldots \wedge \partial s).$$

Thus if $\eta \in T$,

$$\begin{aligned}\psi_k(\alpha) \lrcorner\, \eta \wedge \ldots \wedge \eta &= \tilde{\beta}(\partial s \wedge \ldots \wedge \partial s) \lrcorner\, \eta \wedge \ldots \wedge \eta \\ &= \tilde{\beta}(\bar{\partial} g \wedge \ldots \wedge \bar{\partial} g) \\ &= \bar{\partial}\tilde{\beta}(g \wedge \bar{\partial} g \wedge \ldots \wedge \bar{\partial} g) \\ &= 0 \text{ in } H^{2k}(\Omega^{n-2k} \otimes K).\end{aligned}$$

This proves (iii).

To prove (iv), assume

$$\alpha \in \Lambda^n H^0(L) \otimes H^{2k}\big(\Omega^{n-2k}(L^{-n})\big)$$

is an element of ker $\mu_{0,k}$, i.e.

$$\alpha \lrcorner\, s \wedge \underbrace{\partial s \wedge \ldots \wedge \partial s}_{n-1 \text{ times}} = \bar{\partial} h \qquad h \in \mathcal{A}^{0,2k-1}(\Omega^{n-2k} \otimes \Omega^{n-1})$$

then

$$\mu_{1,k}(\alpha) = \pi_k\Big(\alpha \lrcorner \underbrace{\partial s \wedge \ldots \wedge \partial s}_{n \text{ times}}\Big) \in H^{2k}(\Omega^{n-2k} \otimes K)/\mathrm{im}\, \psi_k.$$

Now

$$\begin{aligned}&\Big(\alpha \lrcorner \underbrace{\partial s \wedge \ldots \wedge \partial s}_{n \text{ times}}\Big) \lrcorner \underbrace{\eta \wedge \ldots \wedge \eta}_{n-2k \text{ times}} \\ &\quad = \alpha \lrcorner \underbrace{\bar{\partial} g \wedge \ldots \wedge \bar{\partial} g}_{n-2k \text{ times}} \wedge \underbrace{\partial s \wedge \ldots \wedge \partial s}_{2k \text{ times}} \qquad \text{mod } s \\ &\quad = \bar{\partial}(\alpha \lrcorner\, g \wedge \bar{\partial} g \wedge \ldots \wedge \bar{\partial} g \wedge \partial s \wedge \ldots \wedge \partial s) \qquad \text{mod } s \\ &\quad = 0 \text{ in } H^n(K)\end{aligned}$$

where we can work mod s as the class lies in $H^n(\Lambda^n(S_1/S_0)^* \otimes L^{-n}) \simeq H^n(K)$.

When $k = 0$, ψ_k and π_k drop out of the picture and the maps (2.c.7) can be written as

$$\begin{cases} \mu_0 : \Lambda^n H^0(X, L) \otimes H^0(X, \Omega^n(L^{-n})) \to H^0(X, \Omega^{n-1} \otimes K) \\ \mu_1 : \ker \mu_0 \to H^0(X, K^2) \\ \tau : \Lambda^{n+1} H^0(X, L) \otimes H^0(X, \Omega^n(L^{-n-1})) \\ \qquad \to \Lambda^n H^0(X, L) \otimes H^0(X, \Omega^n(L^{-n})) \\ \gamma : \Lambda^{n+1} H^0(X, L) \otimes H^0(X, KL^{-n-1}) \to H^0(X, K^2). \end{cases} \tag{2.c.10}$$

It may be easily verified that the image of γ is the Gaussian linear system $\Gamma_{2K} \subset H^0(X, K^2)$ defined in the preceding section. When $k = 0$, $n = 1$ the maps (2.c.10) reduce to those encountered in the Brill-Noether theory.

EXAMPLE: Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d with defining equation

$$F(x^0, x^1, \dots, x^{n+1}) = 0.$$

Denote by $S = \oplus_{k \geqslant 0} S_k$ the graded ring $\mathbb{C}[x^0, x^1, \dots, x^{n+1}]$ and by J $J_F = \oplus_{k \geqslant d-1} J_{F,k}$ the Jacobian ideal

$$J_F = \{F_0, F_1, \dots, F_{n+1}\}, \qquad F_i = \partial F / \partial x^i.$$

Then, as will be discussed in Section 3(b), it is well known (cf. [4], [22]) that there are natural residue isomorphisms

$$S_{d(k+1)-n-2} / J_{F, d(k+1)-n-2} \cong H^{n-k,k}(X). \tag{2.c.11}$$

In particular, assume that

$$\mathrm{d}\left(\left[\frac{n}{d}\right] + 1\right) - (n+2) = l \geqslant 0 \tag{2.c.12}$$

and set

$$k = \left[\frac{n}{d}\right]. \tag{2.c.13}$$

Then

$$H^{n,0}(X) = \dots = H^{n-k+1,k-1}(X) = (0),$$

and the first non-vanishing group is

$$H^{n-k,k}(X) \cong S_l \tag{2.c.14}$$

(since $l \leqslant d-2$, the Jacobian ideal is zero in this degree).

Either from [4] or from the discussion in Section 3 (b) below, we have the

PROPOSITION: (i) *With the above notations and assumptions.*
image $\mu_{l,k}$ = image γ_K.
(ii) *In the dual* (2.c.15)

$$\begin{tikzcd}
\mathrm{Sym}^2 H^{n-k,k} \arrow[rr, "\delta^{*n-2k}"] \arrow[dr, "\nu_k"'] & & \mathrm{Sym}^{n-2k} T^* \\
& H^{2k}(X, K \otimes \Omega^{n-2k}) \arrow[ur, "\lambda_k"'] &
\end{tikzcd}$$

of the diagram (2.a.14) *we have*

$$\ker \delta^{*n-2k} = \nu_k^{-1}(\text{image } \gamma_k),$$

and the right hand side is given by

$$\nu_k^{-1}(\text{image } \gamma_k) = \eta^{-1}(J_{F,2l})$$

where η is the multiplication mapping

$$\eta : \mathrm{Sym}^2(S_l) \to S_{2l}. \tag{2.c.16}$$

As a consequence we have the main result from [4].

COROLLARY: *A general smooth cubic hypersurface $X \subset \mathbb{P}^{3m+1}$ is uniquely determined by its infinitesimal variation of Hodge structure.* (2.c.17)

PROOF: In this situation we have

$$k = m, \qquad l = 1$$

and (2.c.14) together with (2.c.15), (2.c.16) gives

$$\eta : \mathrm{Sym}^2(S_1) \to S_2$$

$$\ker \delta^{*m} = \eta^{-1}(J_{F,2}).$$

Thus, from the infinitesimal variation of Hodge structure of X we may determine the Jacobian ideal $J_F \subset S$. By Lemma (2.b.32) we may then reconstruct a general cubic hypersurface $X \subset \mathbb{P}^{3m+1}$ from its infinitesimal variation of Hodge structure.

REMARK: Referring to (2.b.30) and (2.b.31), the point here is that, in the present case, part (ii) of the data (2.b.30) is not required.

EXAMPLE: We shall give an example where the Gauss linear system $\Gamma_{2K} = (0)$ but $\mu_1 \neq 0$. Let L_1, L_2 be a pair of skew lines in $\mathbb{P}^3$ and X_0 a surface of degree 8 having $\Delta = L_1 + L_2$ as a double curve and no other singularities. An easy Bertini argument shows that such surfaces X_0 exist, and we denote by

$$\pi : X \to X_0 \subset \mathbb{P}^3$$

the normalization and set $L = \pi^* \mathcal{O}_{X_0}(1)$. Then

$$H^0(X, KL^{-k}) \cong H^0(\mathbb{P}^3, I_\Delta(4-k))$$

where I_Δ is the ideal sheaf of Δ. In particular

$$\begin{cases} H^0(X, KL^{-3}) = (0) \\ H^0(X, KL^{-2}) \cong H^0(\mathbb{P}^3, I_\Delta(2)). \end{cases}$$

By the first equation the Gauss linear system $\Gamma_{2\mathrm{K}} = (0)$. We shall compute μ_0 and μ_1.

For this we choose homogeneous coordinates $[x^0, x^1, y^0, y^1]$ so that

$$L_1 = \{y^0 = y^1 = 0\}$$

$$L_2 = \{x^0 = x^1 = 0\}.$$

Then the elements of $H^0(X, KL^{-2})$ are quadrics

$$\sum_{i,j} q_{ij} x^i y^j.$$

Suppose that

$$r = r' + r + r'' \in \Lambda^2 H^0(X, L) \otimes H^0(X, KL^{-2})$$

where

$$\begin{cases} r' = \frac{1}{2} \sum r'_{ijkl} x^i \wedge x^j \otimes x^k y^l, & r'_{ijkl} = -r'_{jikl} \\ r = \sum r_{ijkl} x^i \wedge y^j \otimes x^k y^l & \\ r'' = \frac{1}{2} \sum r''_{ijkl} y^i \wedge y^j \otimes x^k y^l, & r''_{ijkl} = -r_{jikl}. \end{cases}$$

If $\mu_0(r) = 0$ then it is easy to see that $r' = r'' = 0$. Writing $r = r$ we have

$$\mu_0(r) = \sum r_{ijkl}\left(\mathrm{d}x^i y^j x^k y^l - \mathrm{d}y^j x^i x^k y^l\right).$$

The condition that $\mu_0(r) = 0$ is thus

$$r_{ijkl} = -r_{kjil} = -r_{iljk}.$$

It follows that dim ker $\mu_0 = 1$ and image μ_1 is spanned by

$$(x^1\mathrm{d}x^2 - x^2\mathrm{d}x^1) \wedge \left(y^1\mathrm{d}y^2 - y^2\mathrm{d}y^1\right).$$

Example *(continued)*: If instead we require that X_0 have a double curve along $\Delta = L_1 + L_2 + L_3$ where L_1, L_2, L_3 are three skew lines, we may easily see that

$$\ker \mu_0 = (0).$$

In this case the infinitesimal period relations are more difficult to describe as they cannot be detected from the kernel of the iterated differential δ^2.

3. Infinitesimal variations of Hodge structure associated to very ample divisors

(a) The infinitesimal M. Noether theorem

The main result of this section is theorem (3.a.16), which is a strengthening and generalization of a classical result of M. Noether. Its proof introduces some element of commutative algebra into the theory of infinitesimal variations of Hodge structure, and may therefore provide a technique of interest in other contexts.

In this section we will use the following notations:

Y	is a smooth variety of dimension $n + 1 \geqslant 2$;
$L \to Y$	is an ample line bundle with $c_1(L) = \omega$;
$X \in \lvert L \rvert$	is a smooth divisor (thus dim $X = n$);
$s \in H^0(Y, L)$	is a section with $(s) = X$, and using s we make the identification $$H^0\left(\Omega_Y^k((q+1)X)\right) \cong H^0\left(\Omega_Y^k \otimes L^{q+1}\right), \qquad (3.a.1)$$ where $\Omega_Y^k((q+1)X)$ is the sheaf of mermorphis k-forms on Y having a pole of order $(q+1)$ along X.

We begin by recording the following basic cohomology diagram (with $\mathbb{C}$-coefficients):

$$\begin{tikzcd}
H^{n-1}(X) \arrow[r] & H^{n+1}(Y) \arrow[r, "j"] & \mathrm{H}^{\mathrm{n}+1}(Y-X) \arrow[r, "R"] & H^{n}(X) \arrow[r] & H^{n+2}(Y) \arrow[r] & \ldots, \\
H^{n-1}(Y) \arrow[u, "r"] \arrow[ur, "\omega"] & & & H^{n}(Y) \arrow[u, "r"] \arrow[ur, "\omega"] & &
\end{tikzcd} \qquad (3.a.2)$$

concerning which we make following remarks:

(i) the diagram (3.a.2) is obtained by applying Poincaré-Lefschetz duality to the exact homology sequence of the pair (Y, X);

(ii) the restriction mapping r are injections, by the Lefschetz hyperplane theorem;

(iii) the mappings ω are injections, by the "hard" Lefschetz theorem (the right-hand ω is an isomorphism);

(iv) the *residue mapping* R is dual to the "tube over cycle mapping" $\tau: H_n(X) \to H_{n+1}(Y-X)$;

(v) the group $H^{n+1}(Y-X)$ has a *mixed Hodge structure* [10] with 2-stage weight filtration

$$\begin{cases} W_{n+1} = jH^{n+1}(Y) \\ W_{n+2} = H^{n}(Y-X) \end{cases}$$

and where j, R are morphisms of mixed Hodge structures;

and

(vi) the direct sum decompositions hold

$$\begin{cases} H^{n+1}(Y) = \omega \cdot H^{n-1}(Y) \oplus H^{n+1}_{\mathrm{prim}}(Y) \\ \qquad\qquad \text{(Lefschetz decomposition)} \\ H^{n}(X) = rH^{n}(Y) \oplus RH^{n+1}(Y-X). \end{cases} \qquad (3.a.3)$$

Regarding the second direct sum decomposition we set

$$\begin{cases} H^{n}_{f}(X) = rH^{n}(Y) & (f \text{ stands for "fixed"}) \\ H^{n}_{v}(X) = RH^{n+1}(Y-X) & (v \text{ stands for "variable"}). \end{cases}$$

(3.a.4)

Here, fixed refers to the variation of Hodge structure given by $H^{n}(X) =$

$\oplus_{p,q=n} H^{p,q}(X)$ as X varies over smooth divisors in $|L|$. It is a general fact ([10], [17]) that a global variation of Hodge structure is completely reducible; in our case there is a reduction

$$H^n(X) = H_f^n(X) \oplus H_v^n(X) \tag{3.a.5}$$

where $H_f^n(X)$ is a trivial direct summand of the above variation of Hodge structure. As will be proved below, $H_v^n(X)$ is "truly variable". We also note that

$$H_{\mathrm{prim}}^n(X) = \left(H_{\mathrm{prim}}^n(X) \cap H_f^n(X)\right) \oplus \left(H_{\mathrm{prim}}^n(X) \cap H_v^n(X)\right)$$

$$H^{p,q}(X) = \left(H^{p,q}(X) \cap H_f^n(X)\right) \oplus \left(H^{p,q}(X) \cap H_v^n(X)\right), \tag{3.a.6}$$

where the first is an orthogonal direct sum decomposition relative to the polarizing form on $H_{\mathrm{prim}}^n(X)$.

To state our first result, we recall that since $Y - X$ is an affine variety

$$H^n(Y - X) \cong H_{ADR}^n(Y - X),$$

where the right hand side is the *algebraic de Rham cohomology* computed from the complex of regular rational differentials on $Y - X$ (df. [26]).

THEOREM: *If $H^n(\Omega_Y^b(qX)) = 0$ for $p \geqslant 0$, $n > 0$, $q > 0$, in particular if $L \to Y$ is sufficiently ample, then the Hodge filtration $F^q H^{n+1}(Y - X)$ is given by the order of pole along X. In particular* (35)

$$\frac{F^q H^{n+1}(Y - X)}{F^{q+1} H^{n+1}(Y - X)} \cong \frac{H^0\left(\Omega_Y^{n+1}((q+1)X)\right)}{\mathrm{d}H^0\left(\Omega_Y^n(qX)\right) + H^0\left(\Omega_Y^{n+1}(qX)\right)}. \tag{3.a.7}$$

COROLLARY: *There are exact sequences*

$$0 \to H_{\mathrm{prim}}^{n+1-q,q}(Y) \to \frac{H^0\left(\Omega^{n+1}((q+1)X)\right)}{\mathrm{d}H^0\left(\Omega_Y^n(qX)\right) + H^0\left(\Omega_Y^{n+1}(qX)\right)}$$

$$\to H_v^{n-q,q}(X) \to 0. \tag{3.a.8}$$

The corollary follows from the theorem together with the fact that (3.a.2) is a diagram of mixed Hodge structures (all but one of which is a pure Hodge structure) with maps being morphisms of mixed Hodge structures ([10]).

COROLLARY: *If $L \to Y$ is a sufficiently ample and $X = (s) \in |L|$ is smooth, then*[(36)]

$$\mathrm{d}H^0(\Omega_Y^n(qX)) + sH^0(\Omega_Y^{n+1} \otimes L^q) = H^0(\Omega_Y^{n+1} \otimes L^{q+1}),$$

$$q \geqslant n+2. \tag{3.a.9}$$

This result is clear from (3.a.8). It is one of those purely algebraic facts, whose proof however is transcendental.

Sketch of proof of (3.a.7) and (3.a.8): Since this result is essentially contained in [22] we shall only outline the proof. To begin, if we define

$$\Omega_Y^p(\log qX) = \{\varphi \in \Omega_Y^p(qX) : \mathrm{d}\varphi \in \Omega_Y^{p+1}(qX)\},$$

then there are exact sequences

$$0 \to \Omega_Y^p(\log qX) \to \Omega_Y^p(qX) \xrightarrow{d} \frac{\Omega_Y^{p+1}(\log(q+1)X)}{\Omega_Y^{p+1}(qX)} \to 0 \tag{3.a.10}$$

(valid for $q \geqslant 1$), and

$$0 \to \Omega_Y^p \to \Omega_Y^p(\log X) \xrightarrow{R} \Omega_X^{p-1} \to 0$$

where R is the *Poincare residue operator.* From (3.a.10) we infer the pair of sequences

$$0 \to \frac{\Omega_Y^p(\log qX)}{\Omega_Y^p((q-1)X)} \to \frac{\Omega_Y^p(qX)}{\Omega_Y^p((q-1)X)} \xrightarrow{d} \frac{\Omega_Y^{p+1}(\log(q+1)X)}{\Omega_Y^{P+1}(qX)} \to 0$$

valid for $q \geqslant 2$, and

$$0 \to \Omega_Y^p(\log X) \to \Omega_Y^p(X) \xrightarrow{d} \frac{\Omega_Y^{p+1}(\log 2X)}{\Omega_Y^{p+1}(X)} \to 0$$

We assume that $L \to Y$ is ample enough that

$$H^r(\Omega_Y^p(qX)) = (0) \qquad r > 0, q > 0. \tag{3.a.11}$$

It follows that we have:

$$0 \to H^q_{\mathrm{prim}}(\Omega_Y^p) \to H^q(\Omega_Y^p(\log X)) \xrightarrow{R} H^q_v(\Omega_X^{p-1}) \to 0$$

$$H^q(\Omega_Y^p(\log X)) \cong H^{q-1}\left(\frac{\Omega_Y^{p+1}(\log 2X)}{\Omega_Y^{p+1}(X)}\right), \qquad q \geqslant 2,$$

$$H^0(\Omega_Y^p(X)) \xrightarrow{d} \frac{H^0\big(\Omega_Y^{p+1}(\log 2X)\big)}{H^0\big(\Omega_Y^{p+1}(X)\big)} \to H^1(\Omega_Y^p(\log X)) \to 0$$

$$H^r\left(\frac{\Omega_Y^{p+1}(\log(q+1)X)}{\Omega_Y^{p+1}(qX)}\right) \cong H^{r+1}\left(\frac{\Omega_Y^p(\log qX)}{\Omega_Y^p((q-1)X)}\right), \quad r \geqslant 1, q \geqslant 2,$$

$$\frac{H^0(\Omega_Y^p(qX))}{H^0(\Omega_Y^p((q-1)X))} \xrightarrow{d} \frac{H^0\big(\Omega_Y^{p+1}(\log(q+1)X)\big)}{H^0\big(\Omega_Y^{p+1}(qX)\big)}$$

$$\to \frac{H^1(\Omega_Y^p(\log qX))}{\Omega_Y^p((q-1)X)} \to 0.$$

Using these sequences from top to bottom leads to a proof of (3.a.7) and (3.a.8). Q.E.D.

Before stating the infinitesimal M. Noether theorem we need one additional notation. The tangent space $T_s(|L|)$ to the complete linear system $|L|$ at $X = (s)$ is given by

$$\begin{array}{l} T_s(|L|) \cong H^0(Y, L)/\mathbf{C} \cdot s \\ \qquad\qquad\quad \cap \\ \qquad\qquad H^0(X, N) \end{array}$$

where $N = L \otimes \mathcal{O}_X$ is the normal bundle of X in Y (from the exact cohomology sequence of $0 \to \mathcal{O}_Y \xrightarrow{s} L \to N \to 0$, we see that the inclusion is an equality of $h^1(\mathcal{O}_Y) = 0$). We also have a sequence

$$H^0(\Theta_Y) \xrightarrow{\alpha} H^0(\Theta_Y \otimes \mathcal{O}_X) \to H^0(X, N) \tag{3.a.14}$$

arising from the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \Theta_X \arrow[d] & \\
0 \arrow[r] & \Theta_Y(-X) \arrow[r] & \Theta_Y \arrow[r] \arrow[rd, "\sigma"'] & \Theta_Y \otimes \mathcal{O}_X \arrow[r] \arrow[d] & 0, \\
 & & & N \arrow[d] & \\
 & & & 0 &
\end{tikzcd} \tag{3.a.15}$$

and we define

$$T = T_s(|L|)/T_s(|L|) \cap \text{image}\ \sigma$$

(*Explanation*: T is the tangent space to $|L|$ modulo automorphisms induced from Y.) We note that if $L \to Y$ is sufficiently ample, then α in (3.a.14) will be an isomorphism. Moreover, from the exact cohomology sequence of the vertical exact sheaf sequence in (3.a.15) we infer that

$$T \subset H^1(X, \Theta_X)$$

is the image of $T_s(|L|)$ under the Kodaira-Spencer mapping. Thus we may think of T as the tangent space to that part of the moduli of X coming from $|L|$. We denote by $V = \{H_{\mathbb{Z}}, H^{p,q}, T, \Delta\}$ the corresponding infinitesimal variation of Hodge structure; thus

$$\begin{cases} H_{\mathbb{Z}} = H^n(X, \mathbb{Z})/(\text{torsion}) \\ H^{p,q} = H^{p,q}(X) \end{cases}$$

and

$$\delta : T \to \oplus \operatorname{Hom}(H^{p,q}, H^{p-1,q+1})$$

is given by $\delta(\xi) =$ cup-product with the Kodaira-Spencer class $\rho(\xi) \in H^1(X, \Theta_X)$.

DEFINITION: The subspace $H^{p,q}_{\text{i.f.}}(X) \subset H^{p,q}(X)$ of classes that are *infinitesimally fixed* under V is defined by

$$H^{p,q}_{\text{i.f.}}(X) = \{\Psi \in H^{p,q} : \delta(\xi)\psi = 0 \qquad \text{for all} \quad \xi \in T\}.$$

REMARKS: A more accurate terminology would be that $H^{p,q}_{\text{i.f}}(X)$ consists of those classes whose Hodge types does not infinitesimally change. (37)

We then have the (cf. (3.a.4) for the definition of $H^{p,q}_f(X)$).

INFINITESIMAL M. NOETHER THEOREM: *For $L \to Y$ sufficiently ample and any smooth $X \in |L|$* (3.a.16)

$$H^{p,q}_{\text{i.f}}(X) = H^{p,q}_f(X).$$

PROOF: We set

$$M_q = H^0(Y, K_Y L^{q+1})$$

$$M_{\cdot} = \bigoplus_{q \geqslant 0} M_q$$

$$S_{\cdot} = \bigoplus_{k \geqslant 0} \text{image}\{\operatorname{Sym}^k H^0(Y, L) \to H^0(Y, L^k)\}.$$

Then M is a finitely generated S-module (cf. [45]), and consequently M is generated by elements in degree $\leqslant q_0$. Equivalently, we have

$$H^0(Y, L) \otimes M_q \to M_{q+1} \to 0, \qquad q \geqslant q_0.$$

Replacing L by L^{q_0} we may assume that

$$H^0(Y, L) \otimes M_q \to M_{q+1} \to 0, \qquad \text{for} \quad q \geqslant 0. \tag{3.a.18}$$

By Theorem (3.a.7), we infer from (3.a.18) that

$$T \otimes H_v^{n-q,q}(X) \xrightarrow{\delta} H_{\text{prim}}^{n-q-1,q+1}(X) \to 0, \qquad \text{for} \quad q \geqslant 0$$

where the map δ is induced by the differential in the infinitesimal variation of Hodge structure V. Using the Lefschetz decomposition we have

$$H_v^n(X) = \bigoplus_{k \geqslant 0} \omega^k \cdot \left(H_v^{n-2k}(X) \cap H_{\text{prim}}^{n-2k}(X)\right), \tag{3.a.19}$$

and we denote by Q the corresponding direct sum of the polarizing forms on the $H_{\text{prim}}^{n-2k}(X)$. Then using

$$Q(\delta(\xi)\varphi, \psi) = -Q(\varphi, \delta(\xi)\psi),$$

$$\varphi \in H^{p,q}, \qquad \psi \in H^{q+1,n-q-1}, \qquad \xi \in T,$$

(3.a.18) is equivalent to the assertion:

$$Q(\delta(\xi)\varphi, \psi) = 0$$

$$\text{for all} \qquad \psi \in H^{q+1,n-q-1} \qquad \text{and} \qquad \xi \in T \Rightarrow \varphi = 0. \tag{3.a.20}$$

But (3.a.20) is obviously equivalent to the infinitesimal M. Noether theorem. Q.E.D.

COROLLARY (Lefschetz): *Let $S \subset |L|$ be the open dense set of smooth $X \in |L|$. Then the monodromy representation.*

$$\rho : \pi_1(S) \to \operatorname{Aut}(H_v^n(X)) \tag{3.a.21}$$

has no factors on which $\pi_1(S)$ acts as a finite group.

PROOF: Since our considerations are local, we may pass to a finite covering of S and it will suffice to show that $H_v^n(X)$ has no non-trivial

$\pi_1(S)$-invariant subspace H'. Setting $H'^{p,q} = H' \in H^{p,q}(X)$ and using that $H' = \oplus_{p+q} H'^{p,q}$ is a sub-Hodge structure of $H_v^n(X)$ (cf. [10] and [17]), we are reduced to theorem (3.a.16). Q.E.D.

Remark: Using theorem (3.a.7) the proof of (3.a.16) actually gives a statement about the variation of mixed Hodge structure on $H^{n+1}(Y-X)$, but we have not tried to formulate this precisely.

Corollary (cf. [36]): *If $X \in |L|$ is generic and $n = 2m$, then*

$$H^{m,m}(X, \mathbb{Q}) = \text{image of } \{H^{m,m}(Y, \mathbb{Q}) \to H^{2m}(X, \mathbb{Q})\}. \qquad (3.a.22)$$

Proof: If X is generic and $\gamma \in H^{m,m}(X, \mathbb{Z})$ is a Hodge class, then clearly $\gamma \in H^{m,m}_{\text{i.f.}}(X)$. (38)

This corollary implies the well known

Theorem of M. Noether: *Any curve on a generic surface $X \subset \mathbb{P}^3$ of degree $d \geqslant 4$ is a complete intersection.* (3.a.23)

Proof: First, we may rephrase (3.a.16) as follows:

> *If $X \in |L|$ is smooth and $\gamma \in H^{m,m}(X, \mathbb{Z}) \in H_v^{2m}(X)$ is a variable Hodge class (thus we are in the case $n = 2m$), then the set of directions $\xi \in T$ under which γ remains of type (m, m) is a proper linear subspace.* (3.a.24)

Secondly, as (3.a.11) is true for $d \geqslant 0$ and (3.a.14) is true for $d \geqslant 4$, the condition that $\mathcal{O}_{\mathbb{P}^3}(d)$ be sufficiently ample so that (3.a.24) applies is $d \geqslant 4$ (cf. (3.a.11)). We conclude then that:

> *For $X \subset \mathbb{P}^3$ a generic surface of degree $d \geqslant 4$, the Picard number $\rho(X) = 1$.*

Finally, (3.a.23) is a well-known consequence of this fact. Q.E.D. (39)

It is clear that theorem (3.a.16) (in the form (3.a.24)) is strenthening of M. Noether's theorem (3.a.23). However, it should also be possible to improve (3.a.23) in a quantitative manner. To explain this, we remark that for a surface X there are $p_g(= h^{2,0})$ equations

$$\int_\gamma \omega = 0 \qquad \omega \in H^{2,0}(X), \quad \gamma \in H^2(X, \mathbb{Z}) \qquad (3.a.25)$$

expressing the condition that an integral cycle γ be a Hodge class. Thus, in first approximation we expect that the property

$$\rho(X)=\rho$$

should impose $\rho \cdot p_g$ conditions on moduli. For K3 surfaces this is well known: each time the Picard number increases by one, the number of moduli decreases by one.

On the other hand the equations (3.a.25) may not be independent. For example, let $S \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ parametrize the smooth surface of degree d. Since it is $(d+1)$ conditions that a surface X contains a line Λ (if X is defined by $F(x)=0$, then F must vanish at $(d+1)$-points of Λ), and since the Grassmannian of lines in $\mathbb{P}^3$ has dimension 4, the subvariety $S_1 \subset S$ of smooth surfaces containing a line has expected codimension $d-3$. It can be proved that this dimension cannot be correct (even scheme-theoretically), and so in this case when $d \geqslant 5$ the equations (3.a.25) fail to be independent.

Now if we denote by $S_k \subset S$ the variety of the smooth surfaces containing a non-complete intersection curve $C \subset \mathbb{P}^3$ of degree k, then it is geometrically plausible that "the higher the degree of $C \subset \mathbb{P}^3$, the harder it is for a surface X to contain C"; (40) i.e., that

$$\operatorname{codim} S_{k-1} \geqslant \operatorname{codim} S_k .$$

this motivates the following:

CONJECTURE: *For any* $k \geqslant 1$, *codim* $S_k \geqslant d-3$, *with equality holding, only if* $k=1$. (3.a.26)

We will prove the inequality in (3.a.26) in the first non-trivial case $d=5$ of quintic surfaces $X \subset \mathbb{P}^3$. Thus, suppose that there is local piece of hypersurface $R \subset S$ such that every surface X corresponding to a point of R has a primitive Hodge class γ. Let X be the surface corresponding to a smooth point of R, so that the tangent space $T(R)$ is a hyperplane in the tangent space $T(S)=T$ corresponding to all variations of $X \subset \mathbb{P}^3$ (i.e., $T=H^0(X, \mathcal{O}_X(d))$). We denote by $V=\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ the infinitesimal variation of Hodge structure on $H^2_{\text{prim}}(X)$ with tangent splace T, and by $\gamma \in H^{1,1}_{\mathbf{Z}}$ the primitive Hodge class (actually, the condition that γ be integral will not be used in the argument). By assumption

$$\delta(\xi)\gamma \in H^{1,1} \qquad \text{for all} \qquad \xi \in T(R).$$

In other words, the equations in ξ

$$Q(\delta(\xi)\psi, \gamma)=0 \qquad \text{for all} \qquad \psi \in H^{2,0} \tag{3.a.27}$$

define the codimension one linear subspace $T(R)$ of T. We shall show that the condition that the equations (3.a.27) have rank one leads to a contradiction.

Let $[x^0, x^1, x^2, x^3]$ be homogeneous coordinates and $S_{\cdot} = \oplus_{m \geqslant 0} S_m = \mathbb{C}[x^0, x^1, x^2, x^3]$. Then, via Poincaré residues

$$H^{2,0}(X) \cong S_1.$$

If the equations (3.a.27) have rank one, then we may choose coordinates so that

$$Q(\delta(\xi)x^i, \gamma) = 0 \qquad \text{for all} \qquad \xi \in T \quad \text{and} \quad i = 1, 2, 3. \quad (3.a.28)$$

Let $F(X) = 0$ be the defining equation of $X \subset \mathbb{P}^3$ and $J_{\dot{F}} = \oplus_{m \geqslant d-1} J_{F,m}$ the Jacobian ideal. Then by (3.a.8) (cf. (2.c.11))

$$H^2_{\text{prim}}(X) \cong S_6/J_{F,6},$$

and we let γ be represented by a form $P(x) \in S_6$. Setting $F_\alpha = \partial F/\partial x^\alpha$ $(\alpha = 0, 1, 2, 3)$, by the main result in [4] the equations (3.a.28) are equivalent to $^{(41)}$

$$\operatorname{Res}\left\{\frac{P(x)Q(x)x^i \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3}{F_0(x)F_1(x)F_2(x)F_3(x)}\right\} = 0, \qquad \begin{array}{l} P \in S_5 \text{ and} \\ i = 1, 2, 3. \end{array}$$

A contradiction will be obtained if we show that these equations imply $Q \in J_{F,6}$. By the local duality theorem (cf. [4], [20]) this will follow from the assertion:

If $I_{\cdot} = \{F_0, F_1, F_2, F_3; x^1, x^2, x^3\}$ is the ideal generated by the indicated forms, then (3.a.29)

$$I_m = S_m, \qquad m \geqslant 4.$$

Indeed, the ideal $\{x^1, x^2, x^3\}$ generated by x^1, x^2, x^3 has codimension one in S_m for all m. If $I_4 \neq S_4$ then

$$F_\alpha \in \{x^1, x^2, x^3\} \qquad \alpha = 0, 1, 2, 3,$$

which means that all $F_\alpha(1, 0, 0, 0) = 0$ contradicting the smoothness of X.

(b) On the infinitesimal Torelli problem

In this section we will discuss the following conjecture:

> *Let $L \to Y$ be a sufficiently ample line bundle over a smooth variety of dimension $n+1$. Then the infinitesimal Torelli theorem is true for the variation of Hodge structure on $H^n(X)$, where $X \in |L|$ is general.* (3.b.1)

As partial evidence for (3.b.1) we will show now that it is true, in stronger form, when $n = 1$. Suppose that $Y = S$ is a smooth surface and $X = C \in |L|$ is any smooth curve. From the cohomology sequence of the adjunction sequence

$$0 \to K_S \to K_S \otimes L \to K_C \to 0.$$

We first may conclude that the canonical mapping $\varphi_K : C \to \mathbb{P}^{g-1}$ is biregular onto its image; i.e., C is non-hyperelliptic (assuming of course that L is sufficiently ample). From the cohomology diagram of (3.a.15) we next infer that the Kodaira-Spencer map

$$\rho : T \to H^1(C, \Theta)$$

is injective (we are using the notations just below (3.a.15)). Since C is non-hyperelliptic we conclude that the differential of the period mapping

$$\delta : T \to \mathrm{Hom}^{(s)}\big(H^{1,0}(C), H^{0,1}(C)\big)$$

is injective.

In fact, much more is true. At the beginning of Section 2(c) (cf. (2.a.1)) we have discussed the principle that $\mathfrak{W}_d^r$ be used as model for local moduli spaces of higher dimensional varieties. From this point of view the infinitesimal Torelli problem for higher dimensional varities has as curve analogue the following question:

Does the period mapping

$$\varphi : \mathfrak{W}_d^r \to \Gamma \backslash D \tag{3.b.2}$$

have injective differential?

Indeed, reflection shows that the Torelli problem (both infinitesimal and global) for $\mathfrak{W}_d^r$ has much more the flavour of the Torelli problem in higher dimensions than does the Torelli problem for $\mathfrak{M}_g$. (42) If we are at a point $(C, L) \in \mathcal{W}_d^r$ where C is non-hyperelliptic, then (3.b.2) is equivalent to

Is the mapping

$$\mu_0 : H^0(C, L) \otimes H^0(C, KL^{-1}) \rightarrow H^0(C, K) \tag{3.b.3}$$

surjective?

Indeed, by Brill-Noether theory ([2]), (3.b.3) is equivalent to the map

$$\mathcal{W}_d^r \rightarrow \mathfrak{M}_{g,d}^r$$

having injective differential at (C, L) (i.e., the g_d^r given by $|L|$ is unique in the variational sense).

Now suppose that $L \rightarrow S$ is sufficiently ample, and in order to avoid a technically more complicated statement assume also that S is a regular surface. Then we shall prove that:

For $k \geqslant 2$ *and* $C \in |L^k|$ *the mapping*

$$\mu_0 : H^0(C, L) \otimes H^0\big(C, K_C L^{-1}\big) \rightarrow H^0(C, K_C) \tag{3.b.4}$$

is surjective (43)

Proof: By the assumption of regularity we have

$$H^0\big(S, K_S L^k\big) \rightarrow H^0(C, K_C) \rightarrow 0.$$

On the other hand, using $K_C L^{-1} = K_S L^{k-1} \otimes \mathcal{O}_C$ we obtain (using $h^i(L^{1-k}) = h^i(K_S L^{-1}) = 0$ for $i = 0$ and $k \geqslant 2$)

$$\begin{cases} H^0(S, L) \twoheadrightarrow H^0(C, L) \\ H^0\big(S, K_S L^{k-1}\big) \twoheadrightarrow H^0\big(C, K_C L^{-1}\big). \end{cases}$$

Thus it will suffice to show that

$$H^0(S, L) \otimes H^0\big(S, K_S L^{k-1}\big) \rightarrow H^0\big(S, K_S L^k\big) \rightarrow 0$$

is surjective when $k \geqslant 2$ and $L \rightarrow S$ is sufficiently ample, and this is well known. Q.E.D.

In the remainder of this section we will discuss a variant of (3.b.1) where the Hodge structure on $H^n(X)$ is replaced by the mixed Hodge structure on $H^{n+1}(Y - X)$. In this case we may use (3.a.7) to formulate (3.b.1) as a question in commutative algebra (one that is, in a certain sense, dual to the infinitesimal M. Noether theorem (3.a.16)). Following this general discussion we will use the formalism we have developed to verify a couple of examples.

To begin we follow the notations from Section 3(a), especially that just below (3.a.15) and the proof of Theorem (3.a.16). We shall also use the sheaf sequence (*not* exact)

$$\begin{tikzcd}\Theta_Y \arrow[r] \arrow[rr, bend left, "\alpha"] & \Theta_Y \otimes \Theta_X \arrow[r] & L \otimes \mathcal{O}_X\end{tikzcd} \qquad (3.b.5)$$

derived from (3.a.15). With the identification

$$M_q = H^0\big(Y, \Omega_Y^{n+1}((q+1)X)\big),$$

we define

$$E_q \subset M_q$$

by

$$\begin{tikzcd} & H^0(\Omega_Y^n(L^q)) \cong H^0(\Omega_Y^{n+1} \otimes L^q \otimes \Theta_Y) \arrow[d, "\alpha"] \\ H^0(\Omega_Y^{n+1}(L^{q+1})) \arrow[r, "r"] & H^0(\Omega_Y^{n+1} \otimes L^{q+1} \otimes \mathcal{O}_X), \end{tikzcd}$$

where α is induced by (3.b.5) and r is the restriction to X, and then

$$E_q = r^{-1}\big(\alpha H^0(\Omega_Y^n(L^q))\big).$$

It follows that $E_. = \oplus_{q \geqslant 0} E_q$ *is a graded* $S_.$*-submodule* of M. We note that $E_.$ depends on $X \subset Y$. From Theorem (3.a.7) it follows that

$$F^q H^{n+1}(Y-X)/F^{q+1}H^{n+1}(Y-X) \cong M_q/E_q.$$

In particular (cf. (3.a.9))

$$E_q = M_q \qquad \text{for} \qquad q \geqslant n+2,$$

so that $E_.$ is of finite $\mathbb{C}$-codimension in M.

Next, in the diagram

$$\begin{tikzcd} & H^0(Y, \Theta_Y \otimes L^{q-1}) \arrow[d, "\alpha"'] \\ H^0(Y, L^q) \arrow[r, "r"] & H^0(X, L^q), \end{tikzcd} \tag{3.b.6}$$

where α is again induced by (3.b.5) and r is restriction to X, we set

$$J_q = r^{-1}\left(\alpha H^0\left(Y, \Theta_Y \otimes L^{q-1}\right)\right)$$

$$J. = \bigoplus_{q \geqslant 0} J_q$$

Then $J.$ is a graded ideal in $S.$, and moreover

$$J. \otimes M. \to E..$$

Thus $M./J.$ *is a graded* $((S./J.)$*-module.* We remark that, replacing L by a power if necessary, we may assume that

$$J_0 = H^0\left(\Theta_Y \otimes L^{-1}\right) = (0),$$

$$S_1 \otimes J_q \to J_{q+1} \to 0 \qquad \text{for} \quad q \geqslant 1, \text{ and}$$

$$H^i\left(\Theta_Y \otimes L^{-1}\right) = (0) \qquad\qquad i = 0, 1$$

(recall that dim $Y \geqslant 2$). From the third statement and the cohomology diagram (3.b.6) we infer that

$$S_1/J_1 \cong T$$

where T was defined below in (3.a.15), (T is naturally identified with the tangent space to the family of affine varieties $\{Y - X\}$ ($X \in |L|$), modulo automorphisms induced from Y). Combining this with the preceding discussion we have the

PROPOSITION: *The infinitesimal Torelli theorem for the variation of mixed Hodge structure on* $H^{n+1}(Y - X)$ *is equivalent to the pairing*

$$\left(S_1/J_1\right) \otimes \left(M./E.\right) \to \left(M./E.\right) \tag{3.b.7}$$

being non-degenerate in the first factor.

It is a weaker conjecture than (3.b.1) that, for $L \to Y$ sufficiently ample, the pairing in (3.b.7) is for general X non-degenerate in the first factor. Even though we are unable to prove this commutative algebra assertion, we can use the formalism to verify some examples of conjecture (3.b.1)

EXAMPLE: (cf. [4] and [22]). We take $Y = \mathbb{P}^{n+1} = \mathbb{P}V$ with homogenous coordinates $x = [x^0, x^1, \dots, x^{n+1}]$ and $X \subset Y$ a smooth hypersurface of degree d given by

$$F(x) = 0. \tag{3.b.8}$$

We will show that the map

$$\delta : T \to \mathrm{Hom}(H^{n-q,q} \to H^{n-q-1,q+1})$$

is injective whenever the right-hand side is non-zero, or equivalently when

$$d(q+1) \geqslant n+2$$

and

$$d(n-q) \geqslant n+2.$$

To describe the differential forms on Y with poles along X we shall give their lifts under the projection

$$\pi : V - \{0\} \to \mathbb{P}V. \tag{3.b.9}$$

For this we use the notations

$$A_p^k = \left\{ \begin{array}{l} k\text{-forms on } V = \mathbb{C}^{n+2} \text{ whose coefficients} \\ \text{are homogeneous polynomials of degree } p \end{array} \right\}$$

(thus $\psi \in A_p^k$ has total homogeneity $p+k$, in the sense that $\psi(\mu x) = \mu^{p+k}\psi(x)$),

$$e = \sum_{i=0}^{n+1} x^i \partial/\partial x^i$$

is the Euler vector field. It is well known that

$$\exp(te\, x = (\exp t) \cdot x$$

and consequently the orbits of e in $V - \{0\}$ are just the fibres of (3.b.9). From this we infer that the horizontal forms for the fibering (3.b.9) are defined by $^{(44)}$

$$i(e)\psi = 0. \tag{3.b.10}$$

It is an elementary fact that the sequences

$$A^{k-1}_{p+1} \xrightarrow{d} A^{k}_{p} \xrightarrow{d} A^{k+1}_{p-1}, \qquad p+k>0$$

$$A^{k+1}_{p-1} \xrightarrow{i(e)} A^{k}_{p} \xrightarrow{i(e)} A^{k-1}_{p}, \qquad p+k>0 \tag{3.b.11}$$

are exact.

Proof: The Lie derivative $\mathcal{L}_e\psi$ of $\psi \in A^k_p$ with respect to e is given by each of the following formulae:

$$\begin{cases} \mathcal{L}_e\psi = (p+k)\psi & \text{(Euler's formula)} \\ \mathcal{L}_e\psi = di(e)\psi + i(e)d\psi & \text{(Cartan's formula).} \end{cases}$$

The exactness of the sequences (3.b.11) follows by combining the relations (3.b.12). Q.E.D.

The pullback to V of any element in $H^0(\Omega^k_{\mathbb{P}V}((q+1)X))$ is

$$\Psi = \psi/F^{q+1}$$

where (cf. footnote [(44)])

$$\psi \in A^k_{d(q+1)-k}$$

$$i(e)\psi = 0.$$

Using (3.b.11) we write

$$\begin{cases} \psi = i(e)\varphi \\ \varphi \in A^{k+1}_{d(q+1)-(k+1)} \end{cases} \tag{3.b.13}$$

When $k = n+1$ this is

$$\varphi = P(x)\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{n+1}$$

$$\psi = P(x)\sum_{i=0}^{n+1}(-1)^{i-1}x^i\mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x' \wedge \ldots \wedge \mathrm{d}x^{n+1}$$

$$P \in S_{d(q+1)-(n+2)}$$

It will be convenient to set

$$\begin{cases} \Omega_i(\mathrm{d}x) = (-1)^{i-1}\mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^i \wedge \ldots \wedge \mathrm{d}x^{n+1} \\ \Omega_{ij}(\mathrm{d}x) = (-1)^{i+j}\mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^i \wedge \ldots \wedge \mathrm{d}x^j \wedge \ldots \wedge \mathrm{d}x^{n+1} \end{cases} \tag{3.b.14}$$

Then

$$\psi = P(x)\sum x^i\Omega_i(\mathrm{d}x),$$

and a general element in $H^0(\Omega^{n+1}_{\mathbb{P}V}((q+1)X))$ is

$$\Psi_P = \big(P(x)\sum x^i\Omega_i(\mathrm{d}x)\big)/F(x)^{q+1}. \tag{3.b.15}$$

When $k = n$ and the order of pole is q, (3.b.13) is

$$\psi = \tfrac{1}{2}\sum(x^iQ^j - x^jQ^i)\Omega_{ij}(\mathrm{d}x), \qquad Q^i \in S_{\mathrm{d}q-(n+1)},$$

$$\varphi = \sum Q^i(x)\Omega_i(\mathrm{d}x)$$

Accordingly we shall write

$$\Xi_{\{Q^i\}} = \big(\tfrac{1}{2}\sum(x^iQ^j - x^jQ^i)\Omega_{ij}(\mathrm{d}x)\big)/F(x)^q, \qquad Q^i \in S_{\mathrm{d}q-(n+1)}$$

for a general element in $H^0(\Omega^n_{\mathbb{P}V}(qX))$.

Comparing (3.b.15) with this equation and adjusting constants, the condition

$$\mathrm{d}\Xi_{\{Q^i\}} \equiv \Psi_P \text{ modulo } H^0\big(\Omega^{n+1}_{\mathbb{P}V}(q(X)\big)$$

turns out, after an obvious computation, to be

$$\sum Q^iF_i \equiv P \text{ modulo } F, \qquad F_i = \partial F/\partial x^i.$$

By Euler's theorem this is equivalent to

$$\sum \tilde{Q}^iF_i = P, \qquad \tilde{Q}^i \in S_{\mathrm{d}q-(n+1)}$$

To summarize, we denote by

$$\{F_0, \ldots, F_{n+1}\} = J_F = \bigoplus_{q \geqslant d-1} J_{F,q}, \qquad F_i = \partial F/\partial x^i,$$

the graded Jacobian ideal associated to $F(x)$. Then with the notations

$$\begin{cases} \sigma = d(q+1) - (n+2) \\ \rho = (d-2)(n+2) \\ \sigma + \sigma' = \rho \end{cases}$$

we have

$$\begin{cases} M_q \cong S_\sigma \\ E_q \cong J_{F,\sigma} \\ F^q H^{n+1}(\mathbb{P}V - X)/F^{q+1}H^{n+1}(\mathbb{P}V - X) \cong H^{n-q,q}_{\text{prim}}(X) \\ H^{n-q,q}_{\text{prim}}(X) \cong S_\sigma/J_{F,\sigma} \end{cases} \tag{3.b.16}$$

The bilinear form Q on $H^n_{\text{prime}}(X)$ may also be described, as follows: We recall the map

$$\text{Res}: S_k \to \mathbb{C}$$

given by the Grothendieck residue

$$P \to \text{Res}\left\{\frac{P(x)\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{n+1}}{F_0(x) \ldots F_{n+1}(x)}\right\}$$

By homogeneity this map is zero unless $k = \rho$, and in this case the map

$$\text{Res}: S_\rho/J_{F,\rho} \to \mathbb{C}$$

is an isomorphism. More generally, for any $\sigma \geqslant 0$, $\sigma' = \rho - \sigma \geqslant 0$, by Grothendieck's local duality theorem the pairing

$$\text{Res}: S_\sigma/J_{F,\sigma} \otimes S_{\sigma'}/J_{F,\sigma'} \to \mathbb{C}, \tag{3.b.17}$$

given by

$$P \otimes Q \to \text{Res}(PQ)$$

is non-degenerate. One of the results of [4] is that, with the identifications (3.b.16), the Hodge pairing Q is (up to a constant) just (3.b.17).

A consequence of the non-degeneracy of the pairing (3.b.17) is that the pairing

$$S_\sigma/J_{F,\sigma} \otimes S_\tau/J_{F,\tau} \to S_{\sigma+\tau}/J_{f,\sigma+\tau} \qquad \sigma + \tau \leqslant \rho$$

is non-degenerate in each factor (Macaulay's theorem). In particular,

$$S_d/J_{F,d} \otimes S_\sigma/J_{F,\sigma} \to S_{\sigma+d}/J_{F,\sigma+d} \tag{3.b.18}$$

is non-degenerate in the first factor if

$$d(q+1) \geqslant n+2$$

and

$$d(n-q) \geqslant n+2.$$

Suppose we denote by $\mathcal{U}_d \subset \mathbb{P}S_d = |\mathcal{O}_{\mathbb{P}V}(d)|$ the Zariski open set of smooth hypersurfaces and set

$$T = T_F(\mathcal{U}_d)/T_F\,(PGL\text{-orbit through } F).$$

Then

$$T_F(\mathcal{U}_d) \cong S_d/(F),$$

and from

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\mathrm{e}^{tA}x)|_{t=0} = \sum A^i_j x^j \frac{\partial F}{\partial x^i}$$

it follows that

$$T_F(PGL\text{-orbit through } F) \cong J_{F,d}/(F).$$

Thus

$$T \cong S_d/J_{F,d} \tag{3.b.19}$$

On the other hand, the Kodaira Spencer map

$$\rho : T_F(\mathcal{U}_d) \to H^1(X, \Theta_X)$$

induces an injection (cf. [29])

$$\rho : T \to H^1(X, \Theta_X)$$

(which is an isomorphism except when $n = 1$, $d \geqslant 5$ or $n = 2$, $d = 4$). We denote by $\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ the infinitesimal variation of Hodge structure induced by the 1st order variations of X in $\mathbb{P}^{n+1}$. Then we have the identifications (3.b.17), (3.b.18) (for Q), (3.19), and the differential

$$\delta : T \otimes H^{n-q,q} \to H^{n-q-1,q+1}$$

is given by (3.b.18). In particular,

$$\delta : T \to \operatorname{Hom}(H^{n-q,q}, H^{n-q-1,q+1})$$

is injective whenever $h^{n-q,q} \neq 0$, $h^{n-q-1,q+1} \neq 0$. Aside from the case $n = 2$, $d = 3$ this gives a much stronger version of (3.b.1).

EXAMPLE: We let V, W be vector spaces of respective dimensions $m + 1$, $n + 1$ and set

$$Y = \mathbb{P}V \times \mathbb{P}W \cong \mathbb{P}^m \times \mathbb{P}^n.$$

Over Y we denote by $\mathcal{O}(a, b)$ the line bundle $\pi_1^* \mathcal{O}_{\mathbb{P}^m}(a) \times \pi_2^* \mathcal{O}_{\mathbb{P}^n}(b)$; Thus

$$K_y = \mathcal{O}(-m-1, -n-1),$$

Let $v^0, \dots, v^m \in V^*$ and $w^0, \dots, w^n \in W^*$ be respective linear coordinates, and denote by $[v, w] = [v^0, \dots, v^m; w^0, \dots, w^n]$ the corresponding bihomogeneous coordinates on Y. If we define

$$H^0(Y, \mathcal{O}(a, b)) = S_{a,b},$$

then $S_{a,b}$ is the space of forms $F(v, w)$ that are homogeneous of degree a in v and degree b in w, and

$$S_{\cdot,\cdot} = \bigoplus_{a,b \geqslant 0} S_{a,b}$$

is a bigraded ring.

Let $X \in |\mathcal{O}_Y(a, b)|$ be a hypersurface given by $F(v, w) = 0$. If we define $X_V \subset \mathbb{P}V$ by $F(v, 0) = 0$, and $X_W \subset \mathbb{P}W$ by $F(0, w) = 0$, then we assume that each of X, X_V, X_W is smooth. For $a, b > 0$ this is a Zariski open subset of $|\mathcal{O}_Y(a, b)|$. As in the previous example, we consider the projection

$$\pi : (V - \{0\}) \times (W - \{0\}) \to \mathbb{P}V \times \mathbb{P}W = Y \tag{3.b.20}$$

and will describe the differential forms in $H^0(\Omega_Y^k((q + 1)X))$ by their lifts to $V \times W$. For this we consider the two Euler vector fields

$$e_V = \sum_{i=0}^{m} v^i \partial / \partial v^i$$

$$e_W = \sum_{\alpha=0}^{n} w^\alpha \partial / \partial w^\alpha.$$

Together these vector fields generate a $\mathbb{C}^* \times \mathbb{C}^*$-action on $(V-\{0\})\times(W-\{0\})$ whose orbits are the fibers of (3.b.20). It follows that the forms in $H^0(\Omega_Y^k((q+1)X))$ lift under (3.b.20) to forms

$$\Psi = \psi/F^{q+1}$$

on $V\times W$, where ψ is a polynomial differential form in (v, w) that satisfies (cf. footnote $^{(44)}$)

$$\begin{cases} i(e_v)\psi = 0 = i(e_w)\psi \\ \psi \text{ is bihomogeneous of degree } (A, B) = (a(q+1), b(q+1)) \end{cases} \tag{3.b.21}$$

LEMMA: *If* ψ *satisfies* (3.b.21), *then*

$$\psi = i(e_v)i(e_w)\varphi, \tag{3.b.22}$$

where φ *is a bihomogeneous of bidegree* (A, B).

PROOF: We first claim that

$$\mathfrak{L}_{e_w} = f(e_w)d_w + d_w i(e_w), \tag{3.b.23}$$

where d_w is the exterior derivative with respect to the w-variables. This may be proved in the same way as the usual Cartan formula, by showing that each side is a derivation and that these two derivations agree on forms of degrees 0, 1.

It follows that

$$\begin{aligned} AB\psi &= A(\mathfrak{L}_{e_w}\psi) \\ &= A(d_w i(e_w)\psi + i(e_w)d_w\psi) \\ &= i(e_w)d_w(A\psi) \\ &= i(e_w)d_w(\mathfrak{L}_{e_v}\psi) \\ &= i(e_w)d_w i(e_v)d_v\psi \\ &= \pm i(e_w)i(e_v)(d_w d_v\psi) \end{aligned}$$

Q.E.D.

Taking $k = m+n$ and using the notations (3.b.14), the forms in $H^0(\Omega_Y^{m+n}((q+1)X))$ are given by expressions

$$\Psi_P = (P(v, w)\Omega_i(\mathrm{d}v)\wedge\Omega_\alpha(\mathrm{d}w))/F(v, w)^{q+1},$$

$$P \in S_{(A-m-1, B-n-1)}$$

Similarly, the forms in $H^0(\Omega_Y^{m+n-1}(qX))$ are

$$\Xi_{\{Q^i,R^j\}} = \Big(\tfrac{1}{2} \sum_{i,j,\alpha} w^\alpha (Q^i v^j - Q^j v^i) \Omega_{ij}(\mathrm{d}v) \wedge \Omega_\alpha(\mathrm{d}w)$$

$$+ \tfrac{1}{2} \sum_{i,\alpha,\beta} v^i (R^\alpha w^\beta - R^\beta w^\alpha) \Omega_i(\mathrm{d}v) \wedge \Omega_{\alpha\beta}(\mathrm{d}w) \Big) \Big/ F^q, \tag{3.b.25}$$

where Q^i, R^α have appropriate bihomogeneity restrictions. We set

$$F_i = \partial F / \partial v^i, \qquad F_\alpha = \partial F / \partial w^\alpha,$$

and denote by

$$J_F = \bigoplus_{k,l} J_{F,(k,l)} = \{F_i, F_\alpha\}$$

the bihomogeneous Jacobian ideal of $F(v, w)$. As in the ordinary hypersurface case, from (3.b.24) and (3.b.25) we deduce that the condition

$$\Psi_{\mathbb{P}} \equiv d\Xi_{\{Q^i,R^\alpha\}} \text{ modulo } H^0(\Omega_Y^{m+n}(qX))$$

is

$$P \equiv \sum Q^i F_i + \sum R^\alpha F_\alpha \text{ modulo } F.$$

As before, we conclude that (45)

$$M_q = S_{(A-m-1,B-n-1)}$$

$$E_q = J_{F,(A-m-1,B-n-1)},$$

and the pairing (3.b.7) is a part of

$$\big(S_{(a,b)}/J_{F,(a,b)}\big) \otimes \big(S_{\cdot,\cdot}/J_F\big) \to \big(S_{\cdot,\cdot}/J_F\big)\ ^{(46)}.$$

Finally, by applying Grothendieck's local duality theorem to the ideal $\{F_i, F_\alpha\}$ (47) in $S_{\cdot,\cdot} \cong \mathbb{C}[v^0, \dots, v^m, w^0, \dots, w^n]$ we may conclude that:

The pairing

$$S_{(a,b)} \otimes \big(S_{(\alpha,\beta)}/J_{F,(\alpha,\beta)}\big) \to \big(S_{(a+\alpha,b+\beta)}/J_{F,(a+\alpha,b+\beta)}\big)$$

is non-degenerate in the first factor, whenever

$$S_{(\alpha,\beta)}/J_{F,(\alpha,\beta)} \neq (0). \tag{3.b.26}$$

In particular, the infinitesimall Torelli theorem holds for the variation of mixed Hodge structure on $H^{m+n}(\mathbb{P}V\times\mathbb{P}W-X)$ where X is a general member of $|\mathcal{O}(a,b)|(a,b>0)$.

REMARK: When $m+n=2k+1$ is odd, then (cf. (3.a.8))

$$H^{m+n}(\mathbb{P}V\times\mathbb{P}W-X)\cong H_v^{2k}(X).^{(48)}$$

At the other extreme, when $m=n$, if we set

$$a\omega_v+b\omega_w=c_1\mathcal{O}(a,b),$$

then the class

$$\sum_{j=0}^{n}(-1)^j\omega_v^j\omega_w^{n-j}\in H^{2n}_{\text{prim}}(\mathbb{P}V\times\mathbb{P}W)$$

lives in $F^mH^{2n}(\mathbb{P}V\times\mathbb{P}W-X)$, and is therefore represented by a form (3.b.24) when $q=n$. We have not been able to compute what this form is.

REMARK: When $m=1$ the variety $Y=\mathbb{P}^n\times\mathbb{P}^1$ has a natural embedding (Segre embedding)

$$Y\subset\mathbb{P}^{2n+1}$$

as a scroll. Then, for suitable (a,b) a smooth hypersurface $X\in|\mathcal{O}(a,b)|$ is *extremal* in the sense that the Hodge number $h^{n,0}(X)$ is maximal among all non-degenerate varieties $X\subset\mathbb{P}^{2n+1}$ of degree $d=a+b$ (cf. [25]). Concerning extremal varieties we make the following

CONJECTURE: *The global Torelli theorem is true for extremal varieties $X\subset\mathbb{P}^N$, in the strong sense that if $X, X'\subset\mathbb{P}^N$ are extremal varieties with isomorphic Hodge structures then there is a projective automorphism of $\mathbb{F}^N$ taking X to X'.* (3.b.27)

When $n=1$ and $N=g-1$ we have the usual global Torelli theorem for non-hyperelliptic curves, since these are extremal ([20]).

When $n=1$ and $N>g-1$, then the conjecture is true since an extremal g_d^N on a smooth curve is unique ([1] and [2]).

In general the above conjecture includes as a special case the same conjecture for smooth hypersurfaces in $\mathbb{P}^{n+1}$, since there are extremal.

Finally, it also includes the global Torelli theorem for polarized K3 surfaces, since these are also extremal.

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Notes

(1) The precise statement is this: A generic Hodge structure of weight $n \geqslant 2$ with $h^{n,0} \neq 0$ (and $h^{2,0} \geqslant 2$ in case $n = 2$) is not an algebro-geometric motif ([35]). The reason for this result is that due to the infinitesimal period relation (which is non-trivial under the above conditions), the family of Hodge structures coming from any family of algebraic varieties is an integral manifold of a non-trivial Pfaffian differential system, and in particular cannot cover an open set in the family of all polarized Hodge structures.

(2) In particular, "natural" includes holomorphically varying with parameters, which rules out the theta divisor on Weil's intermediate Jacobians (cf. [19] and [33]).

In this regard we remark that the behavior of the Abel-Jacobi map for general intermediate Jacobians also departs radically from the situation for curves. In some ways the difficulties in understanding the period map and the Abel-Jacobi map appear to be parallel.

(3) An infinitesimal variation of Hodge structure is a first order invariant. However, second order conditions must be fullfilled in order that the conditions of its definition be met, so it is not just given by a subspace of the horizontal tangent space to the classifying space for polarized Hodge structures. The point is that the horizontal distribution is *not* integrable, and the "easy" part of the integrability conditions must be satisfied.

(4) One of these, the second fundamental form, is a 2nd order invariant whose definition therefore depends on the data of a 2nd order infinitesimal variation of Hodge structure. We have only defined this invariant in a special case, as giving the general definition would take us too far afield.

(5) The classical Schottky problem is to determine the relations that must be satisfied by a period matrix

$$\begin{cases} Z = X + iY \\ Z = {}^tZ \quad \text{and} \quad Y > 0 \end{cases}$$

in order that it be the period matrix of an algebraic curve. Our infinitesimal Schottky relations are what the name suggests, except that when the weight of the Hodge structure is $\geqslant 2$ we do not determine all of the tangent space to the periods of a family of varieties but only that part which we can geometrically interpret.

(6) For the universal family of curves, $\ker\lambda = (0)$. However, when $\dim X \geqslant 2$ it may well be that, even for the Kuranishi universal family, $\ker\lambda \neq (0)$, and geometrically interpreting the linear subsystem $|\ker\lambda|$ of the bicanonical system $|K^2|$ is the major problem encountered in understanding our first invariant.

(7) The proper concept is a motif [35], and using this theory one can make this definition more precise and therefore more satisfactory. We hope that it will not be misleading to leave matters on an intuitive level.

(8) Briefly, the invariant complex structures on $H_{\mathbf{R}}/H_{\mathbf{Z}}$ are in 1-1 correspondence with the complex structures on the real vector space $H_{\mathbf{R}}$. These are, in turn, in 1-1 correspondence with splittings

$$H_{\mathbf{R}} \otimes \mathbf{C} = E \oplus \bar{E}.$$

In the case at hand we take $E = H''$. To check that we don't want the conjugate structure, take $H_{\mathbf{Z}} = H^1(C, \mathbf{Z})$ where C is a smooth curve. Then the Jacobian $J(C)$ is

$$\begin{aligned} J(C) &= H^1(C, \mathcal{O})/\Lambda \\ &\cong H^{0,1}(C)/\Lambda \end{aligned}$$

where Λ is the projection of $H^1(C, \mathbf{Z})$ to $H^{0,1}(C)$.

(9) We note that $J^m(X)$ varies holomorphically with X.

There are also the intermediate Jacobians of Weil (cf. [33] and [49]), defined by the procedure of footnote [1] by taking

$$E = \bigoplus_k H^{m+2k,m-1-2k}(X).$$

In case (X, Ω) is a polarized algebraic variety these complex tori are naturally polarized abelian varieties. Since they do not in general vary holomorphically with X, the role played by their theta divisors is somewhat mysterious.

(10) For a complex torus J we always identify the dual of $\mathfrak{L}(J)$ with the space $H^{1,0}(J)$ of (translation-invariant) holomorphic differentials on J.

(11) It would seem that Hodge theory of weight $n > 2$ is a *relative theory*, i.e., the objects of interest are not so much the classifying spaces $\Gamma\backslash D$ and $J^m(X)$ but are rather the maps

$$\varphi: S \to \Gamma\backslash D$$

$$u: B \to J^m(X)$$

satisfying the infinitesimal conditions arising from geometry. For example, even though the tautological bundles over $\Gamma\backslash D$ and $J^m(X)$ are not positive, the restriction of their curvature forms to $T_h(D)$ and $\mathfrak{L}_h(J^m(X))$ are positive, and this is what seems to be important for applications.

(12) In this regard we recall that Lefschetz' original proof [32] of his (1,1) theorem had two ingredients. One is the association to a primitive Hodge class of the normal function that would be associated to any algebraic cycle representing the given Hodge class. This is a global step, and as indicated by (1.b.13) it generalizes to higher dimensions. The second ingredient is local in the parameter space, and consists in applying the Jacobi inversion theorem with dependence on parameters to construct the desired algebraic cycle. As is well known this step breaks down in higher dimensions.

As we shall see later, for global reasons there are conditions imposed on the infinitesimal invariant $\delta\nu$ of the normal function associated to any primitive algebraic cycle as in the last example in this section. It may be that a better understanding of these conditions will give insight into questions about higher codimensional cycles.

(13) More precisely, giving the horizontal distribution $T_h(D) \subset T(D)$ is equivalent to giving $T_h(D)^{\perp} \subset T^*(D)$. Over an open set $\mathcal{U} \subset D$ we choose holomorphic 1-forms $\theta^1, \dots, \theta^s$ that give a basis for $T_h(D)^{\perp}$. The equations

$$\theta^\alpha = 0 \qquad \alpha = 1, \dots, s \tag{$*$}$$

define the Pfaffian differential system $I|\mathcal{U}$. An *integral manifold* of $(*)$ is given by a complex manifold S together with a holomorphic mapping

$$f: S \to D$$

such that

$$f^*\theta^\alpha = 0 \qquad \alpha = 1, \dots, s$$

If E is a typical tangent space to a smooth point of $f(S)$, then since

$$f^*\mathrm{d}\theta^\alpha = \mathrm{d}(f^*\theta^\alpha) = 0$$

we have

$$\begin{cases} \theta^{\alpha} | E = 0 \\ d\theta^{\alpha} | E = 0 \end{cases}$$

It can be shown that:

An infinitesimal variation of Hodge structure is given by a point $\{F^p\} \in D$ *and linear map*

$$T \to T_{\{F^p\}}(D)$$

whose image E satisfies the two conditions $(*)$ *and* $(**)$.

In the language of the theory of differential systems, the infinitesimal variations of Hodge structure are identical with the *integral elements* of the Pfaffian system I on D.

(14) Equivalently, we consider the Kodaira-Spencer map (cf. Section 2a)

$$\rho : (m/m^2)^* \to H^1(X, \Theta)$$

where X is the reduced fibre of $\mathfrak{X} \to S$. If we consider the usual map

$$[\ ,\ ] : H^1(X, \Theta) \otimes H^1(X, \Theta) \to H^2(X, \Theta)$$

induced by the Poisson bracket $[\ ,\ ] : \Theta \otimes \Theta \to \Theta$, then the extendability of $\mathfrak{X} \to S$ to second order is equivalent to

$$\left[\rho(\partial/\partial s^i), \rho(\partial/\partial s^j)\right] = 0$$

for all i, j.

In Section 2(a) we will see that

$$\delta(m/m^2)^* \to \oplus \operatorname{Hom}\left(H^q(X, \Omega^p), H^{q+1}(X, \Omega^{p-1})\right)$$

is given by

$$\delta(\partial/\partial s^i) = \kappa_1\left(\rho(\partial/\partial s^i)\right)$$

where

$$\kappa_l : H^l(X, \Theta) \to \bigoplus_{p,q} \operatorname{Hom}\left(H^q(X, \Omega^p), H^{q+l}(X, \Omega^{p-l})\right)$$

is the cup-product mapping. It can be shown that

$$\left[\delta(\partial/\partial s^i), \delta(\partial/\partial s^j)\right] = \kappa_2\left[\rho(\partial/\partial s^i), \rho(\partial/\partial s^j)\right].$$

therefore, the equations

$$\kappa_2\left[\rho(\partial/\partial s^i), \rho(\partial/\partial s^j)\right] = 0$$

are necessary and sufficient that $\mathfrak{X} \to S$ give an infinitesimal variation of Hodge structure.

(15) In this regard we observe that the action of $G_{\mathbb{C}}$ on $T(\check{D})$ is very far from being transitive. For a simpler example consider the Grassmannian $G(k, H)$ of k-plane E in a

complex vector space H. The tangent space at E is naturally identified with $\mathrm{Hom}(E, H/E)$. Choose an isomorphism $H \cong \mathbf{C}^m$ so that $E = \mathbf{C}^k$, and think of $\mathrm{Hom}(E, H/E)$ as given by $k \times (m-k)$ matrices ξ. The isotropy group of E is the group of matrices

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \quad A \in GL_{m-k}, \quad C \in GL_k,$$

and the action of this matrix on ξ is

$$\xi \to A\xi C^{-1}. \tag{$*$}$$

Thus, we are considering linear subspaces

$$T \subset \mathrm{Hom}(\mathbf{C}^k, \mathbf{C}^{m-k})$$

under the action of $GL_k \times GL_{m-k}$ given by $(*)$.

(16) For curves an infinitesimal normal function is given by

$$\begin{tikzcd} \mathcal{L} \arrow[r] & \mathcal{X} \arrow[d] \\ & S_1 = \mathrm{Spec}(\mathbf{C}[s^1, \dots, s^m]/m^2) \end{tikzcd}$$

where $\mathcal{X} \to S_1$ is a 1st order variation of a smooth curve C, and where $\mathcal{L} \to \mathcal{X}$ is an invertible sheaf whose restriction to the reduced fibre C of $\mathcal{J} \to S_1$ is $L \to C$ where $L \in \mathrm{Pic}^{\circ}(\mathrm{C})$ is a line bundle of degree zero.

(17) This invariant will be discussed in Part III of this series of papers. One motivation for the introduction of $\delta\nu$ is the following. Let $\{C_s\}_{s \in S}$ be a family of smooth curves, $\{D_s = \Sigma_i p_i(s) - q_i(s)\}_{s \in S}$ a family of divisors of degree zero) on these curves ($D_s \in \mathrm{Div}^0(C_s)$) and $\{\omega(s) \in H^0(C_s, \Omega^1_{C_s})\}_{s \in S}$ a family of holomorphic 1-forms on the C_s. Thinking of the Jacobian of C_s as

$$J(C_s) = H^0\left(C_s, \Omega^1_{C_s}\right)^* / H_1(C_s, \mathbf{Z}),$$

we may describe the normal function ν corresponding to the divisors $D_s \in \mathrm{Div}^0(C_s 0$ by

$$\nu(s) = \sum_i \int_{q_i(s)}^{p_i(s)} \omega(s). \tag{$*$}$$

When both C_s and $\omega(s)$ are constant in S, it is well known that differentiation of the abelian sum $(*)$ is one of the key steps in the study of algebraic curves (this leads to the *Brill-Noether matrix*, cf. [2]). The definition of $\delta\nu$ was arrived at by trying to make intrinsic sense out of $\mathrm{d}\nu(s)/\mathrm{d}s$ when C_s is variable.

(18) Ideally, to "compute" should mean to interpret geometrically. However, aside from curves, the group $H^1(X, \Theta)$ is not particularly "geometric". One of our main techniques is, in some case, to geometrically interpret classes naturally associated to $\theta \in H^1(X, \Theta)$ cf. (2.a.10) below). Roughly speaking, our best insight seems to come by combining such construction with the heuristic principle (2.c.1).

One example when we may immediately compute the "universal" infinitesimal variation of Hodge structure is when X has trivial canonical bundle. Then

$$\Theta \cong \Omega^1_X$$

$$H^{n,0} = H^0(X, K) = \mathbf{C},$$

and the first piece of the differential

$$\begin{tikzcd}
\delta: H^1(X,\Theta) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(H^{n,0}, H^{n-1,1}) \arrow[d, equal] \\
H^{1,1} \arrow[r] & H^{1,1}
\end{tikzcd}$$

is an isomorphism. In this case we can also say something non-trivial about the differential system given by the infinitesimal period relation.

(19) The cup-product is the map

$$H^1(X,\Theta)\otimes H^q(X,\Omega^p)\to H^{q+1}(X,\Omega^{p-1})$$

induced from the $\mathcal{O}$-linear mapping

$$\Theta\otimes\Omega^p\to\Omega^{p-1}$$

given by

$$\theta\otimes\psi\to i(\theta)\psi.$$

Beginning with the basic map

$$\mu_0: H^0(C,L)\otimes H^0(C,KL^{-1})\to H^0(C,K)$$

of Brill-Noether theory [2], the theory of infinitesimal variations of Hodge structure has throughout a tri-linear algebra aspect.

(20) What this means is: just write out the equation to be verified in local coordinates. We shall only occasionally do this.

(21) The reason for the terminology is that, by iterating the differential, we derive from (2.a.7) a situation in which all higher cohomology dissappears and to some extent we have a formal analogue of curve theory (in particular, quadratic differentials appear).

(22) As we shall try to explain below, the ultimate understanding of $\ker\lambda$ seems to involve a type of geometric question that, even for curves, is new. Although some insight can be gained from Brill-Noether theory [2], obtaining a more satisfactory understanding is one of the main outstanding problems of the theory.

(23) Alternatively, Γ is generated by the ramification divisors of all projections $X\to\mathbb{P}^n$.

PROOF: The Schubert hyperplanes H_Λ defined for Λ a $\mathbb{P}^{N-n-1}\subset\mathbb{P}^N$ by

$$H_\Lambda=\{n\text{-planes } E\subset\mathbb{P}^N: E\cap\Lambda\neq\emptyset\},$$

generate the complete linear system $|\mathcal{O}_{G(n,N)}(1)|$. Since $\gamma^{-1}(H_\Lambda)$ is the ramification divisor of the projection from Λ, our assertion follows.

It is to be emphasized that Γ is an *incomplete* linear system.

(24) In particular, $\Gamma_{2K}=(0)$ unless $H^0(X,KL^{-(n+1)})\neq(0)$. Line bundles with this property seem to show many features of *special divisors* in the case of curves.

(25) In the next section this result will be extended and generalized. However, we feel it is worthwhile to prove the special case first, as it has several special features and also suggests how the general argument should go.

(26) If $\mathbb{P}^N = \mathbb{P}V$ for a vector space V, then the Euler sequence is

$$0 \to \mathcal{O}_{\mathbb{P}^N} \xrightarrow{j} \mathcal{O}_{\mathbb{P}^N} \otimes V^* \to \Theta_{\mathbb{P}^N} \to 0$$

where

$$j(f) = fe$$

with

$$e = \sum x^i \partial/\partial x^i$$

being the Euler vector field.

(27) Using the notation just below (2.b.6), L_p is spanned by $X(u)$ and Θ_p by $X(u)$, $\partial X/\partial u^1, \ldots, \partial X/\partial u^n$.

(28) The degree restriction $d \geqslant 2n+4$ comes about because this is exactly the condition that the mapping (2.b.1) be surjective.

(29) Equivalently, we should be given the Veronese cone $\Xi \subset S_{d-n-2}$ of decomposable elements (i.e., those of the form $(L(x))^{d-n-2}$ where $L(x)$ is a linear form).

(30) We remark that, if $d \neq n+2$, then Aut X is induced by projectivities of $\mathbb{P}^{n+1}$. If $n \geqslant 2$ this follows from Pic $X \cong \mathbf{Z}$, so that any automorphism of X must preserve $K_X = \mathcal{O}_X(d-n-2)$ and therefore also $\mathcal{O}_X(1)$. When $n = 1$ it is still true that any automorphism of X preserves $\mathcal{O}_X(1)$, but this is deeper (cf. [1]).

(31) For $H(x) \in S_d$ and $A = (A^i_j)$, the tangent to the arc

$$H_t(x) = H(e^{tA}x)$$

at $t = 0$ is

$$\sum_{i,j} A^i_j x^j \partial H/\partial x^i \in S_d.$$

Thus, the tangent space to the G-orbit of $H \in S_d$ is $J_{H,d} \subset S_d$.

(32) We shall follow the notations and terminology of [2]. It should be pointed out that the Brill-Noether theory is interesting mainly in case when $L \to C$ is *special* in the sense that

$$\begin{cases} h^0(C, L) \neq 0 \\ h^0(C, KL^{-1}) \neq 0 \end{cases}$$

One higher dimensional analogue of these conditions is:

$$\begin{cases} h^0(X, L) \neq 0 \\ h^0(X, KL^{-n}) \neq 0 \end{cases}$$

($\dim X = n$). The theory of infinitesimal variation of Hodge structure seems especially suited to studying such varieties.

(33) We also would like to remark that the classical characterization of surface S with $p_g(S) \neq 0$ is that for any curve C on S with $h^1(\mathcal{O}_S(C)) = 0$, the characteristic series $\mathcal{O}_C(C)$ is special (this follows from the cohomology sequence of $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$).

(34) In the case of surfaces, making (2.c.1) more precise means the following: Given a very ample line bundle $L \to S$, we consider pairs (C, L_C) where $C \in |L^k|$ is a smooth curve, k is sufficiently large, and $L_C = L \otimes \mathcal{O}_C$. Then $\lambda = (C, L_C) \in \mathcal{W}_d^r$ and there is an isomorphism

$$T_\lambda(\mathcal{U}_d^r)/T_\lambda(|L^k|) \cong H^1(S, \Theta_S).$$

(35) Here we are using the identification (3.a.1) to interpret $H^0(\Omega_Y^{n+1}(qX))$ as the image $s \cdot H^0(\Omega_Y^{n+1} \otimes L^q) \subset H^0(\Omega_Y^{n+1} \otimes L^{q+1})$.

(36) We are again using (3.a.1).

(37) In particular the fixed part of $H^n(X)$, defined to be the maximal direct summand on which the global monodromy acts as a finite group, consists a priori of $H^n_f(X)$ together with a sub-Hodge structure $H' \subset H^n_v(X)$ satisfying

$$H' \cap H^{p,q}(X) \subset H^{p,q}_{\text{i.f.}}(X).$$

We will prove below that $H' = (0)$.

(38) The point is this: Work in a small neighborhood $\mathcal{U} \subset S$ where there is no monodromy. Given $\gamma \in H^{2m}(X, \mathbf{Z}) \cap H^{2m}_v(X)$, the set

$$V_\gamma = \{s \in \mathcal{U} : \gamma \in H^{m,m}(X_s)\}$$

is an analytic subvariety of $\mathcal{U}$. If no $V_\gamma = \mathcal{U}$ then $\cup_\gamma V_\gamma$ is nowhere dense, and in particular a generic $X \in \mathcal{U} - \cup_\gamma V_\gamma$. If $V_\gamma = \mathcal{U}$ then clearly $\gamma \in H^{m,m}_{\text{i.f.}}(X)$.

(39) This condition $\rho(X) = 1$ is equivalent to every line bundle or X being $\mathcal{O}_X(k)$ for same k, and it is well known that

$$H^0(\mathbb{P}^3, \mathcal{O}(k)) \to H^0(X, \mathcal{O}(k))$$

is surjective for every k.

(40) This is too näive on several counts. First the genus of C must be taken into account. Secondly, if C is a plane curve (say a line), then the intersection of X with plane is $C + C'$ where $\deg C' = d - k$.

(41) Here, "Res" denote the Grothendieck residue, given e.g. on page 66 of [4].

(42) For instance it will not always be the case that (2.b.2) holds. Moreover, even when (3.b.2) does hold, the period mapping $\varphi : \mathcal{W}_d^r \to \Gamma \backslash D$ may have degree larger than one. On the other hand, if e.g. $S \subset \mathbb{P}^r$ is a smooth surface and $C \in |\mathcal{O}_S(k)|$ is a smooth curve with $L = \mathcal{O}_C(1)$, then for k sufficiently large the pair (C, L) will be an exceptional special divisor ([2]) and one suspects that in general the $g_d^r (= |L|)$ will be unique.

(43) In the non-regular case we only consider that part of $\mathcal{W}_d^r$ corresponding to pairs (C, L) where $C \in |L^k|$ is smooth. If $T \subset T_{(C,L)}(\mathcal{W}_d^r)$ is the tangent space to this part of $\mathcal{W}_d^r$, then the argument for (3.b.4) may be refined to show that the differential

$$T \to T_C(\mathcal{M}_g)$$

is injective.

(44) We recall that for any situation

$$F: M \to N$$

where M, N are smooth and

$$f_*: T_p(M) \to T_{f(p)}(N)$$

is surjective, the *vertical space* $V_p(M/N) \subset T_p(M)$ is

$$V_p(M/N) = \{\xi \in T_p(M) : f_*(\xi) = 0\},$$

and the *horizontal space*

$$H_p(M/N) \subset \Lambda^* T_p^*(M)$$

is defined by

$$H_p(M/N) = \{\psi \in \Lambda^* T_p^*(M) : i(\xi)\psi = 0 \qquad \text{for all} \qquad \xi \in V_p(M/N)\}.$$

A form ψ on M is $f^*\eta$ for a (unique) form η on N if, and only if, both ψ and $d\psi$ are horizontal.

(45) The condition that $L = \mathcal{O}_Y(a, b)$ be sufficiently ample is satisfied when a, $b > 0$. This follows by proof analysis of theorem (3.a.7), the point being that (3.a.11) holds for such L.

(46) Since Aut($\mathbb{P}V \times \mathbb{P}W$) $\cong$ Aut $\mathbb{P}V \times$ Aut $\mathbb{P}W$, the vector fields on $\mathbb{P}V \times \mathbb{P}W$ are all of the form $\theta = \Sigma A^i_j v^j \partial/\partial v^i + \Sigma A^\alpha_\beta w^\beta \partial/\partial w^\alpha$. Any such vector field θ lifts to an action on $\mathcal{O}(a, b)$, and for $F \in H^0(\mathcal{O}(a, b))$

$$\theta F = \sum A^i_j v^j \partial F/\partial v^i + \sum A^\alpha_\beta w^\beta \partial F/\partial w^\alpha.$$

It follows that a general element of $J_{F(a,b)}$ is $\theta \cdot F$ for a suitable vector field θ as above. Thus, in the notation of our general discussion,

$$\begin{cases} S_1 = S_{(a,b)} \\ J_1 = J_{F(a,b)}. \end{cases}$$

(47) We note that this ideal has $(m + 1)+(n + 1)$ generators which are polynomials on $V \times W \cong \mathbb{C}^{(m+1)+(n+1)}$. Moreover the equations

$$\partial F/\partial v^i = 0 = \partial F/\partial w^\alpha$$

have only the origin as common zero.

PROOF: If these equations are satisfied for some (v, w), then by Euler's theorem $F(v, w) = 0$. If $v \neq 0$, $w \neq 0$ then we obtain a singular point of X. If $v = 0$, $w \neq 0$, then we have a singular point of X_W. Finally, if $v \neq 0$, $w = 0$ we have a singular point of X_V. In all cases we contradict our assumption.

(48) In general, we may describe $H_v^{m+n}(X)$ as the *biprimitive part* of the cohomology, defined by

$$H_v^{m+n}(X) = \{\psi \in H^{m+n}(X) : \omega_V \wedge \psi = 0 = \omega_W \wedge \psi\}.$$

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