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Correction to: “On isospectral deformations of riemannian metrics. II”


<http://www.numdam.org/item?id=CM_1983__50_1_93_0>
The proof of Lemma 3.3: (1) given in the paper in Vol. 47 [p. 201] is incorrect. We here give a complete proof of the lemma.

Define a differential operator $\delta_g^k : S_{k+1} \to S_k$ by

$$\left( \delta_g^k a \right)^{i_1 \cdots i_k} = - (k+1) \nabla^j a^{p_{i_1} \cdots i_k},$$

$\nabla$ being the covariant differentiation with respect to $g$. Then, $\delta_g^k$ is the formally adjoint operator of $\nabla_g^k$ with respect to the inner products in $S_k$'s naturally defined by $g$. Set

$$D_g^k = \frac{1}{k+1} \delta_g^k \nabla_g^k.$$

Then $D_g^k$ is a non-negative, self-adjoint, elliptic differential operator of order 2, and the equation $D_g^k a = 0$ is equivalent to $\nabla_g^k a = 0$ (see [2]).

Next, let us introduce various norms on the space of tensor fields on $M$. A (fixed) $C^\infty$ Riemannian metric $g_0$ naturally defines a norm, $| \cdot |$, on each fibre of the tensor bundle over $M$. Various global norms for a tensor field $T$ are defined by

$$|T|_k = \max_{0 \leq r \leq k} \sup_{x \in M} \left\{ \left| \nabla \cdots \nabla T(x) \right| \right\},$$

$$\|T\|_k^2 = \sum_{r=0}^k \left( \int_M \left| \nabla \cdots \nabla T \right|^2 dV_{g_0} \right),$$

for $k = 0, 1, 2, \ldots$, where $\nabla$ is the covariant differentiation with respect to $g_0$.

Using these notations, we have for every $a \in S_k$,

$$\|D_g^k a - D_{g_0}^k a\|_0 \leq C_1 |g - g_0|_1 \|a\|_2 \quad \text{(when } |g - g_0|_1 < 1),$$

$C_1$ being a constant, because $D_g^k$ is a second order differential operator
whose coefficients consist of $g$ and its first derivatives. On the other hand, since $D^k_{g_0}$ is an elliptic operator of order 2, there is a constant $C_2$ such that

$$\|a\|_2 \leq C_2 \left( \|a\|_0 + \|D^k_{g_0}a\|_0 \right),$$

for every $a \in S_k$.

Now we prove that $\mathcal{R}_k = \{ g \in \mathbb{R}_+ ; (D^k_{g_0})^{-1}(0) = \{0\} \}$ is an open subset of $\mathbb{R}$. Suppose $g_0$ belongs to $\mathcal{R}_k$. Noting that $D^k_{g}$ has a discrete spectrum consisting of non-negative real eigenvalues, we have

$$\|D^k_{g_0}a\|_0 \geq \lambda \|a\|_0 \quad (\lambda > 0),$$

for every $a(\neq 0) \in S_k$, where $\lambda$ is the least eigenvalue. We show $g_0$ is an interior point of $\mathcal{R}_k$. If the contrary holds, there are sequences $(g_n)_{n=1}^{\infty}$ in $\mathbb{R}$ and $(a_n)_{n=1}^{\infty}$ in $S_k$ such that $D^k_{g_n}a_n = 0$, $\|a_n\|_0 = 1$, and $g_n \rightarrow g_0$ with respect to the $C^\infty$ topology (i.e. $|g_n - g_0|_k \rightarrow 0$ for every $k \geq 0$) as $n \rightarrow \infty$. Using (1) and (2), we have

$$\|D^k_{g_0}a_n\|_0 = \|D^k_{g_n}a_n - D^k_{g_0}a_n\|_0 \leq C_1 |g_0 - g_n|_1 \|a_n\|_2$$

$$\leq C_1 C_2 |g_0 - g_n|_1 \left( \|a_n\|_0 + \|D^k_{g_0}a_n\|_0 \right)$$

$$= C_1 C_2 |g_0 - g_n|_1 \left( 1 + \|D^k_{g_0}a_n\|_0 \right).$$

Hence, for sufficiently large $n$,

$$\|D^k_{g_0}a_n\|_0 \leq \frac{C_1 C_2 |g_0 - g_n|_1}{1 - C_1 C_2 |g_0 - g_n|_1}.$$

Therefore, we get $\|D^k_{g_0}a_n\|_0 \rightarrow 0$ as $n \rightarrow \infty$. This contradicts (3).

References
