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PERIODS OF INTEGRALS FOR $SU(n, 1)$

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Introduction

In 1926 Hecke [5] constructed a certain family of holomorphic 1-forms on a modular curve which arise from grossencharacters of an imaginary quadratic field $K/\mathbb{Q}$. He then considered the periods of these 1-forms and showed [6] that, via the Mellin transform, these periods can be expressed as special values at CM-points of certain Eisenstein series of weight 1. Thus, by the theory of complex multiplication, all such periods are, up to a fixed transcendental factor, algebraic numbers lying in a class field over K.

In the present paper we will apply a very general principle in the theory of dual reductive pairs to obtain an extension of Hecke's results to arithmetic quotients of the complex n-ball. This principle is implicit in Hecke's original method and can be applied in many other cases.

We begin with a general picture.

Let $W,\langle,\rangle$ be a vector space over a field k of characteristic 0, with a non-degenerate alternating form; and let $\mathrm{Sp}(W)$ be the symplectic group of $W,\langle,\rangle$. Recall that a dual reductive pair (G, H) in $\mathrm{Sp}(W)$ consists of a pair of reductive subgroups G, H of $\mathrm{Sp}(W)$ such that H is the centralizer of G in $\mathrm{Sp}(W)$ and G is the centralizer of H in $\mathrm{Sp}(W)$.

DEFINITION: A *see-saw dual reductive pair* in $\mathrm{Sp}(W)$ is a *pair* of dual reductive pairs (G, H), (G', H') in $\mathrm{Sp}(W)$ such that $G \supset H'$ and $G' \supset H$. This terminology is suggested by the picture

$$\begin{tikzcd} G \arrow[d, dash] \arrow[dr, dash] & G' \arrow[d, dash] \arrow[dl, dash] \\ H' & H \end{tikzcd} .$$

Now suppose that $k = \mathbb{Q}$ and that the subgroups G, H, G', and H' are $\mathbb{Q}$-rational algebraic subgroups of $\mathrm{Sp}(W)$, viewed as an algebraic group over $\mathbb{Q}$. In this situation the oscillator representation gives rise to a

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correspondence between automorphic forms on the groups (G, H) (resp. (G', H')) in a dual reductive pair, or, more precisely, on certain coverings of these groups ([4], [7], [9]). This correspondence can be described in terms of integral kernels as follows:

Let $\mathrm{Mp}(W)(\mathbb{A})$ be the metaplectic group (two-fold cover of the adele group $\mathrm{Sp}(W)(\mathbb{A})$), and let $\tilde{G}(\mathbb{A})$, $\tilde{H}(\mathbb{A})$, $\tilde{G}'(\mathbb{A})$ and $\tilde{H}'(\mathbb{A})$ be the inverse images in $\mathrm{Mp}(W)(\mathbb{A})$ of the subgroups $G(\mathbb{A})$, $H(\mathbb{A})$, $G'(\mathbb{A})$ and $H'(\mathbb{A})$ of $\mathrm{Sp}(W)(\mathbb{A})$. For any complete polarization (decomposition into maximal isotropic subspaces)

$$W = W' + W'',$$

let $(R, L^2(W''(\mathbb{A})))$ be the associated Schrödinger model of the oscillator representation of $\mathrm{Mp}(W)(\mathbb{A})$. The corresponding theta-distribution on the Schwartz space $\mathcal{S}(W''(\mathbb{A}))$ is given by

$$\Theta(f) = \sum_{\xi \in W''(\mathbb{Q})} f(\xi)$$

for $f \in \mathcal{S}(W''(\mathbb{A}))$. Then for the dual reductive pair (G, H) and for $f \in \mathcal{S}(W''(\mathbb{A}))$ there is a theta-kernel

$$\theta(g, h; f) = \Theta(R(g)R(h)f)$$

where $(g, h) \in \tilde{G}(\mathbb{A}) \times \tilde{H}(\mathbb{A})$. Similarly for the dual reductive pair (G', H') there is a theta-kernel

$$\theta'(g', h'; f) = \overline{\Theta(R(g')R(h')f)}$$

where $(g', h') \in \tilde{G}(\mathbb{A}) \times \tilde{H}'(\mathbb{A})$. Note that these theta-kernels are just restriction to $\tilde{G}(\mathbb{A}) \times \tilde{H}(\mathbb{A})$ (resp. $\tilde{G}'(\mathbb{A}) \times \tilde{H}'(\mathbb{A})$) of the theta-function $\Theta(R(g)f)$ (or its conjugate) where $g \in \tilde{Sp}(W)(\mathbb{A})$. Now if φ is a cusp form on $\tilde{H}(\mathbb{A})$, we obtain an automorphic form $\mathcal{L}(\varphi) = \mathcal{L}(\varphi; f)$ on $\tilde{G}(\mathbb{A})$ by the "lifting" integral:

$$\begin{aligned} \mathcal{L}(\varphi)(g) &= \langle \varphi, \theta(g, \cdot; f) \rangle_H \\ &= \int_{H(\mathbb{Q})\backslash \tilde{H}(\mathbb{A})} \varphi(h) \overline{\theta(g, h; f)} \mathrm{d}h. \end{aligned}$$

Similarly, if φ' is a cusp form on $\tilde{H}'(\mathbb{A})$, we obtain an automorphic form $\mathcal{L}(\varphi') = \mathcal{L}(\varphi'; f)$ on $\tilde{G}'(\mathbb{A})$ by the "lifting" integral:

$$\begin{aligned} \mathcal{L}(\varphi')(g') &= \langle \varphi', \theta(g', \cdot; f) \rangle_{H'} \\ &= \int_{H'(\mathbb{Q})\backslash \tilde{H}'(\mathbb{A})} \varphi'(h') \overline{\theta'(g', h'; f)} \mathrm{d}h'. \end{aligned}$$

Such lifting integrals have been considered by many people; the description here is based on [7].

The see-saw dual reductive pair $\{(G, H), (G', H')\}$ gives rise to the following *fundamental adjointness formula:*

$$\langle \operatorname{Res} \mathcal{L}(\varphi'), \varphi \rangle_H = \langle \varphi', \operatorname{Res} \mathcal{L}(\varphi) \rangle_{H'} \tag{$*$}$$

where Res $\mathcal{L}(\varphi')$ (resp. Res $\mathcal{L}(\varphi)$) denotes the restriction of $\mathcal{L}(\varphi')$ (resp. $\mathcal{L}(\varphi)$) to $\tilde{H}(\mathbb{A})$ (resp. $\tilde{H}'(\mathbb{A})$). Explicitly:

$$\begin{aligned}
\langle \operatorname{Res} \mathcal{L}(\varphi'), \varphi \rangle_H &= \int_{H(\mathbb{Q})\backslash \tilde{H}(\mathbb{A})} \mathcal{L}(\varphi')(h) \overline{\varphi(h)} \mathrm{d}h \\
&= \int_{H(\mathbb{Q})\backslash \tilde{H}(\mathbb{A})} \int_{H'(\mathbb{Q})\backslash \tilde{H}'(\mathbb{A})} \varphi'(h') \overline{\theta'(h, h'; f)} \\
&\quad \times \overline{\varphi(h)} \mathrm{d}h' \mathrm{d}h \\
&= \int_{H'(\mathbb{Q})\backslash \tilde{H}'(\mathbb{A})} \overline{\mathcal{L}(\varphi)(h')} \varphi'(h') \mathrm{d}h' \\
&= \langle \varphi', \operatorname{Res} \mathcal{L}(\varphi) \rangle_{H'}.
\end{aligned}$$

While completely formal in nature, $(*)$ seems to have a number of important applications and actually gives rise to non-trivial identities.

In this paper we will not work adelically, but rather will reformulate $(*)$ in classical language. The particular see-saw pair of interest to us can be constructed as follows:

Let $K/\mathbb{Q}$ be an imaginary quadratic field, let σ be the Galois automorphism, and let $\delta \in K^\times$ be such that $\delta = -\delta^\sigma$. We view K as a subfield of $\mathbb{C}$ and assume that $\operatorname{Im}(\delta) > 0$. Let $V_j, (,)_j$ for $j = 0, 1$, be finite dimensional K vector spaces with non-degenerate σ-Hermitian forms. Let

$$V = V_0 \otimes_K V_1,$$

$$(,) = (,)_0 \otimes (,)_1,$$

and let

$$W = R_{K/\mathbb{Q}} V,$$

and

$$\langle , \rangle = \operatorname{Im}_\delta(,).$$

where $R_{K/\mathbb{Q}}$ denotes restriction of scalars from K to $\mathbb{Q}$ and $\operatorname{Im}_\delta(\alpha) =$

$(2\delta)^{-1}(\alpha - \alpha^{\sigma})$. Then, composing the natural inclusions $U(V_0) \times U(V_1) \to U(V)$ and $U(V) \to \mathrm{Sp}(W)$, we obtain a natural homomorphism

$$\rho\colon U(V_0) \times U(V_1) \to \mathrm{Sp}(W)$$

where $U(V_j)$ and $\mathrm{Sp}(W)$ are the unitary and symplectic groups of these spaces respectively. The image of ρ is a dual reductive pair in the sense of Howe [9]. Next let

$$W_j = R_{K/\mathbf{Q}} V_j$$

and

$$\langle,\rangle_j = \mathrm{Im}_{\delta}(,)_j.$$

Suppose that $U_1 \subset \mathrm{W}_1$ is a maximal isotropic subspace with respect to $\langle,\rangle_1$ such that $U_1 \cap \delta U_1 = \{0\}$. Then there are isomorphisms:

$$V_1 \simeq K \otimes_{\mathbf{Q}} U_1,$$

$$V \simeq V_0 \otimes_{\mathbf{Q}} U_1,$$

and

$$W \simeq W_0 \otimes_{\mathbf{Q}} U_1$$

with

$$\langle,\rangle = \langle,\rangle_0 \otimes (,)_1.$$

We again obtain a natural homomorphism

$$\rho'\colon \mathrm{Sp}(W_0) \times O(U_1) \to \mathrm{Sp}(W)$$

where $O(U_1)$ is the orthogonal group of $U_1,(,)_1|_{U_1}$. Again the image of ρ' is a dual reductive pair. Finally we obtain the following commutative diagram

$$\begin{tikzcd}
U(V_0) \times O(U_1) \arrow[r, "\iota"] \arrow[d, "\iota'"'] & U(V_0) \times U(V_1) \arrow[d, "\rho"] \\
\mathrm{Sp}(W_0) \times O(U_1) \arrow[r, "\rho'"'] & \mathrm{Sp}(W)
\end{tikzcd} \tag{0.1}$$

where ι and ι' are the natural inclusions. Thus we obtain a see-saw-dual reductive pair $\{(\mathrm{Sp}(W_0), O(U_1)), (U(V_1), U(V_0))\}$ as defined above.

Now suppose that the signature of the Hermitian spaces are

$$\mathrm{sig}(V_0, (,)_0) = (n, 0)$$

and

$$\mathrm{sig}(V_1, (,)_1) = (n, 1).$$

Then, extending scalars to $\mathbb{R}$ and choosing appropriate bases, we obtain the commutative diagram:

$$\begin{array}{ccc} U(n) \times O(n,1) & \xrightarrow{\iota} & U(n) \times U(n,1) \\ {\scriptstyle \iota'}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathrm{Sp}(n,\mathbb{R}) \times O(n,1) & \xrightarrow[\rho']{} & \mathrm{Sp}(n(n+1),\mathbb{R}), \end{array} \tag{0.2}$$

or, more graphically:

$$\begin{array}{ccc} \mathrm{Sp}(n,\mathbb{R}) & & U(n,1) \\ | & \times & | \\ U(n) & & O(n,1) \end{array} . \tag{0.3}$$

As explained in Section 1 below, there is a corresponding diagram of equivariant imbeddings of symmetric spaces:

$$\begin{array}{ccc} B & \xrightarrow{\kappa} & D \\ {\scriptstyle \kappa'}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ \mathfrak{H}_n \times B & \xrightarrow[\varepsilon']{} & \mathfrak{H}_{n(n+1)} \end{array} \tag{0.4}$$

where

$$B \simeq \{x \in \mathbb{R}^n \mid 1 - {}^t xx > 0\},$$

$$D \simeq \{z \in \mathbb{C}^n \mid 1 - {}^t\bar{z}z > 0\}$$

and $\mathfrak{H}_k$ is the Siegel space of genus k. Note that the embedding ε' will *not* be holomorphic and that $\kappa'(x) = (\tau_0, x)$ where $\tau_0 \in \mathfrak{H}_n$ is the isolated fixed point of $U(V_0) \subset \mathrm{Sp}(W_0) \simeq \mathrm{Sp}(n, \mathbb{R})$.

We now describe, in classical language, the special case of $(*)$ which yields information about the arithmetic nature of the periods of a certain class as holomorphic $(n, 0)$-forms on D.

Suppose that f is a Siegel modular form on $\mathfrak{H}_{n(n+1)}$ with respect to an arithmetic subgroup $\tilde{\Gamma} \subset \mathrm{Sp}(W)$, such that ε^*f determines a holomorphic

$(n, 0)$-form

$$\eta = f(\varepsilon(z))\mathrm{d}z \tag{0.5}$$

on D. Let

$$\Gamma = \{\gamma \in U(V_1) \mid \rho(\gamma) \in \tilde{\Gamma}\}$$

so that η determines a holomorphic $(n, 0)$-form on the quotient $\Gamma\backslash D$.

For each choice of $U_1 \subset W_1$ as above, let

$$\Gamma_{U_1} = \Gamma \cap O(U_1)$$

and observe that $\kappa = \kappa_{U_1}\colon B \to D$ gives rise to a (possibly non-compact) Lagrangian n-cycle

$$\kappa : \Gamma_{U_1}\backslash B \to \Gamma\backslash D. \tag{0.6}$$

We then consider the period

$$\mathcal{P}(\eta, U_1, \Gamma) = \int_{\Gamma_{U_1}\backslash B} \kappa^*\eta, \tag{0.7}$$

which we assume to be finite. The "see-saw" pair structure yields the following:

$$\begin{aligned} \mathcal{P} &= \int_{\Gamma_{U_1}\backslash B} f(\varepsilon \circ \kappa(x))\mathrm{d}x \\ &= \int_{\Gamma_{U_1}\backslash B} f(\varepsilon' \circ \kappa'(x))\mathrm{d}x \\ &= \int_{\Gamma_{U_1}\backslash B} f(\varepsilon'(\tau_0, x))\mathrm{d}x. \end{aligned}$$

On the other hand, for arbitrary $\tau \in \mathfrak{H}_n$, we define

$$\vartheta(\tau) = \vartheta(\tau, f, U_1, \Gamma) = \int_{\Gamma_{U_1}\backslash B} f(\varepsilon'(\tau, x))\mathrm{d}x \tag{0.8}$$

and obtain the special case of the identity $(*)$:

$$\mathcal{P}(\eta, U_1, \Gamma) = \vartheta(\tau_0, f, U_1, \Gamma) \tag{0.9}$$

which expresses the period as a special value of the function ϑ.

Now it will turn out that, for a suitable choice of f – actually a derivative of a reduced theta-function, see section 4 – the functions $\vartheta(\tau)$ are holomorphic Siegel modular forms with Fourier coefficients in K^{ab}, the maximal abelian extension of K! Therefore Shimura's theorem about special values of arithmetic Siegel modular functions at isolated fixed points applies to the function $p_0(\tau)^{-1}\vartheta(\tau)$ where $p_0(\tau)$ is any Siegel modular form of the same weight as ϑ, with cyclotomic Fourier coefficients and such that $p_0(\tau_0) \neq 0$. Thus we will find (Section 6, Theorem 6.4) that

$$p_0(\tau_0)^{-1}\mathcal{P}(\eta, U_1, \Gamma) \in K^{ab}, \tag{0.10}$$

that is, we obtain, via the "see-saw" pair, a rationality statement about the periods $\mathcal{P}(\eta, U_1, \Gamma)$ of η over a certain class of Lagrangian cycles. It should be noted that, in this special case, the automorphic forms φ and φ' of the general discussion above are essentially trivial, at least at the infinite place.

We now describe the contents of this paper in more detail. In Section 1 we give an intrinsic construction of the see-saw reductive pair described above and of the corresponding equivariant embeddings of symmetric spaces. In Section 2 we find explicit formulas for these embeddings and, in Section 3, we determine the relations between the various automorphy factors. In Section 4 we describe the pullbacks of certain derivatives of reduced theta-functions and thus obtain a family of holomorphic $(n, 0)$-forms on D generalizing Hecke's binary theta series of weight 2. Such forms had previously been constructed by G. Anderson [1]. In Section 5 we use the results of [11] and [12] to prove that the periods of one of our $(n, 0)$-forms is a special value at a CM-point of a holomorphic Siegel modular form with Fourier coefficients in K^{ab}. The main results here are Theorem 5.4 and Corollary 5.6, which follow from the particular case of $(*)$ given in Proposition 5.1. The key fact is that the function $\mathcal{P}'(\tau; \varphi, C_1; \Gamma)$ which occurs there is precisely the holomorphic Siegel modular for considered in [12] and [13]. These Siegel modular forms are generalizations to $SO(n, 1)$ of Hecke's binary theta-functions associated to real quadratic fields, and they are intimately connected with the results of [10]. It should be noted that the constructions of Sections 4 and 5 depend on the choice of an identification of the symmetric space for $\mathrm{Sp}(W)$ with the Siegel space of genus $n(n+1)$ – in short, on the choice of polarization – and that this, in turn, depends on the choice of Lagrangian cycle. In section 6 we show that the K^{ab}-span of the holomorphic $(n, 0)$-forms constructed in section 4 is actually independent of this choice, Corollary 6.3. This allows us to compare the periods of a fixed form over various Lagrangian cycles and gives our main rationality result, Theorem 6.4. Moreover, it follows that there is a natural K^{ab}-vector space $\mathfrak{H}(K^{ab})$ of holomorphic $(n, 0)$-forms associated to the dual pair $(U(V_0), U(V_1))$, and independent of other choices. In Section 7 we compute the Fourier-Jacobi expansion of a form $\eta \in \mathfrak{D}(K^{ab})$, and show that such an η is a cusp form, Corollary 7.5. This implies, in particular, that the period of η of any Lagrangian cycle, compact or not, is finite. We also show, Theorem 7.6, that, up to a uniform transcendental factor all forms $\eta \in \mathfrak{D}(K^{ab})$ are arithmetic in the sense of Shimura. Implicit in the proof of this result is another use of the see-saw pair principle. Finally, in Section 8, we observe that if $\eta \in \mathfrak{D}(K^{ab}) \otimes_{K^{ab}} \mathbb{C}$ is Γ-invariant for some $\Gamma \subset SU(V_1)$, then it extends to a holomorphic $(n, 0)$-form on a smooth compactification $\overline{\Gamma \backslash D}$ of $\Gamma \backslash D$. We then shown, Theorem 8.1, that for any Lagrangian cycle $B \subset D$ associated to a choice of U_0, U_1 etc. as above, there exists a $\Gamma \subset SU(V_1)$ and an $\eta \in \mathfrak{D}(\mathbb{C})$ such that

$$\mathcal{P}(\eta, U_1, \Gamma) \neq 0.$$

In particular every such B eventually becomes non-trivial in the homology of $\overline{\Gamma \backslash D}$ for sufficiently small Γ. This is analogous to a result of Wallach [18].

In the case $n = 1$ and for $V_1,(,)_1$ isotropic, the η's described above coincide with Hecke's family of holomorphic 1-forms. If we identify D with the upper half-plane and take U_1 so that $\kappa(B) = i\mathbb{R}_+^\times$, then the corresponding function ϑ is a holomorphic Eisenstein series of weight 1. Thus the see-saw pair gives, via (0.9), a structural explanation of the otherwise mysterious connection between the periods of binary theta-series of weight 2 and special values of Eisenstein series of weight 1, exploited by Hecke. In particular, we obtain (0.9) without an intervention of the Mellin transform. Moreover, if we choose U_1 so that $\kappa(B)$ is a hyperbolic arc associated to a real quadratic field $F/\mathbb{Q}$, then the corresponding function ϑ is a holomorphic theta-function of weight 1 for an indefinite binary quadratic form associated to F. Such functions were first introduced by Hecke [5], but he did not notice their connection with periods.

Because we do not rely on the Mellin transform we obtain a result about the periods of the analogues of Hecke's binary theta-series of weight 2 when $n = 1$ and $V_1,(,)_1$ is anisotropic. In this case $SU(V_1)$ is isomorphic to the group of elements of norm 1 in a division but indefinite quaternion algebra over $\mathbb{Q}$, and the periods over a hyperbolic arc associated to an embedded real quadratic field F will again be special values of theta-functions of weight 1 associated to F. We hope to return to this example elsewhere.

The method of this paper can be generalized in several ways. For example, let K be any CM-field, $|K:\mathbb{Q}| = 2\mathrm{m}$, let $\mathfrak{k} \subset K$ be the maximal real subfield of K, and let σ be the Galois automorphism of $K/\mathfrak{k}$. Let $V,(,)$ be a σ-Hermitian space over K such that

$$\operatorname{sig} V_\lambda,(,)_\lambda = \begin{cases} (n, 1) & 1 \leqslant \lambda \leqslant r \\ (n+1, 0) & \lambda > r \end{cases}$$

where $V_\lambda,(,)_\lambda$ is the λ^{th} completion of $V,(,)$ for $1 \leqslant \lambda \leqslant m$. Then there is a family of holomorphic $(nr, 0)$-forms on the quotient $\Gamma\backslash D^r$, where $\Gamma \subset SU(V,(,))$ is a suitable arithmetic subgroup, whose periods over a certain types of Lagrangian nr-cycle are expressible, via the see-saw pair construction, as special values of Hilbert-Siegel modular forms of weight $\frac{1}{2}(n+1)$ for $\mathfrak{k}$. Note that this family of examples includes compact quotients $\Gamma\backslash D$ for arbitrary n.

Finally, in his thesis G. Anderson [1] constructed differential forms yielding non-vanishing cohomology for a large class of compact quotients of bounded symmetric domains. It should be possible to apply the see-saw pair construction to obtain information about the arithmetic nature of the periods of these forms.

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§1. Symplectic embeddings: see-saw pairs

In this section we will give an intrinsic construction of a certain type of see-saw pair following Howe [8] and Satake [13].

1.1. Let $K/\mathbb{Q}$ be an imaginary quadratic field. Let σ be the Galois automorphism of $K/\mathbb{Q}$, and choose $\delta \in K^\times$ such that $\delta^\sigma = -\delta$. We view K as a subfield of $\mathbb{C}$ and assume $\mathrm{Im}(\delta) > 0$.

Let V_j, $(,)_j$, $j = 0, 1$, be finite dimensional K-vector spaces with non-degenerate σ-Hermitian forms, so that $(\alpha v, \alpha' v')_j = \alpha^\sigma (v, v')_j \alpha'$ for $\alpha, \alpha' \in K$, $v, v' \in V_j$. Let

$$V = V_0 \otimes_K V_1$$

$$(,) = (,)_0 \otimes (,)_1$$

and let

$$W = R_{K/\mathbb{Q}} V$$

and

$$\langle,\rangle = \mathrm{Im}_\delta(,).$$

Then if we let $G = U(V_1,(,)_1)$ and $H = U(V_0,(,)_0)$, we have a natural homomorphism of algebraic groups over $\mathbb{Q}$:

$$\rho : H \times G \to \mathrm{Sp}(W) \tag{1.0}$$

Also let

$$W_J = R_{K/\mathbb{Q}} V_J$$

$$\langle , \rangle_J = \mathrm{Im}_\delta (,)_J$$

and let

$$G' = \mathrm{Sp}(W_0, \langle , \rangle_0).$$

For a Hermitian space $V,(,)$ as above, let $\Omega(V) = \{U \subset W \mid U$ is maximal isotropic subspace of W for $\langle , \rangle\}$, and let

$$\Omega^+(V) = \{U \in \Omega(V) \mid U \cap \delta U = 0\}.$$

LEMMA 1.1: *Let $U \in \Omega(V)$. Then the following are equivalent:*
(*i*) $U \in \Omega^+(V)$,
(*ii*) $(,)|_U$ *is non-degenerate, symmetric and* $\mathbb{Q}$*-valued,*
(*iii*) *the pair of subspaces U, δU form a complete polarization of $W, \langle , \rangle$.*

PROOF: Immediate □

Now suppose that $U_1 \in \Omega^+(V_1)$. Then we have isomorphisms:

$$V_1 \simeq K \otimes_{\mathbb{Q}} U_1, \tag{1.1}$$

$$V \simeq V_0 \otimes_{\mathbb{Q}} U_1, \tag{1.2}$$

and

$$W \simeq W_0 \otimes_{\mathbb{Q}} U_1.$$

Let

$$H' = O\big(U_1, (,)_1|_{U_1}\big). \tag{1.3}$$

We then obtain a natural commutative diagram of homomorphisms of algebraic groups over $\mathbb{Q}$:

$$\begin{array}{ccc} H \times H' & \xrightarrow{1 \times \iota} & H \times G \\ {\scriptstyle \iota' \times 1}\downarrow & & \downarrow{\scriptstyle \rho} \\ G' \times H' & \xrightarrow[\rho']{} & \mathrm{Sp}(W) \end{array} . \tag{1.4}$$

The pairs ρ: $H \times G \to \mathrm{Sp}(W)$ and ρ': $G' \times H' \to \mathrm{Sp}(W)$ are both dual reductive pairs in the sense of Howe ([4], [9]), and so we have constructed a see-saw pair $\{(G, H), (G', H')\}$ as defined in the introduction.

1.2. We now want to describe the embeddings of symmetric spaces associated to a see-saw pair (1.4).

Assume that

$$\operatorname{sig}(V_0, (,)_0) = (p_0, q_0)$$

and

$$\operatorname{sig}(V_1, (,)_1) = (p_1, q_1),$$

so that

$$\operatorname{sig}(V, (,)) = (p_0 p_1 + q_0 q_1, p_0 q_1 + p_1 q_0).$$

Let

$$D_j = \{\ell \in Gr_{q_j}(V_j(\mathbb{R})) \mid \ (,)_j|_\ell < 0\}$$

and let

$$D = \{\ell \in Gr_q(V(\mathbb{R})) \mid \ (,)|_\ell < 0\}$$

where $q = p_0 q_1 + p_1 q_0$. Then there is a natural embedding

$$D_0 \times D_1 \to D$$

$$(\ell_0, \ell_1) \mapsto \ell_0^\perp \otimes \ell_1 + \ell_0 \otimes \ell_1^\perp \tag{1.5}$$

which is equivariant with respect to the homomorphism

$$H(\mathbb{R}) \times G(\mathbb{R}) \to U(V(\mathbb{R})).$$

Next, viewing $W(\mathbb{C})$ as the complexification of $W(\mathbb{R})$, we define a Hermitian form F on $W(\mathbb{C})$ by

$$F(w, w') = (2\delta)^{-1}\langle \bar{w}, w' \rangle \tag{1.6}$$

where $w \to \bar{w}$ denotes the complex conjugation on $W(\mathbb{C})$ and we have extended $\langle,\rangle$ to a $\mathbb{C}$-bilinear form on $W(\mathbb{C})$. The form F then has

$$\operatorname{sig}(W(\mathbb{C}), F) = (m, m)$$

where $2m = \dim_{\mathbb{R}} W(\mathbb{R}) = 2(p_0 + q_0)(p_1 + q_1)$. Let

$$D^* = \{\ell \in Gr_m(W(\mathbb{C})) \mid \ F|_\ell > 0 \quad \text{and} \quad \langle,\rangle|_\ell \equiv 0\},$$

so that D^* is an intrinsic realization of the symmetric space attached to $G^*(\mathbb{R})$ [13].

Let Δ be the endomorphism of W determined by multiplication by δ on V, and let $W(\mathbb{C})^{\pm}$ be the $\pm\delta$-eigenspaces of Δ in $W(\mathbb{C})$. Define isomorphisms:

$$\varphi^{\pm}: W(\mathbb{R}) \xrightarrow{\sim} W(\mathbb{C})^{\pm}$$

$$w \to \pm\delta w + \Delta w. \tag{1.7}$$

Then

$$\varphi^{\pm}(\Delta w) = \pm\delta\varphi^{\pm}(w),$$

and, if we view $\varphi^{\pm}$ as giving a $\mathbb{C}$-isomorphism (resp. anti-isomorphism) of $V(\mathbb{R})$ with $W(\mathbb{C})^{\pm}$, we have, for $v, v' \in V(\mathbb{R})$:

$$F(\varphi^{+}(v), \varphi^{+}(v')) = -(v, v')$$

and

$$F(\varphi^{-}(v), \varphi^{-}(v')) = \overline{(v, v')}.$$

Moreover,

$$\langle\varphi^{+}(v), \varphi^{+}(v')\rangle = \langle\varphi^{-}(v), \varphi^{-}(v')\rangle = 0,$$

for all v and v' in $V(\mathbb{R})$. Since $\varphi^{-}(v) = \overline{\varphi^{+}(v)}$ we also have:

$$\langle\varphi^{-}(v), \varphi^{+}(v')\rangle = -2\delta(v, v').$$

Thus, if $\ell \in D$ and $\ell^{\perp}$ is the orthogonal complement of ℓ in $V(\mathbb{R})$, we may define an embedding;

$$D \to D^*$$

$$\ell \mapsto \varphi^{+}(\ell) + \varphi^{-}(\ell^{\perp}). \tag{1.8}$$

Finally, we define

$$\varepsilon: D_0 \times D_1 \to D^*$$

$$(\ell_0, \ell_1) \mapsto \varphi^{+}(\ell_0 \otimes \ell_1^{\perp} + \ell_0^{\perp} \otimes \ell_1) + \varphi^{-}(\ell_0 \otimes \ell_1 + \ell_0^{\perp} \otimes \ell_1^{\perp}). \tag{1.9}$$

This embedding is obviously equivariant with respect to (1.0).

Now if $U_1 \in \Omega^+(V_1)$ as above, we have

$$\mathrm{sig}\big(U_1, (,)_1|_{U_1}\big) = (p_1, q_1).$$

Let

$$B = \{\ell \in Gr_{q_1}(U_1(\mathbb{R})) \mid (,)_1|_\ell < 0\},$$

so that B is a realization of the symmetric space associated to $H'(\mathbb{R})$. There is a natural embedding

$$B \to D_1$$

$$\ell \mapsto \ell(\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \ell \tag{1.10}$$

and so we obtain an embedding

$$\kappa : D_0 \times B \to D_0 \times D_1 \tag{1.11}$$

equivariant with respect to the homomorphism

$$1 \times \iota : H(\mathbb{R}) \times H'(\mathbb{R}) \to H(\mathbb{R}) \times G(\mathbb{R}).$$

If we repeat the construction of D^* and (1.8) with V_0 in place of V we obtain

$$D_0^* = \{\ell \in Gr_{m_0}(W_0(\mathbb{C})) \mid F_0|_\ell > 0 \quad \text{and} \quad \langle,\rangle_0|_\ell \equiv 0\}$$

and an embedding

$$\varepsilon_0 : D_0 \to D_0^*$$

$$\ell \mapsto \varphi_0^+(\ell) + \varphi_0^-(\ell^\perp), \tag{1.12}$$

and hence an embedding

$$\kappa' = \varepsilon_0 \times 1 : D_0 \times B \to D_0^* \times B \tag{1.13}$$

equivariant with respect to $\iota' \times 1$: $H(\mathbb{R}) \times H'(\mathbb{R}) \to G'(\mathbb{R}) \times H'(\mathbb{R})$. Note that, if $(p_0, q_0) = (n, 0)$, then D_0 reduces to a point and $\varepsilon_0(D_0)$ is an isolated fixed point of the maximal compact subgroup $\iota'(H(\mathbb{R}))$ in $G'(\mathbb{R})$.

Finally, via (1.3) we have

$$W(\mathbb{C}) \simeq W_0(\mathbb{C}) \otimes_{\mathbb{R}} U_1(\mathbb{R}),$$

$$\Delta = \Delta_0 \otimes 1,$$

and

$$\varphi^{\pm} = \varphi_0^{\pm} \otimes 1;$$

so that we obtain an embedding

$$\varepsilon' : D_0^* \times B \to D^*$$

$$(\ell_0, \ell) \mapsto \ell_0 \otimes \ell^{\perp} + \ell_0^{\perp} \otimes \ell \tag{1.14}$$

which is equivariant with respect to $\rho' : G'(\mathbb{R}) \times H'(\mathbb{R}) \to \mathrm{Sp}(W(\mathbb{R}))$.

Thus we have constructed a natural commutative diagram

$$\begin{array}{ccc} D_0 \times B & \xrightarrow{\kappa} & D_0 \times D_1 \\ {\scriptstyle \kappa'}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ D_0^* \times B & \xrightarrow[\varepsilon']{} & D^* \end{array} \tag{1.15}$$

of embeddings given by (1.9), (1.11), (1.13), and (1.14), and this diagram is equivariant with respect to the see-saw pair (1.4).

§2. Explicit formulas

In this section we will give more explicit expressions for the embeddings of (1.15) in the special case of interest to us. Specifically we assume that

$$\mathrm{sig}(V_0, (,)_0) = (n, 0)$$

and

$$\mathrm{sig}(V_1, (,)_1) = (p, q)$$

with $p \geqslant q > 0$. Later we will take $p = n$ and $q = 1$. As noted in §1, in this case D_0 reduces to a point and the embedding

$$\varepsilon : D_1 \to D^*$$

is holomorphic.

First recall that for any complete polarization

$$W(\mathbb{R}) = W' + W''$$

of the real symplectic space $W(\mathbb{R})$, $\langle,\rangle$ we obtain an unbounded model for D^* as follows. Choose a basis $w_1', \dots, w_m'$ for W' and let $w_1'', \dots, w_m''$ be the dual basis for W'' so that

$$\langle w_i', w_j'' \rangle = \delta_{ij}.$$

Then for the basis $w_1'', \ldots, w_m'', w_1', \ldots, w_m'$ for $W(\mathbb{R})$ we have

$$\langle \cdot \rangle \sim \begin{bmatrix} & -1_m \\ 1_m & \end{bmatrix}$$

and, if $\ell \in D^*$ is spanned by the columns of the matrix

$$\begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix},$$

then $\tau = \omega_2 \omega_1^{-1} \in \mathfrak{H}_m$; and we obtain an isomorphism

$$D^* \xrightarrow{\sim} \mathfrak{H}_m$$

$$\ell \mapsto \tau. \tag{2.1}$$

We now let $U_1 \in \Omega^+(V_1)$ be as in §1 and choose $U_0 \in \Omega^+(V_0)$. Then $U = U_0 \otimes_{\mathbb{Q}} U_1 \in \Omega^+(V)$ and, via Lemma 1.1, we obtain complete polarizations

$$W_0(\mathbb{R}) = U_0(\mathbb{R}) + \Delta_0 U_0(\mathbb{R}) \tag{2.2}$$

and

$$W(\mathbb{R}) = U(\mathbb{R}) + \Delta U(\mathbb{R}) \tag{2.3}$$

where Δ and Δ_0 are as in §1.

Choose $\mathbb{Q}$-bases $\{f_i\}$ for U_0 and $\{e_j\}$ for U_1 and let

$$Q_0 = \left((f_i, f_j)_0\right) \in M_n(\mathbb{Q})$$

and

$$Q_1 = \left((e_i, e_j)_1\right) \in M_{p+q}(\mathbb{Q})$$

be the corresponding matrices for $(,)_0$ and $(,)_1$. Then, with respect to the bases $\Delta_0 f_1, \ldots, \Delta_0 f_n$, $f_1, \ldots, f_n$ for W_0 and $\Delta(f_1 \otimes e_1), \ldots, \Delta(f_n \otimes e_1)$, $\Delta(f_1 \otimes e_2), \ldots, \Delta(f_n \otimes e_2), \ldots, \Delta(f_1 \otimes e_{p+q}), \ldots, \Delta(f_n \otimes e_{p+q})$, $f_1 \otimes e_1, \ldots, f_n \otimes e_1, \ldots, f_1 \otimes e_{p+q}, \ldots, f_n \otimes e_{p+q}$, for W we have the isomorphisms

$$W_0 \simeq \mathbb{Q}^n \times \mathbb{Q}^n \tag{2.4}$$

$$\begin{aligned} W &\simeq \mathbb{Q}^n \otimes \mathbb{Q}^{p+q} \times \mathbb{Q}^n \otimes \mathbb{Q}^{p+q} \\ &\simeq \mathbb{Q}^m \times \mathbb{Q}^m \end{aligned} \tag{2.5}$$

where $m = n(p+q)$. Also the forms $\langle,\rangle_0$ and $\langle,\rangle$ are given by

$$\begin{bmatrix} & -Q_0 \\ Q_0 & \end{bmatrix}$$

and

$$\begin{bmatrix} & -Q \\ Q & \end{bmatrix}$$

respectively, where $Q = Q_0 \otimes Q_1$. Define

$$\Lambda_0 \colon \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q}^n \times \mathbb{Q}^n$$

$$\begin{bmatrix} u'' \\ u' \end{bmatrix} \mapsto \begin{bmatrix} u'' \\ Q_0 u' \end{bmatrix}, \tag{2.6}$$

and

$$\Lambda \colon \mathbb{Q}^m \times \mathbb{Q}^m \to \mathbb{Q}^m \times \mathbb{Q}^m$$

$$\begin{bmatrix} u'' \\ u' \end{bmatrix} \mapsto \begin{bmatrix} u'' \\ Qu' \end{bmatrix}. \tag{2.7}$$

Then if $\ell \in D_0^*$ is spanned by the columns of $\omega \in M_{2n,n}(\mathbb{C})$ we write

$$\Lambda_0 \omega = \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$$

and let

$$\tau = \omega_2 \omega_1^{-1} \in \mathfrak{H}_n.$$

This gives an isomorphism

$$D_0^* \xrightarrow{\sim} \mathfrak{H}_n$$

$$\ell \mapsto \tau. \tag{2.8}$$

Similarly, if $\ell \in D^*$ is spanned by the columns of $\omega \in M_{2m,m}(\mathbb{C})$, we let

$$\Lambda \omega = \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$$

and we obtain an isomorphism:

$$D^* \xrightarrow{\sim} \mathfrak{H}_m$$

$$\ell \mapsto \tau = \omega_2 \omega_1^{-1}. \tag{2.9}$$

Note that we also obtain isomorphisms

$$G' \to \mathrm{Sp}(n, \mathbb{Q})$$

$$g \mapsto \Lambda_0 g \Lambda_0^{-1} = \tilde{g} \tag{2.10}$$

and

$$\mathrm{Sp}(W) \xrightarrow{\sim} \mathrm{Sp}(m, \mathbb{Q})$$

$$g \mapsto \Lambda g \Lambda^{-1} = \tilde{g}, \tag{2.11}$$

and the extension of these

$$G'(\mathbb{R}) \xrightarrow{\sim} \mathrm{Sp}(n, \mathbb{R})$$

and

$$\mathrm{Sp}(W(\mathbb{R})) \xrightarrow{\sim} \mathrm{Sp}(m, \mathbb{R}).$$

Next choose $T_0 \in GL_n^+(\mathbb{R})$ and $T_1 \in GL_{p+q}^+(\mathbb{R})$ such that

$${}^tT_0 Q_0 T_0 = 1_n \tag{2.12}$$

and

$${}^tT_1 Q_1 T_1 = \begin{bmatrix} & & \frac{1}{2}1_q \\ & 1_r & \\ \frac{1}{2}1_q & & \end{bmatrix} = I \tag{2.13}$$

where $r = p - q$. We then have isomorphisms

$$U_1(\mathbb{R}) \simeq \mathbb{R}^{p+q} \xrightarrow[\sim]{T_1^{-1}} \mathbb{R}^{p+q} \tag{2.12$'$}$$

and

$$H'(\mathbb{R}) \simeq O(Q_1) \xrightarrow{\sim} O(I)$$

$$g \mapsto T_1^{-1} g T_1 = \tilde{g}. \tag{2.13$'$}$$

If we let

$$\mathbb{B} = \left\{ X = (x_0, x_1) \in M_q(\mathbb{R}) \times M_{r,q}(\mathbb{R}) \mid \ \tfrac{1}{2}({}^t x_0 + x_0) - {}^t x_1 x_1 > 0 \right\}$$

we obtain an isomorphism

$$\mathbb{B} \xrightarrow{\sim} B$$

$$X \mapsto \operatorname{span} T_1 P'_-(X) = \ell \tag{2.14}$$

where

$$P'_-(X) = \begin{bmatrix} -x_0 \\ x_1 \\ 1_q \end{bmatrix}. \tag{2.15}$$

Note that the positive p-plane $\ell^\perp$ is spanned by the columns of $T_1 P'_+(X)$ where

$$P'_+(X) = \begin{bmatrix} -2\,{}^t x_1 & {}^t x_0 \\ 1_r & 0 \\ 0 & 1_q \end{bmatrix}. \tag{2.16}$$

Similarly, via (1.1) we have

$$V_1(\mathbb{R}) \simeq \mathbb{C}^{p+q} \overset{T_1^{-1}}{\xrightarrow{\sim}} \mathbb{C}^{p+q} \tag{2.17}$$

and

$$G(\mathbb{R}) \simeq U(Q_1) \to U(I)$$

$$g \mapsto T_1^{-1} g T_1. \tag{2.18}$$

If we let

$$\mathbb{D} = \left\{ z = (z_0, z_1) \in M_q(\mathbb{C}) \times M_{r,q}(\mathbb{C}) \mid (2\delta)^{-1}\left(z_0 - {}^t\bar{z}_0\right) - {}^t\bar{z}_1 z_1 > 0 \right\}$$

then we have an isomorphism

$$\mathbb{D} \xrightarrow{\sim} D_1$$

$$z \mapsto \operatorname{span} T_1 P_-(z) = \ell \tag{2.19}$$

where

$$P_{-}(z)=\begin{bmatrix} z_0 \\ z_1 \\ \delta^{-1}1_q \end{bmatrix} \tag{2.20}$$

Again the orthogonal complement $\ell^{\perp}$ is spanned by the columns of $T_1P_+(z)$ where

$$P_{+}(z)=\begin{bmatrix} 2{}^t\bar{z}_1 & {}^t\bar{z}_0 \\ \delta^{-1}\cdot 1_r & 0 \\ 0 & \delta^{-1}\cdot 1_q \end{bmatrix}. \tag{2.21}$$

Using the isomorphisms (2.8), (2.9), (2.14), and (2.19) we obtain a diagram of embeddings

$$\begin{tikzcd} \mathbb{B} \arrow[r, "\kappa"] \arrow[d, "\kappa'"'] & \mathbb{D} \arrow[d, "\varepsilon"] \\ \mathfrak{H}_n \times \mathbb{B} \arrow[r, hook, "\varepsilon'"'] & \mathfrak{H}_m \end{tikzcd} \tag{2.22}$$

from (1.15).

PROPOSITION 2.1. *Fix a choice of data* $\mathcal{C}=(U_0, T_0, U_1, T_1)$ *with* $U_0 \in \Omega^+(V_0)$, $U_1 \in \Omega^+(V_1)$ *and* T_0 (*resp.* T_1) *satisfying* (2.12) (*resp.* (2.13)), *as above. Then the embeddings of* (2.22) *are given as follows*:

(*i*) *If* $X=(x_0, x_1) \in \mathbb{B}$, *then*

$$\kappa(X)=\left(-\delta^{-1}x_0, \delta^{-1}x_1\right).$$

(*ii*) *If* $X \in \mathbb{B}$, *then*

$$\kappa'(X)=\left(-\delta^{-1}Q_0^{-1}, X\right).$$

(*iii*) *If* $z \in \mathbb{D}$, *then*

$$\varepsilon(z)=\omega_2(z)\,\omega_1(z)^{-1}$$

where

$$\omega_2(z)=T_0\otimes T_1P(z)\begin{bmatrix} 1_q & \\ & -1_p \end{bmatrix}$$

and

$$\omega_1(z) = Q_0 T_0 \otimes \delta Q_1 T_1 P(z)$$

with

$$P(z) = \left[P_-(z), -\overline{P_+(z)}\right].$$

and $P_\pm(z)$ *given by* (2.20) *or* (2.21).

(*iv*) *If* $\tau \in \mathfrak{H}_n$ *and* $X \in \mathbb{B}$,

$$\varepsilon'(\tau, X) = \omega_2'(\tau, X)\omega_1'(X)^{-1}$$

where

$$\omega_2'(\tau, X) = \tfrac{1}{2}(\tau + \bar{\tau}) \otimes T_1 P'(X) + \tfrac{1}{2}(\tau - \bar{\tau}) \otimes T_1 P'(X)\begin{bmatrix} -1_q & \\ & 1_p \end{bmatrix}$$

and

$$\omega_1'(X) = 1_n \otimes Q_1 T_1 P'(X)$$

with

$$P'(X) = \left[P'_-(X), P'_+(X)\right].$$

and $P'_\pm(X)$ *given by* (2.15) *or* (2.16).

PROOF: To prove (i) observe that for $X \in \mathbb{B}$ the corresponding $\ell \in B$ is the span of the columns of $T_1 P'_-(X)$ and hence $\kappa(\ell)$ is the $\mathbb{C}$-span of the same columns. Hence the corresponding point of $\mathbb{D}$ is $(-\delta^{-1}x_0, \delta^{-1}x_1)$ as claimed.

With respect to the basis for W_0 chosen above we have

$$\Delta_0 = \begin{bmatrix} & 1_n \\ \delta^2 \cdot 1_n & \end{bmatrix}$$

and so

$$\varphi_0^+ : V_0(\mathbb{R}) \to W_0(\mathbb{C})^+$$

is given by

$$\varphi_0^+(w) = \begin{bmatrix} w \\ \delta w \end{bmatrix}.$$

Therefore, since D_0 is a point,

$$\kappa'(X) = (\varepsilon_0(D_0), X)$$

and

$$\varepsilon_0(D_0) = \text{span of the columns of } \varphi_0^-(T_0).$$

Hence, according to (2.8),

$$\begin{aligned}\begin{bmatrix}\omega_2\\ \omega_1\end{bmatrix} &= \Lambda_0\begin{bmatrix}T_0\\ -T_0\end{bmatrix}\\ &= \begin{bmatrix}T_0\\ -\delta Q_0 T_0\end{bmatrix},\end{aligned}$$

and so $\varepsilon_0(D_0) = -\delta^{-1}Q_0^{-1}$ as claimed.

To prove (iii), observe that, with respect to the basis chosen above for W,

$$\Delta = \begin{bmatrix} & 1_m\\ \delta^2\cdot 1_m & \end{bmatrix}$$

and so, again, $\varphi^+\colon V(\mathbb{R}) \to W(\mathbb{C})^+$ is given by:

$$\varphi^+(w) = \begin{bmatrix}w\\ \delta w\end{bmatrix}.$$

Since the q-plane $\ell \in D_1$ corresponding to $z \in \mathbb{D}$ is spanned by the columns of $T_1P_-(z)$ and $\ell^\perp$ is spanned by the columns of $T_1P_+(z)$, we find that

$$\begin{aligned}\varepsilon(\ell) &= \text{span of the columns of}\\ &\quad\times\left[\varphi^+(T_0\otimes T_1P_-(z)), \varphi^-(T_0\otimes T_1P_+(z))\right].\end{aligned}$$

Then

$$\begin{aligned}&\operatorname{span}\left[\varphi^+(T_0\otimes T_1P_-(z)), \varphi^-(T_0\otimes T_1P_+(z))\right]\\ &\quad= \operatorname{span}\Lambda^{-1}\begin{bmatrix}T_0\otimes T_1P_-(z) & T_0\otimes T_1\overline{P_+(z)}\\ \delta Q_0T_0\otimes Q_1T_1P_-(z) & -\delta Q_0T_0\otimes Q_1T_1\overline{P_+(z)}\end{bmatrix}\\ &\quad= \operatorname{span}\Lambda^{-1}\begin{bmatrix}T_0\otimes T_1\left[P_-(z), \overline{P_+(z)}\right]\\ \delta Q_0T_0\otimes Q_1T_1\left[P_-(z), -\overline{P_+(z)}\right]\end{bmatrix}.\end{aligned}$$

Thus the point of $\mathfrak{H}_m$ corresponding to $\varepsilon(z)$ is given by

$$\varepsilon(z) = \omega_2\omega_1^{-1},$$

where

$$\omega_2 = \omega_2(z) = T_0 \otimes T_1\left[P_-(z), \overline{P_+(z)}\right]$$

and

$$\omega_1 = \omega_1(z) = Q_0T_0 \otimes \delta Q_1T_1\left[P_-(z),\ -\overline{P_+(z)}\right].$$

This proves (iii).

Finally, to obtain (iv), we observe that the point of D_0^* corresponding to $\tau \in \mathfrak{H}_n$ under (2.8) is

$$\operatorname{span} \Lambda_0^{-1}\begin{bmatrix} \tau \\ 1_n \end{bmatrix},$$

and that the orthogonal complement to this in $(W_0(\mathbf{C}), F_0)$ is

$$\operatorname{span} \Lambda_0^{-1}\begin{bmatrix} \bar{\tau} \\ 1_n \end{bmatrix}.$$

Then, by (1.14),

$$\begin{aligned} \varepsilon'(\tau, X) &= \text{span of columns of} \\ &\qquad \times\left[\Lambda_0^{-1}\begin{bmatrix} \bar{\tau} \\ 1_n \end{bmatrix} \otimes T_1P'_-(X),\ \Lambda_0^{-1}\begin{bmatrix} \tau \\ 1_n \end{bmatrix} \otimes T_1P_+(x)\right] \\ &= \operatorname{span}\begin{bmatrix} \bar{\tau} \otimes T_1P'_-(X) & \tau \otimes T_1P'_+(X) \\ Q_0^{-1} \otimes T_1P'_-(X) & Q_0^{-1} \otimes T_1P'_+(X) \end{bmatrix} \\ &= \operatorname{span} \Lambda^{-1}\begin{bmatrix} \bar{\tau} \otimes T_1P'_-(X) & \tau \otimes T_1P'_+(X) \\ 1_n \otimes Q_1T_1P'_-(X) & 1_n \otimes Q_1T_1P'_+(x) \end{bmatrix} \\ &= \operatorname{span} \Lambda^{-1}\begin{bmatrix} \frac{1}{2}(\tau + \bar{\tau}) \otimes T_1P'(X) + \frac{1}{2}(\tau - \bar{\tau}) \otimes T_1P''(X) \\ 1_n \otimes Q_1T_1P'(X) \end{bmatrix} \end{aligned}$$

where

$$P'(X) = \left[P'_-(X), P'_+(X)\right] \tag{2.23}$$

and

$$P''(X)=P'(X)\begin{bmatrix}-1_q & \\ & 1_p\end{bmatrix}.$$

This completes the proof. □

More explicit formulas may be obtained as follows: Let

$$^tP(z)IP(z)=\begin{bmatrix}A & \\ & -B\end{bmatrix} \tag{2.24}$$

and

$$^tP'(X)IP'(X)=\begin{bmatrix}-A' & \\ & B'\end{bmatrix}. \tag{2.25}$$

Since

$$P(\kappa(X))=\delta^{-1}P'(X), \tag{2.26}$$

we have

$$A'(X)=-\delta^{-2}A(\kappa(X)), \tag{2.27}$$

and

$$B'(X)=-\delta^{-2}B(\kappa(X)). \tag{2.28}$$

Also

$$A=(2\delta)^{-1}\left({}^tz_0+z_0\right)+{}^tz_1z_1 \tag{2.29}$$

and

$$B=\begin{bmatrix}-\delta^{-2}1_r & \delta^{-1}z_1\\ \delta^{-1}{}^tz_1 & (2\delta)^{-1}\left(z_0+{}^tz_0\right)\end{bmatrix}. \tag{2.30}$$

Moreover

$$\det B=|\delta|^{-2r}\det A. \tag{2.31}$$

PROPOSITION 2.2: *With the notation of Proposition* 2.1,

(i)

$$\varepsilon(z)=Q_0^{-1}\otimes\varepsilon_0(z)$$

where

$$\varepsilon_0(z) = \omega_2^0(z)\omega_1^0(z)^{-1} \in \mathfrak{H}_{p+q}$$

with

$$\omega_2^0(z) = T_1 P(z)\begin{bmatrix} 1_q & \\ & -1_p \end{bmatrix}$$

and

$$\omega_1^0(z) = \delta Q_1 T_1 P(z).$$

Thus

$$\varepsilon_0(z) = \delta^{-1} T_1 P(z)\begin{bmatrix} A & \\ & B \end{bmatrix}^{-1} {}^tP(z)\,{}^tT_1.$$

(*ii*)

$$\varepsilon'(\tau, X) = \tfrac{1}{2}(\tau + \bar{\tau}) \otimes Q_1^{-1} + \tfrac{1}{2}(\tau - \bar{\tau}) \otimes R_1$$

where

$$R_1 = T_1 P'(X)\begin{bmatrix} A' & \\ & B' \end{bmatrix}^{-1} {}^tP'(X)\,{}^tT_1.$$

Note that R_1 *is a majorant of*

$$Q_1^{-1} = T_1 P'(X)\begin{bmatrix} -A' & \\ & B' \end{bmatrix}^{-1} {}^tP'(X)\,{}^tT_1.$$

PROOF: To prove (i) observe that

$$\omega_2(z) = T_0 \otimes \omega_2^0(z),$$

$$\omega_1(z) = Q_0 T_0 \otimes \omega_1^0(z),$$

and

$${}^t\omega_2^0(z)\omega_1^0(z) = \delta\begin{bmatrix} A & \\ & B \end{bmatrix}.$$

Thus

$$\omega_1^0(z)^{-1} = \delta^{-1}\begin{bmatrix} A & \\ & B \end{bmatrix}^{-1} {}^t\omega_2^0(z),$$

and (i) follows.

Similarly, by (iv) of Proposition 2.1.,

$$\varepsilon'(\tau, X) = \tfrac{1}{2}(\tau + \bar{\tau}) \otimes Q_1^{-1} + \tfrac{1}{2}(\tau - \bar{\tau}) \otimes T_1 P'(X) \begin{bmatrix} -1_q & \\ & 1_p \end{bmatrix}$$
$$\times P'(X)^{-1} {}^tT_1^{-1} {}^tQ_1^{-1}.$$

Since

$${}^tP'(X) {}^tT_1 Q_1 T_1 P'(x) = \begin{bmatrix} -A' & \\ & B' \end{bmatrix},$$

we have

$$P'(x)^{-1} {}^tT_1^{-1} {}^tQ_1^{-1} = \begin{bmatrix} -A' & \\ & B' \end{bmatrix}^{-1} {}^tP'(X) {}^tT_1,$$

and (ii) follows.

3. Automorphy factors

In this section we determine the relations among automorphy factors which arise from (1.4) and (1.15).

First consider $\mathbb{D}$, which is an unbounded realization of the symmetric domain associated to $U(I)$. For $g \in U(I)$ and $z \in \mathbb{D}$, let

$$P(z) = [P_-(z), P_+(z)] \tag{3.1}$$

and write

$$gP(z) = P(gz) \begin{bmatrix} \mu(g, z) & \\ & \overline{\nu(g, z)} \end{bmatrix} \tag{3.2}$$

where $\mu(g, z) \in GL_q(\mathbb{C})$, and $\nu(g, z) \in GL_p(\mathbb{C})$. Explicitly, if

$$g = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

then

$$\mu(g, z) = \delta\big(a_3 z_0 + b_3 z_1 + c_3 \delta^{-1}\big) \tag{3.3}$$

and

$$\nu(g, z) = -\delta \begin{bmatrix} 2\bar{a}_2 {}^tz_1 - \bar{b}_2 \delta^{-1} & \bar{a}_2 {}^tz_0 - \bar{c}_2 \delta^{-1} \\ 2\bar{a}_3 {}^tz_1 - \bar{b}_3 \delta^{-1} & \bar{a}_3 {}^tz_0 - \bar{c}_3 \delta^{-1} \end{bmatrix}. \tag{3.4}$$

We note that, by [17, (1.19) and (1.20)],

$$\det(\nu(g,z)) = \det(g)^{-1}\det\mu(g,z) \tag{3.5}$$

and the Jacobian of the biholomorphic transformation $g\colon \mathbb{D}\to\mathbb{D}$ is given by

$$\operatorname{Jac}(g,z) = \det(g)^{q}\det\mu(g,z)^{-p-q} \tag{3.6}$$

Recall that if $h\in H(\mathbb{R})$ and $g\in G(\mathbb{R})$, then

$$\tilde{h} = T_0^{-1}hT_0 \in U(n)$$

and

$$\tilde{g} = T_1^{-1}gT_1 \in U(I).$$

Then for $g\in G(\mathbb{R})$ and $z\in\mathbb{D}$ we write

$$\mu(g,z) = \mu(\tilde{g},z)$$

and

$$\nu(g,z) = \nu(\tilde{g},z).$$

Similarly, if $h'\in H'(\mathbb{R})$, then $\tilde{h}' = T_1^{-1}h'T_1 \in O(I)$ and we write for $X\in\mathbb{B}$ and $P'(X) = [P'_-(X), P'_+(X)]$, as before,

$$\tilde{h}'P'(X) = P'(h'(X))\begin{bmatrix}\mu'(h',X) & \\ & \nu'(h',X)\end{bmatrix}. \tag{3.7}$$

For

$$g = \begin{bmatrix} a & b\\ c & d\end{bmatrix} \in \mathrm{Sp}(k,\mathbb{R}) \text{ and } \tau\in\mathfrak{H}_k,$$

we let

$$J(g,\tau) = c\tau + d$$

be the usual automorphy factor. If $g'\in G'(\mathbb{R})$ and $g\in\mathrm{Sp}(W(\mathbb{R}))$, then

$$\tilde{g}' = \Lambda_0 g'\Lambda_0^{-1} \in \mathrm{Sp}(n,\mathbb{R})$$

and

$$\tilde{g} = \Lambda g\Lambda^{-1} \in \mathrm{Sp}(m,\mathbb{R})$$

and we write

$$J(g',\tau_0)=J(\tilde{g}',\tau_0)$$

and

$$J(g,\tau)=J(\tilde{g},\tau)$$

for $\tau_0\in\mathfrak{H}_n$ and $\tau\in\mathfrak{H}_m$.

PROPOSITION 3.1: *Fix a choice of data $\mathcal{C}=(U_0, T_0, U_1, T_1)$ as in* §2. *Then*
(*i*) *For $(h,g)\in H(\mathbb{R})\times G(\mathbb{R})$ and $z\in\mathbb{D}$,*

$$J(\rho(h,g),\varepsilon(z))$$
$$=\omega_1(g_1(z))\begin{bmatrix}\tilde{h}'\otimes\mu(g,z) & \\ & {}^t(\tilde{h}')^{-1}\otimes\nu(g,z)\end{bmatrix}\omega_1(z)^{-1},$$

where $\omega_1(z)$ is as in (*iii*) *of Proposition* 2.1.
(*ii*) *For $(g',h')\in G'(\mathbb{R})\times H'(\mathbb{R})$ and for $(\tau,X)\in\mathfrak{H}_n\times\mathbb{B}$, we have*

$$J(\rho'(g',h'),\varepsilon'(\tau,X))$$
$$=\omega_1'(h'(X))\begin{bmatrix}J(g',\tau)\otimes\mu'(h',X) & \\ & J(g',\tau)\otimes\nu'(h',X)\end{bmatrix}$$
$$\times\omega_1'(X)^{-1}$$

where $\omega_1'(X)$ is given by (2.34).

PROOF. To prove (i) we compute

$$\tilde{\rho}(h,g)\begin{bmatrix}\varepsilon(z)\\ 1_m\end{bmatrix}=\Lambda\rho(h,g)\Lambda^{-1}\begin{bmatrix}\omega_2(z)\\ \omega_1(z)\end{bmatrix}\omega_1(z)^{-1}$$

$$=\Lambda\rho(h,g)\left[\varphi^+(T_0\otimes T_1P_-(z)),\varphi^-(T_0\otimes T_1P_+(z))\right]\omega_1(z)^{-1}$$

$$=\Lambda\left[\varphi^+(hT_0\otimes gT_1P_-(z)),\varphi^-(hT_0\otimes gT_1P_+(z))\right]\omega_1(z)^{-1}$$

$$=\Lambda\left[\varphi^+(T_0\tilde{h}\otimes T_1\tilde{g}P_-(z)),\varphi^-(T_0\tilde{h}\otimes T_1\tilde{g}P_+(z))\right]\omega_1(z)^{-1}$$

$$=\Lambda\left[\varphi^+(T_0\otimes T_1P_-(g(z))),\varphi^-(T_0\otimes T_1P_+(g(z)))\right]\omega_1(g(z))^{-1}$$

$$\times \omega_1(g(z)) \begin{bmatrix} \tilde{h} \otimes \mu(g,z) & \\ & {}^t\tilde{h}^{-1} \otimes \nu(g,z) \end{bmatrix} \omega_1(z)^{-1}$$

$$= \begin{bmatrix} \varepsilon(g(z)) \\ 1_m \end{bmatrix} J(\rho(h,g), \varepsilon(z))$$

which gives the expression claimed in (i).

To prove (ii) we compute, similarly,

$$\tilde{\rho}'(g',h') \begin{bmatrix} \varepsilon'(\tau, X) \\ 1_m \end{bmatrix} = \tilde{\rho}'(g,h) \begin{bmatrix} \omega_2'(\tau, X) \\ \omega_1'(X) \end{bmatrix} \omega_1'(X)^{-1}$$

$$= \Lambda \rho'(g',h') \left[\Lambda_0^{-1} \begin{bmatrix} \bar{\tau} \\ 1_n \end{bmatrix} \otimes T_1 P'_-(X), \Lambda_0^{-1} \begin{bmatrix} \tau \\ 1_n \end{bmatrix} \otimes T_1 P'_+(X) \right] \omega_1'(X)^{-1}$$

$$= \Lambda \left[g' \Lambda_0^{-1} \begin{bmatrix} \bar{\tau} \\ 1_n \end{bmatrix} \otimes h' T_1 P'_-(X), g' \Lambda_0^{-1} \begin{bmatrix} \tau \\ 1_n \end{bmatrix} \otimes h' T_1 P'_+(X) \right] \omega_1'(X)^{-1}$$

$$= \Lambda \left[\Lambda_0^{-1} \begin{bmatrix} \overline{g'(\tau)} \\ 1_n \end{bmatrix} \otimes T_1 P'_-(h'(X)), \right.$$

$$\left. \Lambda_0^{-1} \begin{bmatrix} g'(\tau) \\ 1_n \end{bmatrix} \otimes T_1 P'_+(h'(X)) \right] \omega_1'(h'(X))^{-1}$$

$$\times \omega_1'(h'(X)) \begin{bmatrix} \overline{J(g',\tau)} \otimes \mu(h',X) & \\ & J(g',\tau) \otimes \nu(h',X) \end{bmatrix} \omega_1'(X)^{-1}$$

$$= \begin{bmatrix} \varepsilon'(g'(\tau), h'(X)) \\ 1_m \end{bmatrix} J(\rho'(g',h'), \varepsilon'(\tau, X))$$

This gives (ii). □

Corollary 3.2: *With the notation as above,*

$$(i) \quad \det J(\rho(h,g), \varepsilon(z)) = \det \omega_1(g(z)) \det \omega_1(z)^{-1} (\det h)^{q-p}$$
$$\times \det \mu(g,z)^{2n} \det g^{-p}$$

and

$$(ii) \quad \det J(\rho'(g',h'), \varepsilon'(\tau, X)) = \det \omega_1'(h'(X)) \det \omega_1'(X)^{-1}$$
$$\times \det J(g',\tau)^p \det J(g',\bar{\tau})^q$$
$$\times \det \mu'(h',X)^{2n} \det(h')^{-p}$$

Proof. These are immediate consequences of Proposition 3.1 and (3.5). We also observe that, by (2.30),

$$\mu(\iota(h'), \kappa(X)) = \mu'(h', X)$$
$$\nu(\iota(h'), \kappa(X)) = \nu'(h', X) \tag{3.8}$$

where ι: $H'(\mathbb{R}) \to G(\mathbb{R})$ is the natural inclusion. □

It will be useful later to know the following:

Proposition 3.3: *For $z \in \mathbb{D}$ and for $\omega_1(z)$ given in (ii) of Proposition 2.1,*

$$\det \omega_1(z) = (\det Q_0)^{(p+q)/2} |\det Q_1|^{n/2} |\delta|^{2qn} \det A^n$$

where $A = A(z)$ is given by (2.25).

Proof: We have

$$\omega_1(z) = Q_0 T_0 \otimes \omega_1^0(z)$$

with

$$\omega_1^0(z) = \delta Q_1 T_1 \left[P_-(z), -\overline{P_+(z)} \right].$$

Let

$$(*) = \left[P_-(z), -\overline{P_+(z)} \right].$$

Then

$$\det \omega_1(z) = (\det Q_0 T_0)^{p+q} \left(\delta^{p+q} \det Q_1 T_1 \det(*) \right)^n.$$

On the other hand

$${}^t(*) I(*) = \begin{bmatrix} A & \\ & -B \end{bmatrix}$$

where A and B are given by (2.25) and (2.26). Thus

$$\det(*)^2 = (-4)^q \det A (-1)^p \det B$$
$$= 4^q \delta^{-2r} \det A^2$$

using (2.29). Hence

$$\det(*) = 2^q \delta^{-r} \det A$$

where the sign is determined by evaluating both sides at $z = (-\delta^{-1} \cdot 1_q, 0)$. Then we obtain the required expression for det $\omega_1(z)$ by observing that

$$\det Q_0 T_0 = \det Q_0^{1/2}$$

$$\det Q_1 T_1 = (-1)^q 2^{-q} |\det Q_1|^{1/2}.$$

Note that, since $\omega_1(z)$ is invertible we must have

$$\det A(z) \neq 0$$

for $z \in \mathbb{D}$. $\square$

§4. Pullbacks and derivatives of theta-functions

We now specialize to the main case of interest for us, and so assume

$$\operatorname{sig} V_0, (,)_0 = (n, 0)$$

and

$$\operatorname{sig} V_1, (,)_1 = (n, 1).$$

In this section we construct certain holomorphic $(n, 0)$-forms on $\mathbb{D}$ which will be invariant under arithmetic subgroups of $G^1(\mathbb{Q})$ where $G^1(\mathbb{Q}) = \{g \in G(\mathbb{Q}) \mid \det g = 1\}$. Such forms were previously constructed by G. Anderson [1] in a much more general setting. We will then describe the restriction of these forms to $\mathbb{B}$.

4.1. We fix a choice of data $\mathcal{C} = (U_0, T_0, U_1, T_1)$ as in §2 and §3 and hence we obtain isomorphisms

$$U_0(\mathbb{Q}) \otimes U_1(\mathbb{Q}) \simeq \mathbb{Q}^n \otimes \mathbb{Q}^{n+1} \simeq M_{n+1,n}(\mathbb{Q}), \tag{4.1}$$

$$V(\mathbb{R}) \simeq M_{n+1,n}(\mathbb{C}) \tag{4.2}$$

and, via (2.5),

$$W(\mathbb{Q}) \simeq M_{n+1,n}(Q) \times M_{n+1,n}(\mathbb{Q}). \tag{4.3}$$

Note that if $A \otimes B$ is an endomorphism of $U_0(\mathbb{Q}) \otimes U_1(\mathbb{Q})$, then

$$Y \mapsto BY{}^tA$$

is the corresponding endomorphism of $M_{n+1,n}(\mathbb{Q})$.

Next we define a certain linear n-th order differential operator which will play a crucial role. For convenience write $\mathbf{M} = M_{n+1,n}(\mathbb{C})$. Then for $v, w \in \mathbf{M}$ let

$$\begin{aligned} \nabla(w) &= (2\pi i)^{-n} \det\left({}^t w \frac{\partial}{\partial v}\right) \\ &= (2\pi i)^{-n} \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \prod_j {}^t w^j \frac{\partial}{\partial v^{\sigma(j)}} \end{aligned} \tag{4.4}$$

where $w = (w^1, \dots, w^n)$ and $v = (v^1, \dots, v^n)$, $v^i, w^j \in \mathbb{C}^{n+1}$. The following lemma is then easily checked.

LEMMA 4.1: *For $f \in C^\infty(\mathbf{M})$ and for $A, C \in M_n(\mathbb{C})$ and $B, D \in M_{n+1}(\mathbb{C})$, let*

$$g(v) = f(Bv^tA + Dv^tC).$$

Assume that $Dv = 0$. Then

$$(\nabla(w)g)(v) = \det A(\nabla(Bw)f)(Bv^tA + Dv^tC).$$

Now for a smooth function

$$\varphi: \mathbb{C}^m \times \mathfrak{H}_m \to \mathbb{C}$$

we define, for $v \in \mathbf{M}$ and $z \in \mathbb{D}$,

$$f(\varphi, \mathcal{C})(v, z) = \det \omega_1(z)^{-1/2} \varphi(\check{\omega}_1(z)v, \varepsilon(z)) \tag{4.5}$$

where we identify $\mathbb{M} \simeq \mathbb{C}^m$ via (4.2) and where $\check{\omega}_1(z) = {}^t\omega_1(z)^{-1}$ with $\omega_1(z)$ given by Proposition 2.1. Next let

$$\Phi(\varphi, \mathcal{C})(z) = c(\mathcal{C})(\nabla(w)f(\varphi, \mathcal{C}))(0, z), \tag{4.6}$$

where

$$c(\mathcal{C}) = (\det Q_0)^{(n+1)/4} |\det Q_1|^{n/4}$$

and

$$w = \begin{bmatrix} 0 \\ 1_n \end{bmatrix} \in \mathbb{M}. \tag{4.8}$$

The reason for the normalizing factor $c(\mathcal{C})$ will emerge later. Note that since

$$\omega_1(z) = Q_0 T_0 \otimes \omega_1^0(z),$$

Lemma 4.1 implies that

$$\Phi(\varphi, \mathcal{C})(z) = c(\mathcal{C}) \det \omega_1(z)^{-1/2}$$
$$\times \det Q_0^{-1/2}\left(\nabla\left(\check{\omega}_1^0(z) w\right)\varphi\right)(0, \varepsilon(z)).$$

For any smooth function φ: $\mathbb{C}^m \times \mathfrak{H}_m \to \mathbb{C}$ and for $g \in \mathrm{Sp}(m, \mathbb{R})$ let

$$(\varphi \mid g)(v, \tau) = \det J(g, \tau)^{-1/2} \varphi\left(\check{J}(g, \tau) v, g(\tau)\right) \tag{4.10}$$

where we choose a continuous branch of $\det J(g, \tau)^{1/2}$ on $\mathfrak{H}_m$. Also for a smooth function ψ: $\mathbb{D} \to \mathbb{C}$ and for $g \in U(I)(\mathbb{R})$, we let

$$(\psi | g)(z) = \mu(g, z)^{-(n+1)} \psi(g(\tau)). \tag{4.11}$$

Proposition 4.2: *For $g \in G(\mathbb{R})$ and for $\Phi(\varphi, \mathcal{C})$ defined by* (4.6),

$$\Phi(\varphi, \mathcal{C}) \mid \tilde{g} = \varepsilon(g) \det g^{-1-1/2n} \Phi(\varphi \mid \tilde{\rho}(g), \mathcal{C})$$

where $\tilde{g} = T_1^{-1} g T_1$ and $\tilde{\rho}(g) = \Lambda \rho(g) \Lambda^{-1}$ as in §2, *and where the sign*

$$\varepsilon(g) = (\det g)^{-n/2} \mu(g, z)^n \det \omega_1(g(z))^{1/2} \det \omega_1(z)^{-1/2}$$
$$\times \det J(\rho(g), \varepsilon(z))^{-1/2}$$

depends on the choice of branches of the square roots.

Proof: First let

$$j(g, z) = \begin{bmatrix} \mu(g, z) & \\ & \nu(g, z) \end{bmatrix}$$

so that

$$J(\rho(g), \varepsilon(z)) = \omega_1(g(z))(1 \otimes j(g, z)) \omega_1(z)^{-1}.$$

Then

$$f(\varphi | \tilde{\rho}(g), \mathcal{C})(v, z) = \det \omega_1(z)^{-1/2} (\varphi | \tilde{\rho}(g))(\check{\omega}_1(z) v, \varepsilon(z))$$
$$= \det \omega_1(z)^{-1/2} \det J(\tilde{\rho}(g), \varepsilon(z))^{-1/2}$$

$$\times \varphi\big(\check{J}(\rho(g), \varepsilon(z))\check{\omega}_1(z)v, \rho(g)\varepsilon(z)\big)$$

$$= \varepsilon(g)\mu(g, z)^{-n} \det g^{n/2} \det \omega_1(g(z))^{-1/2}$$

$$\times \varphi\big(\check{\omega}_1(g(z))(1 \otimes j(g, z))^{\vee} v, \varepsilon(g(z))\big)$$

$$= \varepsilon(g) \det g^{n/2}\mu(g, z)^{-n} f(\varphi, \mathcal{C})\big(j(g, z)^{\vee} v,$$

$$\times \varepsilon(g(z))\big).$$

Then applying $\nabla(w)$ to both sides and invoking Lemma 4.1 we have

$$\Phi(\varphi \mid \tilde{\rho}(g), \mathcal{C})(z) = c(\mathcal{C})\varepsilon(g) \det g^{n/2}\mu(g, z)^{-n}$$

$$\times \big(\nabla(j(g, z)^{\vee} w) f(\varphi, \mathcal{C})\big)(0, g(z)).$$

But now since

$$j(g, z)^{\vee}\begin{bmatrix} 0 \\ 1_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1_n \end{bmatrix}\nu(g, z)^{\vee}$$

and

$$\nabla(wA) = \det A \nabla(w)$$

we obtain, via (3.5),

$$\Phi(\varphi \mid \tilde{\rho}(g), \mathcal{C})(z) = \varepsilon(g) \det g^{n/2+1}\mu(g, z)^{-n-1}\Phi(\varphi, \mathcal{C})(0, g(z))$$

as claimed. □

We now apply the above construction to reduced theta functions. For $\tau \in \mathfrak{H}_m$, $v \in \mathbf{C}^m$, r, $s \in \mathbf{Q}^m$, and for any lattice $L \subset \mathbf{Q}^m$, let

$$\theta(v, \tau, r, s, L) = \sum_{y \in r+L} e\big(\tfrac{1}{2}{}^t y\tau y + {}^t y(v+s)\big) \tag{4.12}$$

be the classical theta function and let

$$\varphi(v, \tau, r, s, L) = e\big(\tfrac{1}{2}{}^t v(\tau - \bar{\tau})^{-1} v\big)\theta(v, \tau, r, s, L) \tag{4.13}$$

be the corresponding reduced theta function [16]. There then exists an $M \in \mathbf{Z} > 0$ such that for all $\gamma \in \mathrm{Sp}(m, \mathbf{Z})$ with $\gamma \equiv 1_{2m}(M)$,

$$(\varphi|\gamma)(v, \tau) = \lambda(\gamma)\varphi(v, \tau) \tag{4.14}$$

where $\lambda(\gamma)$ is a fourth root of unity depending only on γ and on the choice of branch of $\det J(\gamma, \tau)^{1/2}$.

COROLLARY 4.3: *If* φ *is given by* (4.13) *then* $\Phi(\varphi, \mathcal{C})$ *is a holomorphic function on* $\mathbb{D}$. *Moreover, if*

$$\Gamma_1^* = \{g \in G^1(\mathbb{Q}) \mid \tilde{\rho}(g) \in \Gamma(M)\}$$

with $\Gamma(M) = \{\gamma \in \mathrm{Sp}(M, \mathbb{Z}) \mid \gamma \equiv 1_{2m}(M)\}$, *then* $\forall g \in \Gamma_1^*$,

$$\Phi(\tau, \mathcal{C}) \mid \tilde{g} = \chi(g)\Phi(\varphi, \mathcal{C}).$$

where

$$\chi(g) = \varepsilon(g)\lambda(\tilde{\rho}(g))$$

is a character of finite order of Γ_1^*. *If*

$$\Gamma_1 = \ker \chi,$$

then

$$\Phi(\varphi, \mathcal{C})(z)\,dz$$

defines a Γ_1*-invariant holomorphic* $(n, 0)$*-form on* $\mathbb{D}$.

PROOF: The only point to be checked is the holomorphy of $\Phi(\varphi, \mathcal{C})$, but it easily checked that, for $w \in \mathbb{M}$ and for φ given by (4.13)

$$(\nabla(w)\varphi)(0) = (\nabla(w)\theta)(0) \tag{4.15}$$

so that holomorphy follows from (4.9). □

4.2. We next want to consider the analogous pullbacks with respect to ε'.

For $\tau \in \mathfrak{H}_n$, $X \in \mathbb{B}$ and $v \in \mathbb{M}$ and for an arbitrary smooth function $\varphi : \mathbb{C}^m \times \mathfrak{H}_m \to \mathbb{C}$, let

$$f'(\varphi, \mathcal{C})(v; \tau, X) = \det \omega_1'(X)^{-1/2}\varphi(\check{\omega}_1'(X)v, \varepsilon'(\tau, X)) \tag{4.16}$$

and let

$$\Phi'(\varphi, \mathcal{C})(\tau, X) = c'(\mathcal{C}_1)\det(\mathrm{Im}_\delta\tau)^{1/2}(\nabla(w)f'(\varphi, \mathcal{C}))(0; \tau, X) \tag{4.17}$$

where

$$c'(\mathcal{C}_1) = |\det Q_1|^{n/4}. \tag{4.18}$$

Again the reason for the normalizing factors will become clear later.

PROPOSITION 4.4. *If* $h \in H'(\mathbb{R})$, *with* $\det h = 1$, *and* $g' \in G'(\mathbb{R})$, *then for a suitable choice of* $\det J(g', \tau)^{1/2}$,

$$\begin{aligned}&\Phi'(\varphi \mid \tilde{\rho}'(g', h'), \mathcal{C})(\tau, X)\\&\quad = \det J(g', \tau)^{-(n+1)/2} \mu'(h', X)^{-(n+1)} \Phi'(\varphi, \mathcal{C})(g'(\tau), h'(X)).\end{aligned}$$

PROOF: Let

$$j'(g', h'; \tau, X) = \begin{bmatrix} {}^{t}J(g', \bar{\tau}) \otimes \mu'(h', X) & \\ & J(g', \tau) \otimes \nu'(h', X) \end{bmatrix}$$

so that

$$J(\rho'(g', h'), \varepsilon'(\tau, X)) = \omega_1'(h'(X)) j'(g', h'; \tau, X) \omega_1'(X)^{-1}.$$

Then

$$\begin{aligned}&f'(\varphi | \tilde{\rho}'(g', h'), \mathcal{C})(v; \tau, X)\\&\quad = \det \omega_1'(X)^{-1/2} \det J(\rho'(g', h'), \varepsilon'(\tau, X))^{-1/2}\\&\qquad \times \varphi(\check{J}(\rho' g', h'), \varepsilon'(\tau, X)) \check{\omega}_1'(X) v, \rho'(g', h')\varepsilon'(\tau, X))\\&\quad = \det J(g', \tau)^{-(n-1)/2} |\det J(g', \tau)|^{-1} \mu'(h', X)^{-n}\\&\qquad \times \det \omega_1'(h'(X))^{-1/2}\\&\qquad \times \varphi(\check{\omega}_1'(h'(X)) j(g', h'; \tau, X)^{\vee} v, \varepsilon'(g'(\tau), h'(X)))\\&\quad = \det J(g', \tau)^{-(n-1)/2} |\det J(g', \tau)|^{-1} \mu'(h', X)^{-n} f'(\varphi, \mathcal{C})\\&\qquad \times (v^*; g'(\tau), h'(X))\end{aligned}$$

where

$$v^* = \begin{bmatrix} \mu'(h, X) & \\ & 0 \end{bmatrix}^{\vee} v J(g, \bar{\tau})^{-1} + \begin{bmatrix} 0 & \\ & \nu'(h, X) \end{bmatrix}^{\vee} v J(g, \tau)^{-1}.$$

Here we note that

$$\det \omega_1'(X) = |\det Q_1|^{n/2}(A')^n \tag{4.19}$$

and

$$A' = x_0 - {}^t x_1 x_1 > 0, \tag{4.20}$$

so that the only ambiguity in the choice of roots may be absorbed by a suitable choice for $\det J(g', \tau)^{1/2}$. Now if we apply $\nabla(w)$ to the identity above and use Lemma 4.1, we obtain the desired result. Note that we use the fact that

$$\begin{bmatrix} 0 & \\ & \nu'(h', X) \end{bmatrix}^{\vee} \begin{bmatrix} 0 \\ 1_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1_n \end{bmatrix} \nu'(h', X)^{\vee},$$

as well as (3.8) and (3.5). $\square$

COROLLARY 4.5: *If* φ *is the reduced theta-function given by* (4.13), *and if*

$$\Gamma'^{*} = \{ g \in G'(\mathbb{Q}) \mid \tilde{\rho}'(g) \in \Gamma(M) \}$$

then $\forall g \in \Gamma'^{*}$,

$$\Phi'(\varphi, \mathcal{C})(g(\tau), X) = \lambda(\tilde{\rho}'(g)) \det J(g, \tau)^{(n+1)/2} \Phi'(\varphi, \mathcal{C})(\tau, X).$$

4.3. We now want to compare the pullbacks of $\Phi(\varphi, \mathcal{C})$ and $\Phi'(\varphi, \mathcal{C})$ to $\mathbb{B}$ via κ and κ'. The relation built into our see-saw pair and choice of normalizations is given by:

PROPOSITION 4.6:

$$\Phi(\varphi, \mathcal{C})(\kappa(X)) = |\delta|^n \Phi'(\varphi, \mathcal{C})(\kappa'(X)).$$

PROOF. Let

$$\omega_1'^0(X) = Q_1 T_1 P'(X) \tag{4.21}$$

so that

$$\omega_1'(X) = 1_n \otimes \omega_1'^0(X). \tag{4.22}$$

We want to compare (4.9) to its analogue for $\Phi'(\varphi, \mathcal{C})$:

$$\Phi'(\varphi, \mathcal{C})(\tau, X) = c'(\mathcal{C}_1) \det(\mathrm{Im}_\delta \tau)^{1/2} \det \omega_1'(X)^{-1/2} \times \left(\nabla\left(\check{\omega}_1'^0(X) w\right)\varphi\right)(0, \varepsilon'(\tau, X)). \tag{4.23}$$

From the construction of the see-saw pair we have

$$\varepsilon' \circ \kappa'(X) = \varepsilon \circ \kappa(X).$$

Also, we find that

$$\omega_1(\kappa(X)) = (Q_0 T_0 \otimes 1_{n+1})\omega_1'(X) \tag{4.24}$$

and

$$\omega_1^0(\kappa(X)) = \omega_1'^0(X). \tag{4.25}$$

Therefore, putting $\kappa(X)$ for z in (4.9) we obtain

$$\Phi(\varphi, \mathcal{C})(\kappa(X)) = c(\mathcal{C})(\det Q_0)^{-(n+1)/4} \det \omega_1'(X)^{-1/2}$$
$$\times \det Q_0^{-1/2}\left(\nabla\left(\check{\omega}_1'^0(X)w\right)\varphi\right)(0, \varepsilon' \circ \kappa'(X)).$$

Since

$$c(\mathcal{C})(\det Q_0)^{-(n+1)/4} = c'(\mathcal{C}_1) \tag{4.26}$$

and

$$\det Q_0^{-1/2} = |\delta|^n \det \operatorname{Im}_\delta\left(-\delta^{-1} Q_0^{-1}\right)^{1/2}$$

we obtain the claimed identity. □

§5. The fundamental identity

In this section we will show that the period of the holomorphic $(n, 0)$-form constructed in §4, over the Lagrangian cycle determined by $\kappa : \mathbb{B} \to \mathbb{D}$ can be expressed as a special value of a certain holomorphic Siegel modular form associated with the embedding ε'.

As in §4 we fix a choice of data $\mathcal{C} = (U_0, T_0, U_1, T_1)$. Then for any reduced theta function φ given by (4.13) we have functions $\Phi(\varphi, \mathcal{C})$ and $\Phi'(\varphi, \mathcal{C})$ given by (4.9) and (4.23) respectively. Also, the holomorphic $(n, 0)$-form $\Phi(\varphi, \mathcal{C})(z)\mathrm{d}z$ on $\mathbb{D}$ is invariant with respect to the subgroup $\Gamma_1 \subset G(\mathbb{Q})$ defined in Corollary 4.3.

Let $\Gamma \subset \Gamma_1$ be a torsion free subgroup of finite index and let

$$E = \Gamma \cap H'(\mathbb{Q}) \tag{5.1}$$

where we view $H'(\mathbb{Q})$ as a subgroup of $G(\mathbb{Q})$. Also let

$$E^+ = E \cap H'(\mathbb{R})^0 \tag{5.2}$$

where $H'(\mathbb{R})^0$ is the connected component of the identity in $H'(\mathbb{R})$. Then the embedding

$$\kappa : \mathbb{B} \to \mathbb{D}$$

induces an embedding

$$\kappa : E^+\backslash \mathbb{B} \to \Gamma \backslash \mathbb{D} \tag{5.3}$$

of the orientable n-manifold $E^+\backslash \mathbb{B}$ into the complex manifold $\Gamma\backslash \mathbb{D}$. We want to consider the period:

$$\mathcal{P}(\varphi, \mathcal{C}; \Gamma) = \int_{E^+\backslash \mathbb{B}} \kappa^*(\Phi(\varphi, \mathcal{C})(z)\mathrm{d}z). \tag{5.4}$$

We will show in §7, Corollary 7.5, that $\Phi(\varphi, \mathcal{C})$ is actually a cusp form on $\mathbb{D}$; and it follows easily that $\mathcal{P}(\varphi, \mathcal{C}, \Gamma)$ is always finite. Of course, if $\Gamma' \subset \Gamma$ of finite index, then

$$\mathcal{P}(\varphi, \mathcal{C}; \Gamma') = |E^+ : E'^+|\mathcal{P}(\varphi, \mathcal{C}; \Gamma). \tag{5.5}$$

We also consider the integral

$$\mathcal{P}'(\tau; \varphi, \mathcal{C}_1; \Gamma) = \int_{E^+\backslash \mathbb{B}} \Phi'(\varphi, \mathcal{C})(\tau, X)\mathrm{d}X \tag{5.6}$$

where, since there is no dependence on V_0 here, $\mathcal{C}_1$ denotes the partial choice of data (U_1, T_1).

Our see-saw pair (1.15), or more precisely (2.22) gives:

Proposition 5.1:

$$\mathcal{P}(\varphi, \mathcal{C}; \Gamma) = -(\delta/|\delta|)^{-n}\mathcal{P}'(\tau_0; \varphi, \mathcal{C}_1; \Gamma)$$

where $\tau_0 = -\delta^{-1}Q_0^{-1}$.

Proof: This follows immediately from Proposition 4.6, and the fact that $\kappa^*(\mathrm{d}z) = -\delta^{-n}\mathrm{d}X$. □

Remark: For our special case, Proposition 5.1 is the fundamental adjointness formula $(*)$ of the introduction.

We will now show that the function $\mathcal{P}'(\tau; \varphi, \mathcal{C}_1; \Gamma)$ is, in fact, a holomorphic Siegel modular form of weight $\frac{1}{2}(n+1)$. First we find an explicit formula for the integrand.

Proposition 5.2: *Let ((,)) be the inner product on* $U_1(\mathbb{Q}) \simeq \mathbb{Q}^{n+1}$ *corre-*

sponding to Q_1^{-1}, *and for* $X \in \mathbb{B}$ *let* $((,))_X$ *be the majorant of* $((,))$ *on* $U_1(\mathbb{R}) \simeq \mathbb{R}^{n+1}$ *corresponding to* $R_1(X)$ *given by* (*ii*) *of Proposition* 2.2. *Finally, for* $\tau \in \mathfrak{H}_n$ *let*

$$((,))_{\tau,X} = \tfrac{1}{2}(\tau + \bar{\tau})((,)) + \tfrac{1}{2}(\tau - \bar{\tau})((,))_X.$$

Then, if φ *is given by* (4.13),

$$\Phi'(\varphi, \mathcal{C})(\tau, X) = c'(\mathcal{C}_1)\det\left(\mathrm{Im}_\delta \tau^{1/2}\right)\det \omega_1'(X)^{-1/2}$$
$$\times \sum_{\substack{y \in M_{n+1,n}(\mathbb{Q}) \\ y \in r + L}} \det\left((y, w^{\#})\right) e\left(\tfrac{1}{2}\mathrm{tr}((y,y))_{\tau,X}\right)$$
$$\times e\left(\mathrm{tr}({}^t ys)\right)$$

where

$$w^{\#} = \check{T}_1 \check{P}(X)\begin{bmatrix} 0 \\ 1_n \end{bmatrix}.$$

and $\check{P}'(X) = {}^t P'(X)^{-1}$ *for* $P(X)$ *given by* (2.23). *Here if* $y = (y^1, \ldots, y^n)$ *with* $y^i \in \mathbb{Q}^{n+1}$, *then*

$$((y,y)) = \left(((y^i, y^j))\right) \in M_n(\mathbb{Q}).$$

PROOF: We begin with (4.23). Since

$$(\nabla(w)\varphi)(0) = (\nabla(w)\theta)(0)$$

it is sufficient to compute $\nabla(w)$ applied termwise to $\theta(v, \varepsilon'(\tau, X))$. Now

$$\nabla(w)\{e(\mathrm{tr}({}^t yv))\}_{v=0} = \det({}^t wy). \tag{5.7}$$

Also

$$\check{\omega}_1'^0(X)\begin{bmatrix} 0 \\ 1_n \end{bmatrix} = Q_1^{-1} w^{\#} \tag{5.8}$$

with $\omega_1^0(X)$ given by (4.21), so that

$$\det\left({}^t\left(\check{\omega}_1'^0(X) w\right) y\right) = \det\left((y, w^{\#})\right)$$

as claimed. Finally, by (ii) of Proposition 2.2,

$${}^t y \varepsilon'(\tau, X) y = \mathrm{tr}\left(((y,y))_{\tau,X}\right);$$

and we obtain the claimed expression for $\Phi'(\varphi, \mathcal{C})$. $\square$

We now recall the results of [11] and [12]. Let

$$\check{B} = \{\ell \in U_1(\mathbb{R}) \mid ((\ell, \ell)) = -1\}^0 \tag{5.9}$$

be one component of the hyperboloid of two sheets in $U_1(\mathbb{R})$. As in [11] we may identify the tangent space $T_\ell(\check{B})$ to $\check{B}$ at ℓ with $\ell^\perp$. Then we may define an n-form Ω on $\check{B}$ by

$$\Omega_\ell(w) = \det \operatorname{Im} \tau^{1/2} \sum_{y \in r+L} \det((y, w))\, e\left(\tfrac{1}{2} \operatorname{tr}((y, y))_{\tau,\ell}\right)$$

$$\times e\left(\operatorname{tr}({}^t ys)\right) \tag{5.10}$$

where $w = (w^1, \dots, w^n)$ is an n-tuple in $T_\ell(\check{B}) \simeq \ell^\perp$. Here $((,))_{\tau,\ell}$ is as in Proposition 5.2 but with $((,))_\ell$, the majorant of $((,))$ associated to ℓ, in place of $((,))_X$.

Fix an orientation of $\check{B}$. It was then shown in [11] and [12] that

$$I(\tau) = \int_{E^+\setminus \check{B}} \Omega \tag{5.11}$$

is a holomorphic Siegel modular form of weight $\frac{1}{2}(n+1)$. Explicitly,

THEOREM 5.3: *The integral* (5.11) *of the n-form* Ω *given by* (5.10) *is*

$$2^{n/2} I(\tau) = \tfrac{1}{2}(-1)^n \sum_{\substack{y \in r+L \\ \operatorname{rank} y = n \\ \operatorname{rank}((y,y)) = n-1, ((y,y)) \geqslant 0 \\ y \text{ reduced} \\ \text{mod } E^+}} \varepsilon^*(y)\, e\left(\tfrac{1}{2}\operatorname{tr}(\tau((y,y)))\right) e\left(\operatorname{tr}({}^t ys)\right)$$

$$+ \sum_{\substack{y \in r+L \\ ((y,y)) > 0 \\ \text{mod } E^+}} \varepsilon(y)\, e\left(\tfrac{1}{2}\operatorname{tr}(\tau((y,y)))\right) e\left(\operatorname{tr}({}^t ys)\right)$$

where $\varepsilon(y) = \pm 1$ *and* $\varepsilon^*(y) \in \mathbb{Q}$ *are defined in* [12].

Note that, when $n = 1$, this result coincides with Hecke's construction of theta functions of weight 1 for real quadratic fields or of Eisenstein series of weight 1 if $((,))$ is an anisotropic (resp. isotropic) over $\mathbb{Q}$. Here for $y \in M_{n+1,n}(\mathbb{Q})$, $((y, y)) \in M_n(\mathbb{Q})$ as in Proposition 5.2. Also, in the first part of the sum, y is reduced mod E^+ as defined in [12].

THEOREM *Let* $\mathcal{P}'(\tau;\, \varphi, \mathcal{C}_1;\, \Gamma)$ *be given by* (5.6). *Then*

$$\mathcal{P}'(\tau;\, \varphi, \mathcal{C}_1;\, \Gamma) = (-1)^n 2^{1-n/2} \vartheta(\tau;\, \varphi, \mathcal{C}_1;\, \Gamma)$$

where

$$\vartheta(\tau; \varphi, \mathcal{C}_1; \Gamma) = 2^{n/2} I(\tau)$$

with $2^{n/2}I(\tau)$ given by Theorem 5.3. Note that by [12], the Fourier coefficients of ϑ are cyclotomic numbers with bounded denominators.

PROOF. We have a parameterization

$$\lambda : \mathbb{B} \xrightarrow{\sim} \check{B}$$

$$X \mapsto (A')^{-1/2} Q_1 T_1 P_-(X). \tag{5.12}$$

Note that this determines one of the two sheets and an orientation of $\check{B}$. We now need:

LEMMA 5.5:

$$d\lambda\left({}^t\left(\frac{\partial}{\partial x_1}\right), -2\frac{\partial}{\partial x_0}\right) = (A')^{-1/2} w^{\#}$$

where $w^{\#}$ is as above.

PROOF: First we observe that

$$A' = x_0 - {}^t x_1 x_1.$$

We then compute

$$d\lambda\left(\frac{\partial}{\partial x_0}\right) = -\tfrac{1}{2}(A')^{-3/2} Q_1 T_1 \begin{bmatrix} x_0 - 2{}^t x_1 x_1 \\ x_1 \\ 1 \end{bmatrix}$$

and

$$d\lambda\left({}^t\left(\frac{\partial}{\partial x_1}\right)\right) = (A')^{-3/2} Q_1 T_1 \begin{bmatrix} -x_0 {}^t x_1 \\ x_1 {}^t x_1 + A' \cdot 1_{n-1} \\ {}^t x_1 \end{bmatrix}.$$

On the other hand, we compute $w^{\#}$. By (iv) of Proposition 2.1,

$$\check{P}'(X) = I P'(X) \begin{bmatrix} -A' & \\ & B' \end{bmatrix}^{-1}$$

and

$$B' = \begin{bmatrix} 1_{n-1} & -x_1 \\ -{}^t x_1 & x_0 \end{bmatrix}$$

so that

$$(B')^{-1} = (A')^{-1} \begin{bmatrix} x_1{}^t x_1 + A' \cdot 1_{n-1} & x_1 \\ {}^t x_1 & 1 \end{bmatrix}.$$

Using this we obtain

$$\check{P}'(X) = (A')^{-1} I \begin{bmatrix} x_0 & -x_0{}^t x_1 & x_0 - 2{}^t x_1 x_1 \\ -x_1 & x_1{}^t x_1 + A' \cdot 1_{n-1} & x_1 \\ 1 & {}^t x_1 & 1 \end{bmatrix}.$$

Since $\check{T}_1 \mathrm{I} = Q_1 T_1$ we obtain the claimed identity. □

By this lemma and Proposition 5.2 we conclude that, if Ω is the n-form on $\check{B}$ defined by (5.10), then

$$\lambda^*\Omega = \tfrac{1}{2}(-1)^n \det \omega_1'(X)^{-1/2} (A')^{-n/2} c'(\mathcal{C}_1)^{-1}$$
$$\times \Phi'(\varphi, \mathcal{C})(\tau, X) \mathrm{d} X.$$

By Theorem 5.3 and the fact that E^+ is torsion free

$$\mathcal{P}'(\tau; \varphi, \mathcal{C}_1; \Gamma) = \det \omega_1'(X)^{-1/2} (A')^{n/2} 2^{1-n/2} (-1)^n$$
$$\times c'(\mathcal{C}_1) \vartheta(\tau; \varphi, \mathcal{C}_1; \Gamma)$$

with ϑ as in the Theorem. Since

$$\det \omega_1'(X) = |\det Q_1|^{n/2} (A')^n$$

we obtain the desired result. □

Using Proposition 5.1 we obtain the following *fundamental identity* which gives an "explicit" formula for the period $\mathcal{P}(\varphi, \mathcal{C}; \Gamma)$:

COROLLARY 5.6:

$$\mathcal{P}(\varphi, \mathcal{C}; \Gamma) = -i^n 2^{1-1/2} \vartheta(\tau_0; \varphi, \mathcal{C}_1; \Gamma)$$

where $\tau_0 = -\delta^{-1} Q_0^{-1}$.

§6. Rationality of periods

In this section we will first show that the K^{ab}-vector space spanned by the holomorphic $(n, 0)$-forms constructed in Section 4 does not depend on the choice of data $\mathcal{C}$. We will then shown that the ratio of the periods of an $(n, 0)$-form in this space, over the cycles arising from various $U_1 \in \Omega^+(V)$, all lie in the field K^{ab}.

We must first determine how our constructions depend on the choice of data $\mathcal{C} = (U_0, T_0, U_1, T_1)$. Suppose that $\mathcal{C}^* = (U_0^*, T_0^*, U_1^*, T_1^*)$ is another choice of data. Then there exist $\alpha_0 \in GL_n(K)$ and $\alpha_1 \in GL_{n+1}(K)$ such that the diagrams

$$\begin{array}{ccc} V_0(\mathbf{Q}) & \xrightarrow[\sim]{i_0(\mathcal{C})} & K^n \\ \| & & \downarrow \alpha_0 \\ V_0(\mathbf{Q}) & \xrightarrow[i_0(\mathcal{C}^*)]{\sim} & K^n \end{array} \tag{6.1}$$

and

$$\begin{array}{ccc} V_1(\mathbf{Q}) & \xrightarrow[\sim]{i_1(\mathcal{C})} & K^{n+1} \\ \| & & \downarrow \alpha_1 \\ V_1(\mathbf{Q}) & \xrightarrow[i_1(\mathcal{C}^*)]{\sim} & K^{n+1} \end{array} \tag{6.2}$$

commute, where $i_0(\mathcal{C})$, $i_1(\mathcal{C})$ (resp. $i_0(\mathcal{C}^*)$, $i_1(\mathcal{C}^*)$) are the isomorphisms determined by our choices of data. We let

$$\tilde{\alpha}_0 = (T_0^*)^{-1} \alpha_0 T_0 \tag{6.3}$$

and

$$\tilde{\alpha}_1 = (T_1^*)^{-1} \alpha_1 T_1 \tag{6.4}$$

so that, as in (2.13)′ and (2.18), $\tilde{\alpha}_0 \in U(n)$ and $\tilde{\alpha}_1 \in U(I)$. Also we have commutative diagrams:

$$\begin{array}{ccc} W_0(\mathbf{Q}) & \xrightarrow[\sim]{i_0(\mathcal{C})} & \mathbf{Q}^n \times \mathbf{Q}^n \\ \| & & \downarrow \alpha_0 \\ W_0(\mathbf{Q}) & \xrightarrow[i_0(\mathcal{C}^*)]{\sim} & \mathbf{Q}^n \times \mathbf{Q}^n \end{array} \tag{6.5}$$

and

$$\begin{tikzcd}
W(\mathbb{Q}) \arrow[r, "i(\mathcal{C})", "\sim"'] \arrow[d, equal] & Q^m \times \mathbb{Q}^m \arrow[d, "\alpha"] \\
W(\mathbb{Q}) \arrow[r, "\sim", "i(\mathcal{C}^*)"'] & \mathbb{Q}^m \times \mathbb{Q}^m
\end{tikzcd} \tag{6.6}$$

We let

$$\tilde{\alpha}_0 = \Lambda_0^* \alpha_0 \Lambda_0^{-1} \in \mathrm{Sp}(n, \mathbb{Q}) \tag{6.7}$$

and

$$\tilde{\alpha} = \Lambda^* \alpha \Lambda^{-1} \in \mathrm{Sp}(m, \mathbb{Q}). \tag{6.8}$$

The following lemma is then immediate from the definitions:

LEMMA 6.1: *For $\tilde{\alpha}_0$, $\tilde{\alpha}_1$, $\tilde{\alpha}_0$ and $\tilde{\alpha}$ as above, the following diagrams compute*:

(*i*)

$$\begin{tikzcd}
\mathbb{D} \arrow[rrr, "\varepsilon"] \arrow[ddd, "\tilde{\alpha}_1"'] & & & \mathfrak{H}_m \arrow[ddd, "\tilde{\alpha}"] \\
& D \arrow[ul, "i_1"] \arrow[r, "\varepsilon"] \arrow[dl, "i_1^*"'] & D^* \arrow[ur, "i_0"'] \arrow[dr, "i_0^*"] & \\
& & & \\
\mathbb{D} \arrow[rrr, "\varepsilon^*"] & & & \mathfrak{H}_m
\end{tikzcd}$$

where $i = i(\mathcal{C})$, $i^* = i(\mathcal{C}^*)$, $i_1 = i_1(\mathcal{C})$ and $i_1^* = i_1(\mathcal{C}^*)$ are given by (2.9) *and* (2.19) *respectively, and ε and ε^* are given by Proposition* 2.1.

(*ii*)

$$\begin{tikzcd}
D_0^* \arrow[r, "i_0(\mathcal{C})", "\sim"'] \arrow[d, equal] & \mathfrak{H}_n \arrow[d, "\tilde{\alpha}_0"] \\
D_0^* \arrow[r, "\sim", "i_0(\mathcal{C}^*)"'] & \mathfrak{H}_n
\end{tikzcd}$$

where $i_0(\mathcal{C})$ and $i_0(\mathcal{C}^*)$ are given by (2.8)

PROPOSITION 6.2. *Let $\mathcal{C}$ and $\mathcal{C}^*$ be two choices of data as above, and let $\tilde{\alpha}_1 \in U(I)$ and $\tilde{\alpha} \in \mathrm{Sp}(m, \mathbb{Q})$ be given by* (6.4) *and* (6.8). *Then with the* *notation of* (4.10) *and* (4.11),

$$\Phi(f \mid \tilde{\alpha}; \mathcal{C}) = c(\mathcal{C}, \mathcal{C}^*)\Phi(f; \mathcal{C}^*) \mid \tilde{\alpha}_1$$

with

$$c(\mathcal{C}, \mathcal{C}^*) = \pm(\det \alpha_0)^{(n+1)/2}(\det \alpha_1)^{n/2},$$

where the sign is determined by the choice of branches.

PROOF: This is essentially the same as that of Proposition 4.2 so we omit it. □

Proposition 6.2 has several consequences. First let $\mathfrak{T}(K^{ab})$ be the K^{ab} vector space generated by the functions $\varphi(v, \tau, r, s)$ for $r, s \in \mathbb{Q}^m$. Observe that, if $g \in \mathrm{Sp}(m, \mathbb{Q})$ and $\varphi \in \mathfrak{T}(K^{ab})$, then $\varphi \mid g \in \mathfrak{T}(K^{ab})$. For a choice of data $\mathcal{C}$ and for $\varphi \in \mathfrak{T}(K^{ab})$ we have the holomorphic $(n, 0)$-form on D:

$$\eta(\varphi; \mathcal{C}) = i(\mathcal{C})^*(\Phi(\varphi; \mathcal{C})\mathrm{d}z)$$

where $i(\mathcal{C})\colon D \hookrightarrow \mathbb{D}$ is given by (2.19). Let $\mathfrak{D}(K^{ab}, \mathcal{C})$ be the K^{ab}-vector space generated by the $\eta(\varphi; \mathcal{C})$'s for $\varphi \in \mathfrak{T}(K^{ab})$.

COROLLARY 6.3: *The space $\mathfrak{D}(K^{ab}, \mathcal{C})$ is independent of the choice of $\mathcal{C}$. In particular, if $\varphi \in \mathfrak{T}(K^{ab})$,*

$$\eta(\varphi \mid \tilde{\alpha}; \mathcal{C}) = c(\mathcal{C}, \mathcal{C}^*)\eta(\varphi; \mathcal{C}^*)$$

where $\tilde{\alpha} \in \mathrm{Sp}(m, \mathbb{Q})$ is determined by (6.8) *and where $c(\mathcal{C}, \mathcal{C}^*)$ is as in Proposition* 6.2. *Note that $c(\mathcal{C}, \mathcal{C}^*)$ lies in a quadratic extension of K.*

PROOF: Suppose that $\eta = \eta(\varphi, \mathcal{C}^*) \in \mathfrak{D}(K^{ab}, \mathcal{C}^*)$ with $\varphi \in \mathfrak{T}(K^{ab})$. Then by Proposition 6.2 and diagram (i) of Lemma 6.1 we have

$$\begin{aligned}
\eta &= i_1(\mathcal{C}^*)^*(\Phi(\varphi; \mathcal{C}^*)\mathrm{d}z) \\
&= i_1(\mathcal{C})^*(\tilde{\alpha}_1)^*(\Phi(\varphi; \mathcal{C}^*)\mathrm{d}z) \\
&= i_1(\mathcal{C})^*(\Phi(\varphi; \mathcal{C}^*) \mid \tilde{\alpha}_1\mathrm{d}z) \\
&= c(\mathcal{C}, \mathcal{C}^*)^{-1} i_1(\mathcal{C})^*(\Phi(\varphi \mid \tilde{\alpha}; \mathcal{C})\mathrm{d}z) \\
&= c(\mathcal{C}, \mathcal{C}^*)^{-1}\eta(\varphi \mid \tilde{\alpha}; \mathcal{C}).
\end{aligned}$$

Now $\eta(\varphi \mid \tilde{\alpha}; \mathcal{C}) \in \mathfrak{D}(K^{ab}, \mathcal{C})$ since $\varphi \mid \tilde{\alpha} \in \mathfrak{T}(K^{ab})$. □

By Corollary 6.3 we have an intrinsic K^{ab} vector space $\mathfrak{D}(K^{ab})$ of holomorphic $(n, 0)$-forms on D determined only by the pair (V_0, V_1). Note that the space $\mathfrak{D}(K^{ab})$ is actually invariant under the action of $G(\mathbb{Q})$. For any torsion free subgroup $\Gamma \subset G^1(\mathbb{Q})$ and commensurable with the subgroup Γ_1 of Corollary 4.3, let

$$\mathfrak{D}(K^{ab})^\Gamma = \{\eta \in \mathfrak{D}(K^{ab}) \mid \gamma^*\eta = \eta \quad \forall \gamma \in \Gamma\}.$$

If $\eta \in \mathfrak{D}(K^{ab})^\Gamma$ and $U_1 \in \Omega^+(V_1)$, we may define the period

$$\mathcal{P}(\eta, U_1, \Gamma) = \int_{E^+\backslash B} \kappa^*\eta$$

where $E^+ = \Gamma \cap H'(\mathbb{R})^0$.

We can now state our main rationality result:

THEOREM 6.4: *Let $\Gamma \subset G^1(\mathbb{Q})$ be commensurable with the subgroup Γ_1 of Corollary 4.3 and let $\eta \in \mathfrak{D}(K^{ab})^\Gamma$. Let $U_1, U_1^* \in \Omega^1(V_1)$ and assume that $\mathcal{P}(\eta, U_1^*, \Gamma) \neq 0$. Then*

$$\frac{\mathcal{P}(\eta, U_1, \Gamma)}{\mathcal{P}(\eta, U_1^*, \Gamma)} \in K^{ab}.$$

PROOF: Choose data $\mathcal{C} = (U_0, T_0, U_1, T_1)$ and $\mathcal{C}^* = (U_0, T_0, U_1^*, T_1^*)$ and write

$$\eta = \eta(\varphi; \mathcal{C}^*)$$

with $\varphi \in \mathfrak{T}(K^{ab})$. Then by Corollary 5.6.

$$\begin{aligned}\mathcal{P}(\eta, U_1^*, \Gamma) &= \mathcal{P}(\varphi, \mathcal{C}^*, \Gamma)\\ &= -i^n 2^{1-1/2n} \vartheta\left(-\delta^{-1} Q_0^{-1}; \varphi, \mathcal{C}_1^*, \Gamma\right).\end{aligned}$$

On the other hand, by Corollary 6.3, we may write

$$\eta = c(\mathcal{C}, \mathcal{C}^*)^{-1} \eta(\varphi \mid \tilde{\alpha}, \mathcal{C})$$

so that

$$\mathcal{P}(\eta, U_1, \Gamma) = c(\mathcal{C}, \mathcal{C}^*)^{-1}(-i^n) 2^{1-1/2n} \vartheta\left(-\delta^{-1} Q_0^{-1}; \varphi \mid \tilde{\alpha}, \mathcal{C}_1, \Gamma\right).$$

Thus

$$\frac{\mathcal{P}(\eta, U_1, \Gamma)}{\mathcal{P}(\eta, U_1^*, \Gamma)} = c(\mathcal{C}, \mathcal{C}^*)^{-1} \frac{\vartheta(\tau_0; \varphi \mid \tilde{\alpha}, \mathcal{C}_1, \Gamma)}{\vartheta(\tau_0; \varphi, \mathcal{C}_1^*, \Gamma)}$$

with $\tau_0 = -\delta^{-1}Q_0^{-1}$.

Now, as in Shimura [15], let

$$\mathfrak{M}_k(K^{ab}) = \{f \mid f \text{ is a Siegel modular form of weight } k \text{ on } \mathfrak{H}_n \text{ with Fourier coefficients in } K^{ab}\}$$

and let

$$\mathcal{A}_0(K^{ab}) = \{h = f/g \mid f, g \in \mathfrak{M}_k(K^{ab}), g \not\equiv 0\}.$$

By [15, p. 266, (10)],

$$\mathcal{A}_0(K^{ab}) = \mathfrak{K}$$

where $\mathfrak{K}$ is the field of arithmetic automorphic functions in the sense of canonical models [14].

We observe that, by the same argument as in Shimura [16, Prop. 1.5]

$$f(\tau) = \frac{\vartheta(\tau; \varphi \mid \tilde{\alpha}, \mathcal{C}_1; \Gamma)}{\vartheta(\tau, \varphi, \mathcal{C}_1^*; \Gamma)}$$

lies in $\mathcal{A}_0(K^{ab})$. In fact, if $\varphi \in \mathfrak{T}(K^{ab})$, then by [11, Prop. 1.1], for all γ in some congruence subgroup of $\mathrm{Sp}(n, \mathbb{Z})$, we have:

$$\vartheta(\gamma\tau; \varphi, \mathcal{C}_1; \Gamma) = \psi(\gamma) \det J(\gamma, \tau)^{(n+1)/2} \vartheta(\tau, \varphi; \mathcal{C}_1; \Gamma)$$

with a certain root of unity $\psi(\gamma)$. But then if φ and φ^* are in $\mathfrak{T}(K^{ab})$,

$$\vartheta(\gamma\tau; \varphi, \mathcal{C}_1; \Gamma)\vartheta(\gamma\tau; \varphi^*, \mathcal{C}_1^*; \Gamma)$$
$$= \chi(\gamma) \det J(\gamma, \tau)^{n+1} \vartheta(\tau; \varphi, \mathcal{C}_1; \Gamma)\vartheta(\tau; \varphi^*, \mathcal{C}_1^*; \Gamma)$$

where $\gamma \mapsto \chi(\gamma)$ is a character of finite order of some congruence subgroup of $\mathrm{Sp}(n, \mathbb{Z})$. If $n > 1$, then the congruence subgroup property for $\mathrm{Sp}(n, \mathbb{Z})$ implies that the kernel of χ is again a congruence subgroup. Thus such products lie in $\mathfrak{M}_{n+1}(K^{ab})$ since the ϑ's here have Fourier coefficients in K^{ab}. If $n = 1$, then the functions $\vartheta(\tau; \varphi, \mathcal{C}_1; \Gamma)$ are K^{ab} linear combinations of Hecke's binary theta-functions attached to real quadratic fields [11], and his Eisenstein series of weight 1, and hence lie in $\mathfrak{M}_1(K^{ab})$. Thus we may write

$$f(\tau) = \frac{\vartheta(\tau; \varphi \mid \tilde{\alpha}, \mathcal{C}_1; \Gamma)\vartheta(\tau; \varphi, \mathcal{C}_1^*; \Gamma)}{\vartheta(\tau; \varphi, \mathcal{C}_1^*; \Gamma)^2}$$

which proves our claim.

Now the point $\tau_0 = -\delta^{-1}Q_0^{-1} \in \mathfrak{H}_n$ is an isolated fixed point in the sense of [14] associated to the data $(M_n(K), K, y \mapsto Q_0^{-1}\,{}^t\bar{y}Q_0, \iota')$ where ι' is the natural extension of the corresponding homomorphism in (1.4). Hence by [14, (2.5.4) of the Main Theorem],

$$f(\tau_0) \in K^{ab}$$

as claimed.

7. The Fourier Jacobi Expansion; Arithmeticity

In this section we assume that the Hermitian space $V_1,(,)_1$ is isotropic, and we determine the Fourier-Jacobi expansion of the function $\Phi(\varphi, \mathcal{C})$. We then show that, up to a constant independent of φ, the coefficients in this expansion are arithmetic theta-functions as defined by Shimura [16] and hence that $\Phi(\varphi, \mathcal{C})$ is an arithmetic automorphic form in the sense of [17].

Since $V_1,(,)_1$ is isotropic we may choose a Witt decomposition

$$V_1 = V_1' + V_1^0 + V_1'' \tag{7.1}$$

so that $\dim_K V_1' = \dim_K V_1'' = 1$, $\dim_K V_1^0 = n-1$ and $(,)_1|_{V_1'} = 0, (,)_1|_{V_1''} = 0$ and $(V_1^0)^\perp = V_1' + V_1''$. Next choose $U_1 \in \Omega^+(V_1)$ so that there is a Witt decomposition

$$U_1 = U_1' + U_1^0 + U_1'' \tag{7.2}$$

which gives (7.1) when tensored with K. Choose a $\mathbb{Q}$-basis $e_0, \dots, e_n$ for U_1 with $U_1' = \mathbb{Q}e_0$, $U_1'' = \mathbb{Q}e_n$ and $U_1^0 = \text{span}\{e_1, \dots, e_{n-1}\}$, so that, for this basis,

$$Q_1 = \begin{bmatrix} & & \frac{1}{2} \\ & R_1 & \\ \frac{1}{2} & & \end{bmatrix}; \tag{7.3}$$

and we may assume that

$$T_1 = \begin{bmatrix} 1 & & \\ & T_1' & \\ & & 1 \end{bmatrix} \in GL_{n+1}(\mathbb{R}) \tag{7.4}$$

with ${}^tT_1'R_1T_1' = 1_{n-1}$. Finally, complete the choice of data $\mathcal{C}$ by taking any $U_0 \in \Omega^+(V_0)$ with $\mathbb{Q}$-basis $f_1, \dots, f_n$ and T_0 as before.

In order to obtain the Fourier-Jacobi expansion of the form $\Phi(\varphi, \mathcal{C})$

we consider the complete polarization of W defined by

$$W = W'' + W' \tag{7.5}$$

with

$$W' = V_0 \otimes V_1' + \Delta U_1^0 \tag{7.6}$$

and

$$W'' = V_0 \otimes V_1'' + U_1^0. \tag{7.7}$$

Thus W'' has $\mathbb{Q}$-basis:

$$f_1 \otimes e_n, \dots, f_n \otimes e_n; f_1 \otimes e_1, \dots, f_n \otimes e_1, \dots, f_1 \otimes e_{n-1}, \dots, f_n \otimes e_{n-1};$$
$$\Delta(f_1 \otimes e_n), \dots, \Delta(f_n \otimes e_n);$$

and W' has $\mathbb{Q}$-basis:

$$f_1 \otimes e_0, \dots, f_n \otimes e_0; \Delta(f_1 \otimes e_1), \dots, \Delta(f_n \otimes e_1), \dots, \Delta(f_1 \otimes e_{n-1})$$
$$, \dots, \Delta(f_n \otimes e_{n-1}); \Delta(f_1 \otimes e_0), \dots, \Delta(f_n \otimes e_0).$$

With respect to the basis $f_1 \otimes e_n, \dots, \Delta(f_n \otimes e_0)$ for W we find that

$$\langle , \rangle \sim \begin{bmatrix} & -Q^* \\ {}^tQ^* & \end{bmatrix} \tag{7.8}$$

with

$$Q^* = \begin{bmatrix} & & -\tfrac{1}{2}Q_0 \\ & -Q_0 \otimes R_1 & \\ \tfrac{1}{2}Q_0 & & \end{bmatrix}. \tag{7.9}$$

Let $\Lambda^* = \begin{bmatrix} 1 & \\ & Q^* \end{bmatrix}$.

Then the procedure of Section 2, (2.9), yields an isomorphism

$$D^* \xrightarrow{\sim} \mathfrak{H}_m. \tag{7.10}$$

The proof of the following result is then analogous to that of Proposition 2.2.

PROPOSITION 7.1: *Under the identifications* (7.10) *and* (2.19) *determined by* $\mathcal{C}$,

$$\varepsilon^* : \mathbb{D} \to \mathfrak{H}_m$$

is given by

$$\varepsilon^*(z) = Q_0^{-1} \otimes \varepsilon^{*0}(z)$$

with

$$\varepsilon^{*0}(z) = 2|\delta|^2 \begin{bmatrix} z_0 - {}^t z_1 z_1 & {}^t z_1^t T_1' & {}^t z_1 z_1 \\ T_1' z_1 & -\frac{1}{2}\delta^{-1} R_1^{-1} & -\delta^{-1} T_1' z_1 \\ {}^t z_1 z_1 & -\delta^{-1t} z_1 {}^t T_1' & |\delta|^{-1}\left(z_0 + \delta^t z_1 z_1\right) \end{bmatrix}.$$

In particular

$$\varepsilon^*(z) = \omega_2^*(z)\omega_1^*(z)^{-1}$$

with

$$\omega_2^*(z)\mathfrak{X} = Q_0^{-1} \otimes \omega^{*0}_2(z),$$

$$\omega_2^*(z)\mathfrak{X} = Q_0^{-1} \otimes \omega^{*0}_2(z),$$

$$\omega_1^*(z)\mathfrak{X} = 1_n \otimes \omega^{*0}_1(z),$$

where

$$\omega^{*0}_2(z) = \begin{bmatrix} \delta z_0 & -2\delta^t z_1 & -\delta^{-1} z_0 \\ \delta T_1' z_1 & T_1' & 0 \\ z_0 & 2^t z_1 & z_0 \end{bmatrix},$$

$$\omega^{*0}_1(z) = \frac{1}{2} \begin{bmatrix} -\delta^{-1} & 0 & \delta^{-1} \\ -2\check{T}_1' z_1 & 2\delta^{-1}\check{T}_1' & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

and

$$\mathfrak{X} = \begin{bmatrix} 1_n & & \\ & {}^t T_0 \otimes 1_{n-1} & \\ & & 1_n \end{bmatrix}.$$

Finally,

$$\det \omega_1^*(z) = (-1)^n \delta^{-n^2} (\det Q_0)^{(n-1)/2} |\det Q_1|^{n/2}.$$

We next want to compare this formula with that given in Proposition 2.2. Let

$$\alpha = \left[\begin{array}{ccc|ccc} & & 1_n & 0 & & \\ & 0 & & & 1_r & \\ 0 & & & & & 1_n \\ \hline 1_n & & & & & 0 \\ & 1_r & & & 0 & \\ & & 0 & 1_n & & \end{array}\right] \tag{7.11}$$

with $r = n(n-1)$ and let

$$\tilde{\alpha} = \Lambda \alpha (\Lambda^*)^{-1}. \tag{7.12}$$

Also let

$$S = \begin{bmatrix} 1_n & & \\ & 1_r & \\ & & Q_0 \end{bmatrix}. \tag{7.13}$$

LEMMA 7.2: *Let ε and ε^* be the embeddings of $\mathbb{D}$ into $\mathfrak{H}_m$ given by Proposition 2.2 and Proposition 7.1 respectively. Let*

$$\beta = \begin{bmatrix} S & \\ & \check{S} \end{bmatrix}^{-1} \circ \tilde{\alpha} \in \mathrm{Sp}(m, \mathbb{Q}).$$

Then the following diagram commutes:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\varepsilon^*"] \arrow[d, equal] & \mathfrak{H}_m \arrow[d, "\beta"] \\ \mathbb{D} \arrow[r, "\varepsilon"'] & \mathfrak{H}_m \end{tikzcd}.$$

PROOF: Immediate from the definitions. □

Now for a function $\varphi\colon \mathbb{C}^m \times \mathfrak{H}_m \to \mathbb{C}$ define

$$\Phi^*(\varphi, \mathcal{C})(z) = c(\mathcal{C}) \det \omega_1^*(z)^{-1/2} \left(\nabla\left(\check{\omega}^{*0}_1(z) w\right) \varphi\right)(0, \varepsilon^*(z)) \tag{7.14}$$

with $w = \begin{bmatrix} 0 \\ 1_n \end{bmatrix}$.

PROPOSITION 7.3: *Let β be as in Lemma 7.2. Then*

$$\Phi^*(\varphi|\beta, \mathcal{C}) = \pm \det Q_0 \Phi(\varphi, \mathcal{C})$$

where the sign depends on the choice of square roots.

PROOF: This is analogous to the proof of Proposition 6.2 and Proposition 4.2 so we omit it. □

Thus to obtain qualitative information about the Fourier-Jacobi expansion of the function $\Phi(\varphi, \mathcal{C})$ we need only compute the Fourier-Jacobi expansion of the function $\Phi^*(\varphi, \mathcal{C})$ where φ is an elementary reduced theta-function given by (4.13); since, for an arbitrary $\varphi \in \mathfrak{T}(K^{ab})$, $\varphi|\beta$ is a K^{ab}-rational linear combination of such functions. In fact it will be sufficient to consider the case where $L = \mathbb{Z}^m$.

For φ the reduced theta-function given by (4.13) with $r, s \in M_{n+1,n}(\mathbb{Q})$, and $L = \mathbb{Z}^m$, write

$$r = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix}.$$

Let

$$\tilde{L} = \{a + \delta^{-1}b \mid a, b \in M_{1,n}(\mathbb{Z})\} \subset M_{1,n}(K).$$

For such a φ, the Fourier-Jacobi expansion of $\Phi^*(\varphi, \mathcal{C})$ is given by:

PROPOSITION 7.4: *For φ given by* (4.6), *and $z = (z_0, z_1) \in \mathbb{D}$,*

$$\Phi^*(\varphi, \mathcal{C})(z) = \left((-1)^n \delta^{n^2}\right)^{1/2} \det Q_0^{1/2} \sum_{t \in \mathbb{Q}_+^\times} g_t(z_1) e(tz_0)$$

where

$$g_t(z_1) = \sum_{\substack{\alpha \equiv \rho(\tilde{L}) \\ \bar{\alpha} Q_0^{-1\,t}\alpha = |\delta|^{-2}t}} g_{t,\alpha}(z_1)$$

with $\rho = r_0 - \delta^{-1} r_2$, and

$$g_{t,\alpha}(z_1) = e\left(\delta^{3\,t} z_1 z_1 \alpha Q_0^{-1\,t}\alpha\right) e\left(\tfrac{1}{2}\mathrm{tr}_{K/\mathbb{Q}}\left(\alpha^t \bar{\eta}\right)\right)$$

$$\times \sum_{y_1 - r_1 \in M_{n-1,n}(\mathbb{Z})} \det \begin{bmatrix} -\delta^2 z_1 \alpha + \delta^t T_1' y_1 \\ \delta \bar{\alpha} \end{bmatrix}$$

$$\times e\left(\tfrac{1}{2}\delta \operatorname{tr}\left(Q_0^{-1}\, {}^t y_1 R_1^{-1}\, {}^t y_1\right) + 2|\delta|^2 \alpha Q_0^{-1}\, {}^t y_1 T_1' z_1\right) e\left(\operatorname{tr}\left({}^t y_1 s_1\right)\right)$$

with $\eta = s_0 + \delta s_2$.

PROOF: We begin with (7.14) which, by the same argument as in the proof of Proposition 5.2, gives

$$\Phi^*(\varphi, \mathcal{C})(z) = c(\mathcal{C}) \det \omega_1^*(z)^{-1/2} \sum_{y - r \in M_{n+1,n}(\mathbf{Q})} \det({}^t y w)$$

$$\times e\left(\tfrac{1}{2} \operatorname{tr}\left({}^t y \varepsilon^{*0}(z) y Q_0^{-1}\right) + \operatorname{tr}\left({}^t y s\right)\right) \tag{7.15}$$

with

$$w = \check{\omega}^{*0}_1(z)\begin{bmatrix} 0 \\ 1_n \end{bmatrix}. \tag{7.16}$$

Write

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}.$$

Now

$$w = \begin{bmatrix} -\delta^2\, {}^t z_1 & \delta \\ \delta T_1' & 0 \\ \delta\, {}^t z_1 & -1 \end{bmatrix}, \tag{7.17}$$

so that

$$\det({}^t w y) = \det\begin{bmatrix} |\delta|^2 z_1 \alpha + \delta\, {}^t T_1' y_1 \\ \delta \bar{\alpha} \end{bmatrix} \tag{7.18}$$

where $\alpha = y_0 - \delta^{-1} y_2$. Also

$$c(\mathcal{C}) \det \omega_1^*(z)^{-1/2} = (\det Q_0)^{1/2}\left((-1)^n \delta^{n^2}\right)^{1/2}. \tag{7.19}$$

By a straightforward calculation starting from the formula for $\varepsilon^{*0}(z)$ given in Proposition 7.1, we find that

$$\tfrac{1}{2}\operatorname{tr}\left({}^t y \varepsilon^{*0}(z) y Q_0^{-1}\right)$$

$$= |\delta|^2 \operatorname{tr}\left[z_0 \bar{\alpha} Q_0^{-1}\, {}^t\alpha - \delta\, {}^t z_1 z_1 \alpha Q_0^{-1}\, {}^t\alpha + 2\alpha Q_0^{-1}\, {}^t y_1 T_1' z_1\right.$$

$$\left. - \tfrac{1}{2}\delta^{-1} Q_0^{-1}\, {}^t y_1 R_1^{-1}\, {}^t y_1\right].$$

Then a suitable arrangement of terms in (7.15) gives the desired formula for $\Phi^*(\varphi, \mathcal{C})$. Note that if $\eta = s_0 + \delta s_2$, then

$$\mathrm{tr}_{K/\mathbb{Q}}(\alpha^t\bar{\eta}) = y_0\,{}^ts_0 + y_2\,{}^ts_2. \quad \square$$

REMARK: If we let

$$\tau = \delta Q_0^{-1} \otimes R_1^{-1} \in \mathfrak{H}_{n(n-1)}, \tag{7.21}$$

and

$$u = 2T_1' z_1 \alpha Q_0^{-1} \in M_{n-1,n}(\mathbb{C}) \simeq \mathbb{C}^{n(n-1)}, \tag{7.22}$$

and view

$$r_1 \in M_{n-1,n}(\mathbb{Q}) \simeq \mathbb{Q}^{n(n-1)},$$

then

$$g_{t,\alpha}(z_1) = \det R_1^{1/2} e\left(\tfrac{1}{2}{}^tu(\tau - \bar{\tau})^{-1}u\right)$$
$$\times \sum_{x \equiv r_1(\mathbb{Z}^{n(n-1)})} \det\begin{bmatrix} \frac{1}{2}|\delta|^2 uQ_0 + \delta R_1^{-1}x \\ \delta\bar{\alpha} \end{bmatrix} e\left(\tfrac{1}{2}{}^tx\tau x + {}^txu\right). \tag{7.23}$$

Since the constant term $g_0(z_1) \equiv 0$ we have the following:

COROLLARY 7.5: *For all $\varphi \in \mathcal{S}(K^{ab})$ and for any $\mathcal{C}$, $\Phi(\varphi, \mathcal{C})$ is a cusp form.*

We now recall Shimura's definition [17] of arithmetic automorphic forms of $\mathbb{D}$, specialized to our case.

The unipotent radical of the parabolic subgroup of $G(\mathbb{Q})$ determined by the Witt decomposition (7.1) is

$$N(\mathbb{Q}) = \left\{ \gamma(x,y) = \begin{bmatrix} 1 & -2{}^t\bar{x}R_1 & \delta y - {}^t\bar{x}R_1x \\ & 1_{n-1} & x \\ & & 1 \end{bmatrix} \;\middle|\; \begin{matrix} x \in K^{n-1}, \\ y \in \mathbb{Q} \end{matrix} \right\}.$$

Let $\Gamma \subset G^1(\mathbb{Q})$ be an arithmetic subgroup commensurable with Γ_1 of Corollary 4.3, and assume that

$$\Gamma \cap N(\mathbb{Q}) \supset \{\gamma(x,y) \mid x \in \mathfrak{L}, y \in J'\}$$

for lattices $\mathfrak{L} \subset K^{n-1}$ and $J' \subset \mathbb{Q}$. Let J be the dual lattice of J'. If $f(z)$ is an automorphic form of weight $n+1$ on $\mathbb{D}$ with respect to Γ, i.e. $f \mid \gamma = f$ for all $\gamma \in \Gamma$ where $f \mid \gamma$ is defined by (4.11), let

$$f(z) = \sum_{t \in J} g_t(z_1) e(tz_0)$$

be the Fourier-Jacobi expansion of f. Also, for $x \in K^{n-1}$ let

$$(f \mid \gamma(x, 0))(z) = \sum_{t \in J} h_t(z_1) e(tz_0)$$

be the Fourier-Jacobi expansion of $f \mid \gamma(x, 0)$. Then f is *arithmetic* if for all $x \in K^{n-1}$ and for all $t \in J$,

$$h_t(0) \in K^{ab}.$$

We may now state the main result of this section.

THEOREM 7.6: *For $\tau \in \mathfrak{H}_{n-1}$, let $\theta(\tau) = \theta(0, \tau; 0, 0)$ be the theta function defined by* (4.5) *in the case $m = n-1$. Then for any $\varphi \in \mathfrak{I}(K^{ab})$ the function*

$$\theta\left(\delta R_1^{-1}\right)^{-(n+2)} \Phi(\varphi, \mathcal{C})$$

is arithmetic, i.e.

$$p^{-1} \cdot \Phi(\varphi, \mathcal{C}) \in \mathfrak{M}_{n+1}(K^{ab})$$

in the notation of [17, §2, *p.* 580] *with* $p = \theta(\delta R_1^{-1})^{n+2}$. *Here R_1 is given by* (7.3).

PROOF: By Proposition 4.2 and writing $\gamma = \gamma(x, 0)$ for $x \in K^{n-1}$, we have

$$\Phi(\varphi, \mathcal{C}) \mid \gamma = \pm \Phi(\varphi \mid \check{\rho}(\gamma), \mathcal{C}).$$

Thus it is sufficient to prove that for all $r, s \in M_{n+1,n}(\mathbb{Q})$ and for all t and α as in Proposition 7.4,

$$p^{-1} \cdot g_{t,\alpha}(0) \in K^{ab}.$$

But now

$$g_{t,\alpha}(0)$$

$$= \sum_{y_1 = r_1} \det\begin{bmatrix} \delta {}^t T_1' y_1 \\ \delta \bar{\alpha} \end{bmatrix} e\left(\tfrac{1}{2}\mathrm{tr}\left(Q_0^{-1}\, {}^t y_1 R_1^{-1} y_1\right) + \mathrm{tr}\left(y_1 {}^t s_1\right)\right)$$

$$= \delta^n \det R_1^{-1/2} \sum_{y_1 \equiv r_1} \det\begin{bmatrix} y_1 \\ \bar{\alpha} \end{bmatrix} e\left(\tfrac{1}{2}\operatorname{tr}\left(y_1 Q_0^{-1}\, {}^t y_1 \delta R_1^{-1}\right) + \operatorname{tr}\left(y_1 {}^t s_1\right)\right).$$

We may also write

$$\det\begin{bmatrix} y_1 \\ \bar{\alpha} \end{bmatrix} = \sum_{i=1}^{n} (-1)^{n-i} \bar{\alpha}_i \det\left(y_1^i\right)$$

where

$$y_1^i = y_1 Q_0^{-1} \xi_i$$

and

$$\xi_i = Q_0 \begin{bmatrix} 1_{i-1} & 0 \\ 0 & 0 \\ 0 & 1_{n-i} \end{bmatrix}.$$

Thus, up to the factor $\delta^n \det R_1^{-1/2}$, $g_{i,\alpha}(0)$ is the value at the point $\delta R_1^{-1} \in \mathfrak{H}_{n-1}$ of a K-linear combination of theta functions with spherical harmonic:

$$\sum_{x \equiv r_1} \det\left(x Q_0^{-1} \xi_i\right) e\left(\tfrac{1}{2}\operatorname{tr}\left({}^t x Q_0^{-1} x \tau\right) + \operatorname{tr}\left(x {}^t s_1\right)\right)$$

associated to the positive definite quadratic form Q_0^{-1}. These functions have weight $\frac{1}{2}n + 1$ with respect to a certain congruence subgroup of $\mathrm{Sp}(n-1, \mathbb{Z})$ and have Fourier coefficients in $\mathbb{Q}^{ab}$ – see, for example, [2].

Therefore we find that

$$\theta\left(\delta R_1^{-1}\right)^{-(n+2)} g_{i,\alpha}(0) \in K^{ab}$$

and the theorem is proved.

8. A non-vanishing result

In this section we will prove the non-vanishing of certain of the periods $\mathcal{P}(\eta, U_1, \Gamma)$. We suppose that $V_0, (,)_0$, $V_1, (,)_1$ are given and we fix a choice of data $\mathcal{C} = (U_0, T_0, U_1, T_1)$.

THEOREM 8.1: *Let $\mathfrak{D}(\mathbb{C}) = \mathfrak{D}(K^{ab}) \otimes_{K^{ab}} \mathbb{C}$. Then for any $U_1 \in \Omega^+(V_1)$ there exist $\Gamma \subset G^1(\mathbb{Q})$, commensurable with Γ_1 of Corollary* 4.3, *and $\eta \in \mathfrak{D}(\mathbb{C})^\Gamma$ such that*

$$\mathcal{P}(\eta, U_1, \Gamma) \neq 0.$$

REMARK: Let $\overline{\Gamma\backslash D}$ be a smooth compactification of $\Gamma\backslash D$ as constructed in [3, Chapt. 4], and let $\kappa: \Gamma_{U_1}\backslash B \to \Gamma\backslash D$ be the Lagrangian cycle corresponding to U_1. Let $\overline{\kappa(\Gamma_{U_1}\backslash B)}$ be the closure of the image of $\Gamma_{U_1}\backslash B$ in $\overline{\Gamma\backslash D}$ and let $[U_1]_\Gamma \in H_n(\overline{\Gamma\backslash D}, \mathbb{Z})$ be the corresponding homology class. Also observe that by Proposition 7.4 and the criterion of [3, Chapt. 4] and $\eta \in \mathfrak{D}(\mathbb{C})$ extends to a holomorphic $(n, 0)$-form on $\overline{\Gamma\backslash D}$. Thus Theorem 8.1 says that for any U_1 there exists a Γ such that $[U_1]_\Gamma \neq 0$ and in particular $[U_1]_\Gamma$ defines a non-zero linear functional on the space $\mathfrak{D}(\mathbb{C})^\Gamma \subset H^{n,0}(\Gamma\backslash \overline{D})$. This non-vanishing result is analogous to the result of Wallach [18].

Before proving Theorem 8.1 we must introduce certain spaces of functions. Since we have fixed $\mathcal{C}$ we obtain isomorphisms

$$W(\mathbb{Q}) \simeq \mathbb{Q}^m \simeq M_{n+1,n}(\mathbb{Q}) \simeq U_0(\mathbb{Q}) \otimes U_1(\mathbb{Q}). \tag{8.1}$$

Fix a lattice $L \subset U_1(\mathbb{Q})$ such that the dual lattice L^* with respect to the form ((,)) on U_1, defined in Proposition 5.2, contains L; $L^* \supset L$. Also let $E(L) = \{h \in SO(U_1) \mid hL = L$ and h acts trivially in $L^*/L\}$ and fix a torsion free congruence subgroup $E \subset E(L)$ such that

$$E \subset E(L) \cap H_1^+(\mathbb{R}). \tag{8.2}$$

This can be done by [10] Proposition 6.1. For each $M \in \mathbb{Z} > 0$ let

$$E(M) = \{h \in E \mid h \text{ acts trivially in } M^{-1}L^*/ML\}. \tag{8.3}$$

For convenience let $\tilde{L}^* = (L^*)^n$ and $\tilde{L} = L^n$, viewed as lattices in $W(\mathbb{Q})$ via (8.1). Then for $r, s \in M^{-1}\tilde{L}^*/M\tilde{L}$ and $\tau \in \mathfrak{H}_n$, let $\vartheta(\tau, r, s, M)$ be given by Theorem 5.3 with $M\tilde{L}$ in place of L and $E(M)$ in place of E^+ in the summation. Let

$$\mathfrak{D}'(M) = \mathbb{C}\text{-linear span of the functions } \vartheta(\tau, r, s, M)$$
$$\text{for } r, s \in M^{-1}L^*/M\tilde{L}.$$

Note that if $M \mid M'$, then we have the "distribution relation":

$$\vartheta(\tau, r, s, M) = |E(M): E(M')|^{-1} \sum_{\substack{r' \equiv r(M\tilde{L}) \\ \operatorname{mod} M'\tilde{L}}} \vartheta(\tau, r', s, M')$$

and so there is a natural inclusion

$$\mathfrak{D}'(M) \hookrightarrow \mathfrak{D}'(M').$$

Define

$$\mathfrak{Y}' = \lim_{\substack{\rightarrow \\ M}} \mathfrak{Y}'(M). \tag{8.6}$$

The following is then easily checked:

LEMMA 8.2: *The space $\mathfrak{Y}'$ does not depend on the choice of L or E, and is stable under the action of* $\mathrm{Sp}(n, \mathbb{Q})$ *given by*

$$(\vartheta|g)(\tau) = \det J(g_1, \tau)^{-(n+1)/2} \vartheta(g(\tau)).$$

Next view $\tilde{L}$ as a lattice in $M_{n+1,n}(\mathbb{Q})$ and for $M \in \mathbb{Z} > 0$, $\tau \in \mathfrak{H}_m$, $v \in \mathbb{C}^m$ and r, $s \in M^{-1}\tilde{L}^*/M\tilde{L}$ let $\varphi(v, \tau, r, s, M)$ be given by (4.13) with $M\tilde{L}$ in place of L. Let

$$\mathfrak{T}(M) = \mathbb{C}\text{-linear span of the functions } \varphi(v, \tau, r, s, M) \tag{8.7}$$
$$\text{for } r, s \in M^{-1}\tilde{L}^*/M\tilde{L}$$

and define, again for the natural inclusions,

$$\mathfrak{T} = \lim_{\substack{\rightarrow \\ M}} \mathfrak{T}(M). \tag{8.8}$$

Again this space is independent of the choice of L and stable under the action of $\mathrm{Sp}(m, \mathbb{Q})$ defined by (4.10).

Finally let $\mathfrak{Y}(\mathbb{C}) = \mathfrak{Y}(K^{ab}) \otimes_{K^{ab}} \mathbb{C}$ with $\mathfrak{Y}(K^{ab})$ defined in Section 6. Since we have fixed $\mathcal{C}$, we may identify $\mathfrak{Y}(\mathbb{C})$ with the space of $\Phi(\varphi, \mathcal{C})$dz's for $\varphi \in \mathfrak{T}$, and so we have a surjective linear map

$$\Phi : \mathfrak{T} \to \mathfrak{Y}(\mathbb{C})$$
$$\varphi \mapsto \Phi(\varphi, \mathcal{C})\mathrm{d}z \tag{8.9}$$

which is ρ-equivariant by Proposition 4.2. For fixed M there is a surjective linear map

$$I_M : \mathfrak{T}(M) \to \mathfrak{Y}'(M)$$
$$\varphi(v, \tau, r, s, M) \mapsto \mathrm{vol}(E(M)\backslash \check{B})^{-1} \vartheta(\tau, r, s, M) \tag{8.10}$$

which is well defined by Theorem 5.3.

Since by (8.1) the diagram

$$\begin{array}{ccc} \mathfrak{T}(M) & \xrightarrow{I_M} & \mathfrak{Y}'(M) \\ \downarrow & & \downarrow \\ \mathfrak{T}(M') & \xrightarrow{I_{M'}} & \mathfrak{Y}'(M') \end{array} \tag{8.11}$$

commutes, we may define a surjective linear map

$$I: \mathfrak{T} \to \mathfrak{D}' \tag{8.12}$$

which is ρ'-equivariant by Proposition 4.4.

The fundamental identity, Corollary 5.6, now says that for any $\Gamma \subset G^1(\mathbb{Q})$ as in Section 5 and for any $\varphi \in \mathfrak{T}^{\rho(\Gamma)}$,

$$\mathcal{P}(\Phi(\varphi), U_1, \Gamma) = c \operatorname{vol}(E_\Gamma \backslash \check{B}) I(\varphi)(\tau_0) \tag{8.13}$$

where $E_\Gamma = \{\gamma \in \Gamma \cap H'(\mathbb{Q}) \mid \det \gamma = 1\}$, $\tau_0 = -\delta^{-1} Q_0^{-1}$ and $c = -i_2^{n1-1/2n}$.

Proof of Theorem 8.1: By Theorem A of the appendix we may choose $\vartheta \in \mathfrak{D}'$ with $\vartheta \neq 0$. Since the $\mathrm{Sp}(n, \mathbb{Q})$ orbit of $-\delta^{-1}Q_0^{-1} \in \mathfrak{H}_n$ is dense in $\mathfrak{H}_n$ and since $\mathfrak{D}'$ is stable under $\mathrm{Sp}(n, \mathbb{Q})$ we may assume that $\vartheta(\tau_0) \neq 0$. Choose $\varphi \in \mathfrak{T}$ such that $I(\varphi) = \vartheta$. Choose a subgroup $\Gamma \subset G^1(\mathbb{Q})$ as in Section 5 and such that $\varphi \in \mathfrak{T}^{\rho(\Gamma)}$ and $E_\Gamma = \Gamma \cap H_1(\mathbb{Q}) \subset E$ where E is as in (8.2) above. Then for $\eta = \eta(\varphi; \mathcal{C})$,

$$\begin{aligned}\mathcal{P}(\eta, U_1, \Gamma) &= \mathcal{P}(\Phi(\eta), U_1, \Gamma)\\ &= \text{const. } \vartheta(\tau_0) \neq 0\end{aligned}$$

as claimed.

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Appendix

We want to prove

THEOREM A.1: *With the notation of Section* 8:

$\mathfrak{D}' \neq 0.$

The proof will be based on the following proposition which complements Lemma 11.4 of [10]: Let $\mathfrak{k}$ be a totally real field with $|\mathfrak{k}:\mathbb{Q}| = r$ and let $\mathcal{O}$ be the ring of integers of $\mathfrak{k}$. Let $V,(,)$ be a $\mathfrak{k}$-vector space with a nondegenerate symmetric bilinear form and assume that $V,(,)$ is indefinite at at least one infinite place of $\mathfrak{k}$. Let $L \subset V$ be an $\mathcal{O}$-lattice such that the dual lattice

$$L^* = \{ v \in V(\mathfrak{k}) \mid \mathrm{tr}_{\mathfrak{k}/\mathbb{Q}}(v, v') \in \mathbb{Z} \qquad \forall v' \in L \}$$

contains L. Let $G = SO(V,(,))$ viewed as an algebraic group over $\mathfrak{k}$ and let

$$G(L) = \{ g \in G(\mathfrak{k}) \mid gL = L \quad \text{and} \quad g \text{ acts trivially in } L^*/L \}.$$

Let $\Gamma \subset G(L)$ be a (torsion free) congruence subgroup. Finally, for any integral $\mathcal{O}$-ideal $\mathfrak{a}$, let

$$R_{\mathfrak{a}} : L^* \to L^*/\mathfrak{a}L$$

be the reduction map.

Let $m = \dim_{\mathfrak{k}} V$, and for any $\beta = {}^t\beta \in GL_{m-1}(\mathfrak{k})$, let

$$\tilde{L}^*_{\beta} = \{ X \in (L^*)^{m-1} \mid (X, X) = \beta \}.$$

PROPOSITION A.2. *Assume that there exists a compact open subgroup $S \subset G(\mathbb{A}_f)$, the finite adèles of G, such that*

(i) $\Gamma = G(\mathfrak{k}) \cap G_{\infty} \cdot S$

and

(ii) $S = \prod_{\substack{v \\ \text{finite}}} S_v$

with S_v compact open in $G(\mathfrak{k}_v)$. Then there exists an integral $\mathcal{O}$-ideal $\mathfrak{a}$ such that the induced map

$$R_{\mathfrak{a}} : \Gamma \backslash \tilde{L}^*_\beta \to \Gamma \backslash \tilde{L}^* / \mathfrak{a} \tilde{L}$$

on Γ-orbits is injective.

PROOF: It is sufficient to prove that if $X_1, X_2 \in \tilde{L}^*_\beta$ are such that $\Gamma \cdot X_1 \cap \Gamma \cdot X_2 = \phi$, then there exists an ideal α such that

$$R_{\mathfrak{a}}(\Gamma \cdot X_1) \cap R_{\mathfrak{a}}(\Gamma \cdot X_2) = \phi.$$

Suppose that no such $\mathfrak{a}$ exists. Then by the first part of the proof of Lemma 11.4 of [10] there exists an $\eta \in G(\mathfrak{k})$ such that $\eta X_1 = X_2$. Let $\Sigma = \{v \text{ finite place of } \mathfrak{k} \mid \eta \notin S_v\}$. Then as in the proof of Lemma 11.4 of [10] there exists an element $\mu \in S$ such that $\forall v \in \Sigma$,

$$\mu_v X_1 = X_2.$$

But since η and μ_v are unique, we must have $\eta = \mu_v \in S_v$ for $v \in \Sigma$. Thus $\Sigma = \phi$ and $\eta \in G(\mathfrak{k}) \cap G_\infty S = \Gamma$ which contradicts our assumption about X_1 and X_2, and the lemma is proved. □

COROLLARY A.3. *For any $X \in \tilde{L}^*_\beta$ there exists an integral $\mathcal{O}$ ideal $\mathfrak{a}$ such that*

$$\Gamma \cdot X = \coprod_{\mu \in R_{\mathfrak{a}}(\Gamma \cdot X)} \{Y \in \tilde{L}^*_\beta \mid R_{\mathfrak{a}}(Y) = \mu\}.$$

Now by specializing to the case $\mathfrak{k} = \mathbb{Q}$, and the $\mathbb{Q}$-space $U_1,((,))$, with lattice L as in Section 8, we can give:

PROOF OF THEOREM A.1: Take $X \in \tilde{L}^*$ with $(X, X) = \beta > 0$. Also by Corollary A.3 choose $M \in \mathbb{Z} > 0$ so that

$$E \cdot X = \coprod_{\mu \in R_M(E \cdot X)} \{Y \in \tilde{L}^*_\beta \mid R_M(Y) = \mu\}$$

where E is the congruence subgroup chosen in Section 8. Note that we may assume the E is defined by a compact open subgroup as in Proposition A.2. Then consider the coefficient $a(\beta)$ of $e(\frac{1}{2}\mathrm{tr}(\tau\beta))$ in the Fourier expansion of the function

$$\theta(\tau) = \sum_{\mu \in R_M(E \cdot X)} \vartheta(\tau, \mu, 0, M) \in \mathfrak{D}'.$$

Then

$$\begin{aligned} a(\beta) &= \sum_{\mu \in R_M(E \cdot X)} \sum_{\substack{Y \in \tilde{L}^* \\ R_M(Y) = \mu \\ (Y, Y) = \beta \\ \mathrm{mod}\, E}} \varepsilon(Y) \\ &= \varepsilon(X) \cdot (\text{const.}) \\ &\neq 0, \end{aligned}$$

since for $\gamma \in E$, $\varepsilon(\gamma X) = \varepsilon(X)$. Thus, in particular, $\theta \neq 0$ and the theorem is proved.

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