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Some non-semi-simple Iwasawa modules

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The purpose of this note is to show that the semisimplicity result of [2] may fail to be true in the cases not covered by the theorem in section 2. We base the examples on the idea of J.F. Jaulent [4] although our method in §1 is somewhat different. Theorem 1 of this note gives an alternate proof of Theorem 1 of [2] and Theorem 9 of [4]. We follow the notation in [2].

Let \( k/Q \) be a totally complex abelian extension, and denote by \( \Delta = \text{Gal}(k/Q) \). Let \( J \in \Delta \) be the automorphism given by complex conjugation (under some fixed embedding of an algebraic closure of \( k \) into the complex numbers). Fix a prime \( p \), such that \( \delta^p = 1 \) for all \( \delta \in \Delta \). Let \( \hat{\Delta} = \text{Hom}(\Delta, \mu_{p-1}) = \text{Hom}(\Delta, \mathbb{Z}^*_p) \) and denote by \( V \) the set of characters \( \chi \) of \( \Delta \) which are either odd or trivial, i.e. \( V = \{ \chi \in \hat{\Delta} | \chi(J) = -1 \) or \( \chi = \chi_0 \} \).

For each \( \chi \in V \), there exists a (unique) \( \mathbb{Z}_p \)-extension (see [1]) \( K_\chi/k \), \( \text{Gal}(K_\chi/k) = \Gamma_\chi \) such that \( K_\chi/Q \) is normal and \( \text{Gal}(K_\chi/Q) \cong \Gamma_\chi \cdot \Delta \) a semidirect product with \( \delta \sigma \delta^{-1} = \sigma^{\chi(\delta)} \) for all \( \sigma \in \Gamma_\chi, \delta \in \Delta \).

Let \( L/K_\chi \) be the maximal abelian unramified \( p \)-extension of \( K_\chi \) so that \( \text{Gal}(L/K_\chi) = X \simeq \lim_{\rightarrow} A_n \) (where \( A_n \) is the \( p \)-primary subgroup of the ideal class group of \( k_n \subseteq K_\chi \), and the limit is taken as usual with respect to the norm maps). Then, as usual, \( X \) is a noetherian torsion \( \Delta \)-module, so we have

\[
X/TX \simeq \tau X \simeq \tau X_0 \simeq X_0/TX_0
\]

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where $TX = \{ x \in X | T x = 0 \}$ and $X_0 = \{ x \in X | T^k x = 0 \text{ some } k \geq 1 \}$ and "~" here denotes pseudo-isomorphism.

Since $\Gamma_x$ acts trivially on $X/TX$ and on $\tau X$ we have a natural action of $\Delta$ on these groups and following [4], we study their $\Delta$-decompositions.

If $M$ is a $\Delta$-module which is also a (pro) $p$-group then for $\phi \in \Delta$ write

$$M_\phi = \{ m \in M | \delta(m) = \phi(\delta) \cdot m \text{ for } \delta \in \Delta \}$$

and call this the $\phi$-component of $M$.

Now $X_0 \sim \Lambda/T^{a_1} + \ldots + \Lambda/T^{a_r}$ for integers $a_1, \ldots, a_r \geq 1$. We say $X_0$ is semi-simple $\iff a_1 = a_2, \ldots = a_r = 1$ and in this case it is clear that

$$\tau X \sim X_0 \sim X/TX \text{ as } \Delta \text{-modules}.$$
**Lemma 1:** If $M$ is the compositum of all $\mathbb{Z}_p$-extensions of $F$, and $G = \text{Gal}(M/F)$, then $G \cong \mathbb{Z}_p^{[D] + 1}$ and we have the $D$-decomposition of $G$ for all $\phi \in \hat{D}$,

$$G_{\phi'} \cong \mathbb{Z}_p \quad \text{if} \quad \phi' \neq \chi_0$$

$$\cong \mathbb{Z}_p + \mathbb{Z}_p \quad \text{if} \quad \phi' = \chi_0.$$

**Lemma 2:** $F_{\phi}F_{x}/F_{x}$ is unramified if and only if either (a) $F_{\phi} = F_{x}$ or (b) $F_{nr} \subseteq F_{\phi}F_{x}$, where $F_{nr}$ is the unique non-ramified $\mathbb{Z}_p$-extension of $F$ and is equal to $F \cdot \mathbb{Q}_p^{nr}$, the compositum of $F$ with the non-ramified $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$.

**Theorem 1:** $K_{\phi}K_{x}/K_{x}$ is unramified if and only if

$$\phi \in \hat{V} \quad \text{and} \quad \phi|D = \chi|D.$$  

**Proof:** Suppose $K_{\phi}K_{x}/K_{x}$ is unramified so that for each $p$ over $p$, we have $F_{\phi}F_{x}/F_{x}$ is unramified. Hence by Lemma 2, either (a) $F_{\phi} = F_{x}$ so that $\phi|D = \chi|D$ or (b) $F_{nr} \subseteq F_{\phi}F_{x}$. In this case, (b), we must have $\text{Gal}(F_{\phi}F_{x}/F)_{\chi_0}$ is non-trivial since $\text{Gal}(F_{nr}/F)_{\chi_0} \cong \mathbb{Z}_p$. Since both $F_{\phi}/F$ and $F_{x}/F$ are infinitely ramified it follows that only the $\chi_0$ component of $\text{Gal}(F_{\phi}F_{x}/F)$ is non-zero and so $\phi|D = \chi_0|D = \chi|D$. Hence in either case, $\phi \in \hat{V}$ and $\phi|D = \chi|D$.

Conversely, suppose $\phi \in \hat{V}$, and $\phi|D = \chi|D$. If $\chi|D \neq \chi_0|D$ then $F_{\phi} = F_{x}$ by Lemma 1, and so $K_{\phi}K_{x}/K_{x}$ is unramified at primes over $p$.

If $\phi|D = \chi|D = \chi_0|D$ then again by Lemma 1 either $F_{\phi} = F_{x}$; or $F_{nr} \subseteq F_{\phi}F_{x}$, so again $K_{\phi}K_{x}/K_{x}$ is unramified at primes over $p$. Since $K_{\phi}/k$ in unramified outside of primes over $p$, the conclusion of the theorem follows.

**Corollary:** $(X/TX)_{\phi} \sim \mathbb{Z}_p$ for $\phi \in \hat{V}$, $\phi|D = \chi|D$, $\phi \neq \chi$, and $\sim 0$ otherwise.

**Remark:** This corollary furnishes another proof of Theorem 1 in [2] and Theorem 9 of [4].

We also note if for any $\phi$ we have $F_{\phi} = F_{x}$, then it follows that $K_{\phi}K_{x} \subseteq L$ in the notation of [2] and for each $\phi$, $(X'/TX')_{\phi}$ has non-zero $\mathbb{Z}_p$-rank. This gives many examples of $\mathbb{Z}_p$-extensions where $X'/TX'$ and $\tau X'$ are infinite.
§2. $\Delta$-structure of $\tau X$

Let $\Gamma = \Gamma_x = \text{Gal}(K_x/k)$, and so $\tau X = \lim_{\leftarrow} A_n^\Gamma$. Since the limit is taken with respect to the norm maps $N_{m,n}$ and since $\delta N_{m,n} = N_{m,n} \delta$ for all $\delta \in \Delta$, it follows that

$$(\tau X)_\phi = \lim_{\leftarrow} (A_n^\Gamma)_\phi \text{ for } \phi \in \hat{\Delta}.$$ 

We consider the usual exact sequences

$$1 \to P_n \to I_n \to C_n \to 1$$
$$1 \to E_n \to k_n^* \to P_n \to 1$$

where $I_n, C_n, P_n, E_n$ are the ideal group, class group, group of principal ideals, and unit group of the $n$th layer $k_n$ of $K_x$ respectively.

We obtain the exact sequence

$$1 \to P_n^\Gamma \to I_n^\Gamma \to C_n^\Gamma \xrightarrow{f} NP_n/P_n^{\gamma -1} \simeq E_0 \cap Nk_n^*/NE_n \to 1$$

where the map $f$ is given below. Choose a fixed generator $\gamma$ of $\Gamma_x$. Then for $x \in C_n^\Gamma$, $\gamma x = x$ and so $\frac{\gamma A}{A} = (\alpha) \in P_n$ for an ideal $A \in x$, define $f(x) = (\alpha) \mod P_n^{\gamma -1}$. This is a group homomorphism which is not a $\Delta$-map, (c.f. [4]), but satisfies

$$f: (A_n^\Gamma)_\phi \to (\gamma P_n/P_n^{\gamma -1})_\phi.$$

Also the isomorphism is given by:

$$NP_n/P_n^{\gamma -1} \simeq E_0 \cap N(k_n^*)/NE_n$$
$$(\alpha) \mod P_n^{\gamma -1} \to N(\alpha) \mod N(E_n)$$

where $N$ denotes the norm map $N_{n,0}$ from $k_n$ to $k = k_0$. Hence we obtain the exact sequence

$$1 \to \frac{P_n \cap I_0}{I_0} \to P_n^\Gamma/P_0 \to I_n^\Gamma/I_0 \to C_n^\Gamma/j(C_0) \to \frac{E_0 \cap N(k_n^*)}{N(E_n)} \to 1$$

where $j(C_0) \subseteq C_n^\Gamma$ is the subgroup generated by the ideals of $k = k_0$. We shall compute the $\phi$-components of the groups $E_0 \cap N(k_n^*)/E_0^{\gamma}$ and $I_n^\Gamma/I_0$. Since the groups on either side of $C_n^\Gamma/j(C_0)$ are (at worst) quotients
of these, this will describe the set of \( \phi \)-components of \( A_n^r \sim C_n^r/j(C_0) \) which are possibly non-trivial. (As in [2], we use the notation \( A_n \sim B_n \) for sequences of groups \( \{A_n\} \) and \( \{B_n\} \) to mean there are homomorphisms \( \phi_n : A_n \to B_n \) whose kernels and cokernels have orders bounded independently of \( n \).)

For each prime \( p \) of \( k \) dividing \( p \), let \( p = A(p) \in I_n \) where \( e_p \sim p^\kappa \) is the ramification index of \( p \) for \( k_n/k \). Since \( \Delta \) permutes the primes of \( k \) over \( (p) \) transitively it follows that

\[
I_n^r/I_0 \simeq \bigoplus_{p \mid (p)} \langle A(p) \rangle/\langle p \rangle \sim \mathbb{Z}/p^n\mathbb{Z}[\Delta/D]
\]

where \( \langle A(p) \rangle, \langle p \rangle \) are the multiplicative subgroups of \( I_n^r \) generated by \( A(p) \) and \( p \) respectively.

Hence it follows that \( (I_n^r/I_0)_\phi \sim \mathbb{Z}/p^n\mathbb{Z} \) if \( \phi \mid D = \chi_0 \mid D \sim 0 \) otherwise.

On the other hand by [2, Lemma 1] we have

\[
(E_0 \cap N(k^*_n)/E_0p^n)_{\phi_1} \sim \mathbb{Z}/p^n\mathbb{Z} \text{ if } \phi_1 \text{ even, } \phi_1 \neq \chi_0 \text{ and } \phi_1 \mid D \neq \chi \mid D \sim 0 \text{ otherwise.}
\]

Since \( (A_n)_\phi \to (E_0 \cap N(k^*_n)/N_{E_n})_{\phi_2} \), the possible \( \phi \)-components of \( A_n^r \) which have non-trivial image in this group are among those \( \phi \),

\[
\phi(J) = \chi(J), \phi \neq \chi^{-1} \text{ and } \phi \mid D \neq \chi_0 \mid D.
\]

Hence the non-trivial \( \phi \)-components of \( A_n^r \) are among

\[
\{\phi \mid \phi \mid D = \chi_0 \mid D\} \cup \{\phi \mid \phi(J) = \chi(J), \phi \neq \chi^{-1}, \phi \mid D \neq \chi_0 \mid D\}.
\]

This provides no restriction in the case that \( D \subseteq \ker \chi \) when in fact \( X_0 \) is semisimple [2].

If \( \chi \mid D \neq \chi_0 \mid D \), then we see that the \( \chi^{-1} \) component of \( A_n^r \) and that of \( TX \) must be pseudo-null.

### §3. Examples

We now describe a set of characters \( \chi \) so that for the \( \mathbb{Z}_p \)-extensions \( K_\chi/k \) the groups \( X/\tau X \) and \( \tau X \) have different \( \Delta \)-decompositions. This implies that the corresponding \( X_0 \) is not semi-simple.

By Corollary of §1, we see that \( (X/\tau X)_{\chi^{-1}} \sim \mathbb{Z}_p \) if \( \chi^{-1} \neq \chi \) and \( \chi^{-1} \mid D = \chi \mid D \), i.e. if \( \chi^2 = \chi_0 \) and \( \chi_0 \mid D = \chi_0 \mid D \). On the other hand §2 implies that \( (\tau X)_{\chi^{-1}} \sim 0 \) if \( \chi \mid D \neq \chi_0 \mid D \) so we have:
For any character $\chi$, such that $\chi^2 \neq \chi_0$, $\chi|D \neq \chi_0|D$ and $\chi^2|D = \chi_0|D$ we have $(\tau X)_{x^{-1}} \sim 0$ and $(X/\tau X)_{x^{-1}} \sim \mathbb{Z}_p$.

The examples of Jaulent [4] are of this type.

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