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EQUIVALENCE OF GENERIC MAPPINGS AND C^∞ NORMALIZATION

Terence Gaffney* and Leslie Wilson*

Introduction

It is important in singularity theory to know when two mappings $f, g: N \to P$ are equivalent. We say f and g are equivalent if there are diffeomorphisms h and k such that $f = k \circ g \circ h$ and we say f and g are right-equivalent if $f = g \circ h$ (unless we say otherwise, all maps and map-germs will be assumed C^∞). One wants to have canonical forms for various kinds of singular behavior, both local and global. For example, suppose one wants to show that any pair of singular points of a certain kind can be removed by homotopy. First one constructs a polynomial canonical form having such a pair of singular points for which one can give the homotopy explicitly. Then, for some C^∞ map f having such a pair of singular points, one finds a connected, open neighborhood of the singular points on which f is equivalent to the canonical form. The homotopy of f is then induced from that of the canonical form. This procedure is discussed in more detail in [24].

For stable germs, we have long had Mather's theorem: f and g are equivalent if, and only if, their algebras $Q(f)$ and $Q(g)$ are isomorphic. What can be said for mappings? One approach would be to try to patch together the highly nonunique local equivalences supplied by Mather's Theorem – that seems hard. This problem was apparently first dealt with in Wilson's Thesis (see [23]) in the case of stable mappings between surfaces. The approach there was as follows: first, we assume there is a diffeomorphism k which maps the singular values of g onto those of f;

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second, we prove that f and $k \circ g$ are locally right equivalent; third, we note that these local right equivalences are essentially unique, and hence can easily be pieced together to get a global right equivalence.

Mather pointed out that an analytic stable map with $\dim N < \dim P$ is a normalization of its image and hence that two such maps with the same image are right equivalent by the Uniqueness of Normalization Theorem. Mather conjectured that there should be a C^∞ theory of normalization as well. We will develop such a theory in Section 1. The arguments in Section 1 were known to us in 1975. However, these arguments required a generalization of Glaeser's Theorem (IX.1.1 of [21]). This generalization was finally established by Merrien in [17], then further generalized by Bierstone and Milman (see [1]). We wish to thank them for liberating us from this problem at last.

Actually, this approach works for $\dim N \geq \dim P$ as well. Let $S(f)$ denote the singular set of f. We will call $f(S(f))$ the discriminant of f (see for example [20] for the motivation behind this terminology). If f is a stable germ, its restriction to $S(f)$ is a normalization of its discriminant. So if two stable germs have the same discriminant, their restrictions to their singular sets are right equivalent. An additional argument is then needed to show that the stable germs themselves are right equivalent. Gaffney supplied this additional argument in the case $\dim N = \dim P$ in his Thesis ([12]). After generalizations by Wirthmüller ([25]) and Pham ([19]), a rather complete result on local right equivalence was finally obtained in [9]. This result is stated in Section 2, along with results describing the set of right equivalences between two fixed map-germs.

In Section 3, we present global equivalence theorems for the case $\dim N \leq \dim P$. One consequence of these global equivalence theorems is that stable maps in these dimensions are C^∞ right equivalent if, and only if, they are topologically right equivalent.

We have given a survey of the results of this paper and of [9] and [10] in [13].

1. Uniqueness of C^∞ normalizations

The purpose of this section is to discuss the notion of a C^∞ normalization and to prove a uniqueness of normalization theorem. Our C^∞ normalizations will be C^∞ map-germs which are equivalent to analytic normalizations. (We still lack a good notion of C^∞ normalization for the entire C^∞ category.) Our main result is Theorem 1.11, which says that if two C^∞ normalizations have the same image, then they are right equivalent. This is a corollary of Proposition 1.10, which says that a map-germ g with normal domain can be factored through any simple map-germ f having the same image (i.e. there exists a C^∞ h such that $g = f \circ h$).

The proof of Proposition 1.10 has two parts. Without loss of generality f can be assumed analytic with image variety V. First we show that, given any analytic universal denominator d for V and any C^∞ function-germ a defined on the domain of f, there is a C^∞ b such that $a = (b \circ f)/(d \circ f)$; b/d has properties which make it what we call a weakly C^∞ function-germ. We then prove that $(b/d) \circ g$ defines a C^∞ function-germ α defined on the domain of g. The transformation $a \to \alpha$ is h^* for some C^∞ map-germ h, and $g = f \circ h$ as desired. For these proofs, we must pass from the realm of the complex analytic, to the real analytic, to the formal, and finally to the C^∞. We begin with definitions, notation, and a review of more or less well-known relationships between these categories.

Let O_p denote the ring of convergent power series in p variables over $k = \mathbb{R}$ or $\mathbb{C}$; if we want to specify that these are centered at $y \in k^p$, we write $O_{p,y}$ or just O_y. We will identify O_y with the space of germs of analytic functions at y. If V is the germ of an analytic variety at y, $I(V)$ denotes the ideal of analytic germs which vanish on V, and $O(V) = O_y/I(V)$. $Q(O(V))$ denotes the ring of fractions $(O(V) - \{\text{zero divisors}\})^{-1}O(V)$ and $O^w(V)$ is the integral closure of $O(V)$ in $Q(O(V))$. Elements of $O^w(V)$ are called *weakly analytic function-germs on V*. Suppose the irreducible components of V are $V_1, \ldots, V_r$; $Q(O(V))$ and $O^w(V)$ are direct sums of the corresponding rings for the V_i's, but $O(V)$, in general, is not. If $O^w(V) = O(V)$, V is said to be *normal*.

Let S_i be the germ of an analytic variety at x_i in k^{n_i}, for $i = 1, \ldots, r$, let S_i^1 be any representative of S_i, and let S be the germ at $\{x_1, \ldots, x_r\}$ of $\cup S_i^1$ in the disjoint union of the k^{n_i}'s. Each of $O(S)$, $O^w(S)$ and $Q(O(S))$ can be defined as before, and each is the direct sum of the corresponding rings for the S_i's. Consequently, S is normal if, and only if, each S_i is.

Consider an analytic map-germ $g: S \to V$. g is *finite* if $O(S)$ is a finite $g^*O(V)$ module; this is equivalent to each $g_i: S_i \to V$ being finite. If $k = \mathbb{C}$, then the image of a finite g is a variety-germ by the Proper Mapping Theorem. g is *bimeromorphic* if $g^*: Q(O(V)) \to Q(O(S))$ is an isomorphism; this is equivalent to V being the union of variety-germs $V_1, \ldots, V_r$ with $g(S_i) \subset V_i$ and $g|S_i: S_i \to V_i$ bimeromorphic, for all i; if $k = \mathbb{C}$, this is equivalent to the existence of nowhere dense subvarieties of S and V off of which g is bianalytic. g is *simple* if it is both finite and bimeromorphic. A simple g is a *normalization* if S is normal.

If, for each i, S_i is irreducible and $g(S_i)$ is not contained in any proper subvariety of any irreducible component of V, then g pulls back non zero divisors to non zero divisors, hence $g^*O^w(V) \subset O^w(S)$; if S is also normal, then $g^*O^w(V) \subset O(S)$. If g is simple, then $O(S) \subset g^*O^w(V)$, since $O^w(V)$ contains all finite extensions of $O(V)$ in $Q(O(V))$. Thus, if g is a normalization, then $g^*: O^w(V) \to O(S)$ is an isomorphism.

For the relationship between real and complex analytic spaces and maps, see [18]. Let $V_{\mathbb{C}}$ denote the complexification of a real variety-germ V, and let $g_{\mathbb{C}}: S_{\mathbb{C}} \to V_{\mathbb{C}}$ be the complexification of $g: S \to V$. $V_1, \ldots, V_r$ are the irreducible components of V if, and only if, $(V_1)_{\mathbb{C}}, \ldots, (V_r)_{\mathbb{C}}$ are the irreducible components of $V_{\mathbb{C}}$. S is normal if, and only if, $S_{\mathbb{C}}$ is normal. g is finite (respectively, simple; respectively, a normalization) if, and only if, $g_{\mathbb{C}}$ is.

$D(V)$ denotes the ideal in $O(V)$ of *universal denominators*, i.e. those d in $O(V)$ such that $dO^w(V) \subset O(V)$. For any complex variety-germ V, there is a d which is a universal denominator and a non zero divisor at each $z \in V$ (since d and V are germs, this statement means that for any representatives d^1 of d and V^1 of V, the germ of d^1 at z is a universal denominator and non zero divisor of V_z^1 for all z sufficiently near y). For a real variety-germ V, $\mathrm{D}(\mathrm{V}_{\mathbb{C}}) = D(V)O(V_{\mathbb{C}})$.

For our passage to the C^∞ category, we will need some semi-local properties. Over $\mathbb{C}$, if S is normal at x, then it is normal at each point near x; if $g: S \to V$ is finite (respectively, bimeromorphic), then it is so at $g^{-1}(z)$ for all $z \in V$ also. Now let's consider the real analytic case. The *maximal part of* V is $V^m = \{z \in V : \dim V_z = \dim V_y\}$ (recall that we mean: take a representative V^1 of V, form $(V^1)^m$, and then take its germ at y). For S the germ of a variety at $\{x_1, \ldots, x_r\}$, S^m again denotes the set of points at which the dimension of S is maximal. If V^1 is a representative of V and z lies in $(V^1)^m$, then $(V_z^1)^\#$ denotes the variety-germ spanned by the germ of $(V^1)^m$ at z. Note that $(V_z^1)^\#$ is the union of all irreducible components of V_z^1 of dimension equal to $\dim V_y$. We speak of a property as holding at $V_z^\#$ for all z if it holds for representatives for all z sufficiently near y. There is $d \in D(V)$ such that d_z is a universal denominator and a non zero divisor for $V_z^\#$ for all z in V^m. If $g: S \to V$ is simple, then $g(S^m) = V^m$ and $g_z: S^\#_{g^{-1}(z)} \to V_z^\#$ is simple for all z in V^m. If S is normal, then $S_{\mathbb{C}}$ is normal and, in particular, irreducible at each point, so $S_z^\# = S_z$ is normal for all z in S^m.

The following examples should help explain our use of S^m, V^m and $V^\#$.

EXAMPLE 1.1.a: Consider the map $f(w,x,y,z) = (w,x,y,(3w(x^2+y^2)z - z^3)/2)$. Its singular set S is given by $z^2 = w(x^2+y^2)$ and $V = f_{\mathbb{C}}(S_{\mathbb{C}}) \cap \mathbb{R}$ is given by $z^2 = w^3(x^2+y^2)^3$. Note that $S^m = S \cap \{w \geq 0\}$, $V^m = V \cap \{w \geq 0\}$, and $f(S^m) = V^m$. $f|S: S \to V$ is a normalization over each point in V^m. Every $d \in D(V)$ vanishes identically on $V - V^m$ and $d \circ f$ vanishes identically on $S - S^m$. Because of this, our later proofs will fail for f off of points of S^m and V^m.

EXAMPLE 1.1.b: Consider the stable map $f(u, v, w, x) = (u, v, w, x^3 + ux = y, vx + wx^2 = z)$, which is a normalization of its image variety V. Its restriction from the ux-plane to the uy-plane is the cusp map $(u, x) \to (u, x^3 + ux)$. Suppose Q is a point in the real uy-plane satisfying $4u^3 > -27y^2$. Q has one real and two imaginary preimages under f, labeled P_1, P_2 and P_3 respectively. f is an immersion at each P_i, so $V_i = f(\mathbb{C}^4, P_i)$ are complex 4-manifold-germs in general position at Q. Complex conjugation takes P_2 to P_3, takes V_2 onto V_3 and leaves $V_2 \cap V_3$ invariant. Thus $V_2 \cap \mathbb{R}^5 = V_3 \cap \mathbb{R}^5 = (V_2 \cap V_3) \cap \mathbb{R}^5$ is a 3-manifold-germ transverse to the 4-manifold-germ $V_1 \cap \mathbb{R}^5$. V_Q is the union of these two manifold-germs and $V_Q^\#$ is $V_1 \cap \mathbb{R}^5$. The germ of f at $P_1 = f^{-1}(Q) \cap \mathbb{R}^4$ is a normalization of $V_Q^\#$, but not of V_Q.

Next we consider the passage from the analytic to the formal category. Let F_p denote the formal power series ring in p variables, with the same conventions as for O_p.

F_p is the completion of O_p with respect to the M-adic topology, where M is the unique maximal ideal in O_p. For the definition of the completion A^Δ of a topological ring A, see Chapter VIII of [26]. Let V be an analytic variety-germ at $y \in \mathbb{R}^p$. By Theorem 6 of Chapter VIII of [26], $O(V)^\Delta$ equals $F_y/I(V)F_y$, which we label $F(V)$. Let $F^w(V)$ be the integral closure of $F(V)$ in $Q(F(V)) = (F(V) - \{\text{zero divisors}\})^{-1}F(V)$. Since $F^w(V)$ is a finite $F(V)$ module, it is complete by Theorem 15 of Chapter VIII of [26], and hence it contains $O^w(V)^\Delta$. Let $f: S \to V$ be a normalization of V (they exist by [18]). Applying $\otimes_{O_y} F_y$ to the isomorphism $g^*: O^w(V) \to O(S)$ gives an isomorphism $g^*: O^w(V)^\Delta \to F(S)$ (see I.8.2 of [21]). This g^* extends to $g^*: F^w(V) \to F^w(S)$. By III.4.5 of [21], $F^w(S) = F(S)$. But g^* is monic since $g(S)$ is not contained in any proper subvariety of V. Thus $F^w(V) = O^w(V)^\Delta$ and g is a formal normalization. A formal denominator is a $d \in F(V)$ such that $dF^w(V) = F(V)$. By I.4.8 of [21], $D(V)F(V)$ is the ideal of formal denominators.

Suppose X is a set-germ at y in $\mathbb{R}^p$. Following Malgrange ([16]), we define $J(X)$ to be the set of Taylor series at y of C^∞ functions whose restrictions to X have a zero of infinite order at y. $T_y\alpha$ will denote the Taylor series at y of a C^∞ function or germ α. $K(X)$ denotes the set of C^∞ germs which vanish on X.

PROPOSITION 1.2: *If V is an analytic variety-germ at y, then* $T_yK(V^m) = J(V^m) = J(V_y^\#) = I(V_y^\#)F_y$.

PROOF: Theorem VI.3.5 of [16] states that $J(V_y^{\#}) = I(V_y^{\#})F_y$. Since $I(V_y^{\#})F_y \subset T_y K(V^m) \subset J(V^m)$, it is enough to prove that $I(V_y^{\#})F_y = J(V^m)$.

Let $V_1, \ldots, V_r$ be the irreducible components of $V_y^{\#}$. These all have the same dimension at y. In the proof of his Theorem VI.3.5, Malgrange shows that $I(V_y^{\#})F_y = I(V_1)F_y \cap \ldots \cap I(V_r)F_y$. Note that $J(V^m) = J(V_1^m) \cap \ldots \cap J(V_r^m)$: in fact, if α_i has a zero of infinite order on V_i^m, for each i, and $T_y\alpha_1 = \ldots = T_y\alpha_r$, then $T_y(\alpha_1 - \alpha_i) = 0$ for all i and hence α_1 has a zero of inifinite order on V^m at y. Thus it is sufficient to prove the proposition in the case $r = 1$. So we write V instead of $V_y^{\#}$.

Malgrange's proof now carries over almost without change. One only needs to represent V as a branched covering π over $\mathbb{R}^l$, $l = \dim V$, such that the following holds: there is an analytic function Δ on $\mathbb{R}^l$ such that $d = \Delta \circ \pi | V$ is a nonzero universal denominator for V and $\pi(V^m - \{d = 0\})$ contains a connected component of $\mathbb{R}^l - \{\Delta = 0\}$. This can be done, as is seen by Proposition III.5 of [18] (we use that $V - V^m$ has dimensions less than l, hence is contained in the complex singular set, hence is in $\{d = 0\}$). □

COROLLARY 1.3: *Suppose V and W are variety-germs. Suppose k is a diffeomorphism-germ such that $k(W^m) = V^m$. Then k induces an isomorphism $T_z k$ from $F(W_z^{\#})$ onto $F(V_{k(z)}^{\#})$ for all $z \in W^m$. Hence $T_z k$ maps $D(W_z^{\#})F(W_z^{\#})$ onto $D(V_{k(z)}^{\#})F(V_{k(z)}^{\#})$ for all $z \in W^m$.*

Suppose X (respectively, Y) is the germ of a set at $x \in \mathbb{R}^n$ (respectively, at $y \in \mathbb{R}^p$). We say $f: X \to Y$ is a C^∞ *map-germ* if it is the germ at x of the restriction to X of some C^∞ map $F: \mathbb{R}^n \to \mathbb{R}^p$. A map-germ $h: X_1 \to X_2$ is called a C^∞ *diffeomorphism-germ* if it is C^∞ and has a C^∞ inverse. We say that $f_i: X_i \to Y_i$, $i = 1, 2$, are C^∞ *equivalent* if there are C^∞ diffeomorphism-germs $h: X_1 \to X_2$ and $k: Y_1 \to Y_2$ such that $k \circ f_1 = f_2 \circ h$. Note that we do not require that the ambient spaces of X_1 and X_2 (or of Y_1 and Y_2) have the same dimension. A C^∞ map-germ $f: X \to Y$ is C^∞ *simple* (respectively, a C^∞ *normalization*) if it is C^∞ equivalent to $g: S^m \to V^m$, where $g: S \to V$ is an analytic simple map-germ (respectively, an analytic normalization).

Let E_p denote the space of germs of C^∞ functions at 0 in $\mathbb{R}^p$, with E_y its translate to $y \in \mathbb{R}^p$. For any set germ X at y, $E(X)$ denotes $E_y/K(X)$. If V is a variety-germ, $E^w(V^m)$ is defined to be $\{a/b : a \in E(V^m), b \in O(V)$, and $T_z(a/b) \in F^w(V_z^{\#})$ for all $z \in V^m\}$: elements of $E^w(V^m)$ are called *weakly C^∞ function-germs* on V^m.

PROPOSITION 1.4: *If V is normal, then $E^w(V^m) = E(V^m)$.*

PROOF: Choose some $a/b \in E^w(V^m)$. At each z in V^m, $V_z = V_z^{\#}$ is formally normal, hence $T_z b$ divides $T_z a$ in $F(V)$. Pick a representative V^1 of V, a neighborhood U of y and $a^1 \in E(U)$, $b^1 \in O(U)$ such that the germ at y of $a^1 | (V^1)^m$ is a and of $b^1 | V$ is b. Then, shrinking U if necessary, $T_z a^1$ is in $(b^1 + K((V^1)^m))F_z$ for all z in U. By Whitney's Spectral Theorem, a^1 lies in the closure of $b^1 E(U) + K((V^1)^m)$. A special case of Theorem 6.8 of [2] is that, for any Nash subanalytic set $X \subset U$, the ideal generated in $E(X)$ by $a_1, \ldots, a_s$ in $O(U)$ is closed. By the definition of the quotient topology, this means that $\{a_1, \ldots, a_s\}E(U) + K(X)$ is closed in $E(U)$. Since $(V^1)^m$ is semianalytic and hence Nash subanalytic, a^1 is in $b^1 E(U) + K((V^1)^m)$. Since $b^1 = 0$ is nowhere dense in $(V^1)^m$, a/b gives a well-defined element of $E(V^m)$. □

COROLLARY 1.5: *Suppose $g: S \to V$ is analytic, S normal, $g(S^m) \subset V^m$ and, at each $x \in S^m$, $g(S_x) = g(S_x^\#)$ is not contained in any proper subvariety of any irreducible component of $V_{g(x)}^\#$. Then $g^* E^w(V^m) \subset E(S^m)$.*

PROOF: At each point $x \in S^m$, g pulls back non zero divisors of $V_{g(x)}^\#$ to non zero divisors of S_x, so $g^* E^w(V^m)$ is contained in $E^w(S^m)$. Since S is normal, the conclusion follows from Proposition 1.4. □

PROPOSITION 1.6: *Let $g: S \to V$ be an analytic normalization. Then $g^*: E^w(V^m) \to E(S^m)$ is an isomorphism.*

PROOF: By Corollary 1.5, $g^* E^w(V^m) \subset E(S^m)$. Since g is finite, the Malgrange Preparation Theorem implies that $E(S^m)$ is generated over $g^* E(V^m)$ by $O(S)$. But $O(S) = g^* O^w(V) \subset g^* E^w(V^m)$. Thus g^* is surjective. Since $g(S^m) = V^m$, g^* is injective. □

Let V be a variety-germ. The ideal of *C^∞ denominators for V^m* is $D^\infty(V^m) = \{d \in E(V^m): dE^w(V^m) \subset E(V^m)\}$. Let $D^\infty(V^m)^\sim$ denote $\{d \in E(V^m)$: for all $z \in V^m$, $T_z d \in D(V_z^\#)F(V_z^\#)\}$.

PROPOSITION 1.7: $D^\infty(V^m) = D^\infty(V^m)^\sim$.

PROOF: Let $g: S \to V$ be a normalization. By Proposition 1.6, d is in $D^\infty(V^m)$ if, and only if, d is a relative denominator for g, i.e. $d \circ g E(S^m) \subset g^* E(V^m)$. Let $(g^* E(V^m))^\sim = \{\alpha \in E(S^m): \forall z \in V^m, \exists \beta \in F_z$ such that, $\forall x \in S^m$ with $g(x) = z$, $T_x \alpha = \beta \circ T_x g\}$. Note that

$$D^\infty(V^m)^\sim = \{d \in E(V^m): d \circ g E(S^m) \subset (g^* E(V^m))^\sim\}.$$

By Theorem 3.2 of [1], for any semiproper analytic g from a subanalytic set S to a Nash subanalytic set X, $(g^* E(X))^\sim = g^* E(X)$. Some representative of our g satisfies these hypotheses, so $(g^*E(V^m))^\sim = g^*E(V^m)$. Thus $D^\infty(V^m)^\sim = D^\infty(V^m)$. □

COROLLARY 1.8: *Suppose V and W are analytic variety-germs. If k is a C^∞ diffeomorphism-germ of W^m onto V^m, then $k^*D^\infty(V^m) = D^\infty(W^m)$.*

PROOF: This follows immediately from Corollary 1.3 and Proposition 1.7. □

PROPOSITION 1.9: *Suppose V is an analytic variety-germ and $f : X \to V^m$ is a C^∞ simple map-germ. Then $f^*E^w(V^m) \supset E(X)$.*

PROOF: There is an analytic simple map-germ $g: S \to W$ and C^∞ diffeomorphism-germs $k: W^m \to V^m$ and $h: S^m \to X$ such that $f \circ h = k \circ g$. Let $d \in O(V)$ be an analytic denominator and non zero divisor of $V_z^\#$ for all z in V^m. Since d is in $D^\infty(V^m)$, $d \circ k$ is in $D^\infty(W^m)$ by Corollary 1.8.

Since g is simple, $g^*(F^w(W_z^\#)) \supset F(S^\#_{g^{-1}(z)})$ for each z in W^m. Thus, $d \circ k \circ gE(S^m) \subset (g^*E(W^m))^\sim$, which equals $g^*E(W^m)$ by Theorem 3.2 of [1]. Pick any a in $E(X)$. There exists $b \in E(V^m)$ such that $(d \circ k \circ g)(a \circ h) = b \circ k \circ g$. At each z in W^m, $T_z(b \circ k/d \circ k)$ lies in $F^w(W_z^\#)$. Thus b/d lies in $E^w(V^m)$ and $(b/d) \circ f = a$. □

PROPOSITION 1.10: *Suppose f is a C^∞ simple map-germ and g is a C^∞ map-germ which is equivalent to an analytic map-germ $G: S \to V$. Suppose S is normal, $G(S^m) \subset V^m$ and, at each x in S^m, $G(S_x) = G(S_x^\#)$ is not contained in any proper subvariety of any irreducible component of $V^\#_{G(x)}$. If f and g have the same image, then there is a C^∞ map-germ h such that $f \circ h = g$.*

PROOF: There are diffeomorphisms k and r such that $k \circ g = G \circ r$. Let $F = k \circ f$. Let x_i be one of the coordinate functions on the ambient space of the domain X of F. By Proposition 1.9, there is an α_i in $E^w(V^m)$ such that $\alpha_i \circ F = x_i | X$. By Corollary 1.5, $H_i = \alpha_i \circ G$ is in $E(S^m)$. Let H be the map-germ whose component functions are the H_i's. Then $F \circ H = G$. Let $h = H \circ r$. Then $f \circ h = g$. □

Our main result follows immediately.

THEOREM 1.11: *Suppose f and g are C^∞ normalizations having the same image. Then f and g are right equivalent.*

If the ambient spaces of the domains of f and g are of the same dimension, then we can strengthen Theorem 1.11 in a way which is needed in Section 2.

THEOREM 1.12: *Suppose, for $i = 1$ and 2, that X_i is a set-germ at x_i in $\mathbb{R}^n$ and $f_i: X_i \to Y$ is a C^∞ normalization. Then there exists a diffeomorphism-germ $h:(\mathbb{R}^n, x_1) \to (\mathbb{R}^n, x_2)$ such that $h(X_1) = X_2$ and $f_1 = f_2 \circ (h|X_1)$.*

PROOF: By Theorem 1.11, there are C^∞ map-germs $H: X_1 \to X_2$ and $K: X_2 \to X_1$ such that $f_2 \circ H = f_1$ and $f_1 \circ K = f_2$. We want to show that H is the restriction to X_1 of a diffeomorphism-germ h. The argument comes from Mather III, which also contains the proof of the following Lemma.

LEMMA 1.13: *Let A and B be $n \times n$ matrices with entries in E_n. Then there exists an $n \times n$ matrix C with entries in E_n such that $C(I - AB) + B$ is invertible.*

Let A be $DK(x_2)$, B be $DH(x_1)$, C be as in Lemma 1.13 and h be $C(Id - K \circ H) + H$. Since $Dh(x_1)$ is invertible, h is a diffeomorphism-germ. Clearly $h|X_1 = H$. □

2. Existence and uniqueness of local right equivalences

Let $f:(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a C^∞ map-germ. Let $S(f)$ denote the critical set of f, that is, the set of points at which df has rank less than p. Let $J(f)$ denote the ideal generated by $p \times p$ minors of the Jacobian matrix of f in the ring of C^∞ germs, and let $I(S(f))$ denote the ideal of all C^∞ germs vanishing on $S(f)$. We say f is a *critical normalization* if $J(f) = I(S(f))$ and $f|S(f): S(f) \to f(S(f))$ is a C^∞ normalization.

Critical normalizations are studied in [9]. In particular, it is shown there that if f is equivalent to an analytic germ g whose complexification $g_{\mathbb{C}}$ is one-to-one on $S(g_{\mathbb{C}})$ off a codimension one subvariety of $S(g_{\mathbb{C}})$ and $j^1 g_{\mathbb{C}}$ is transverse to the first-order Thom-Boardman singularities off a codimension two subvariety of $S(g_{\mathbb{C}})$, then f is a critical normalization. Thus critical normalizations are common. The collection of critical normalizations includes all stable germs and all those Thom-Mather topologically stable germs (germs which are multitransverse to Mather's canonical stratification) which are equivalent to analytic germs. Furthermore, finitely A-determined germs (see [22], especially Section 2) are critical normalizations as long as $p > 2$ and the critical set is not an isolated point.

The quadratic differential $d^2f(0)$ of f is a quadratic form from the kernel to the cokernel of $df(0)$ (see [4]). Suppose f has rank $p - 1$ at 0.

We choose an orientation of cok $df(0)$, which is one-dimensional. Then $d^2f(0)$ has a well-defined index (the dimension of the space spanned by those eigenvectors corresponding to negative eigenvalues). The following theorem is proven in [9] using Theorem 1.12 of this paper.

THEOREM 2.1: *Suppose f and g are critical normalizations with $f(S(f)) = g(S(g))$. If either f or g has rank unequal to $p - 1$ at 0, then f and g are right equivalent. Suppose f and g have rank $p - 1$ at 0. Then $df(0)$ and $dg(0)$ have the same image and $d^2f(0)$ and $d^2g(0)$ have the same rank. We give* cok $df(0)$ *and* cok $dg(0)$ *the same orientation. If $d^2f(0)$ and $d^2g(0)$ have the same index, then f and g are right equivalent.*

Define $\mathrm{Iso}_R(f)$ (respectively $\mathrm{Iso}_R{}^0(f)$) to be the set of diffeomorphism-germs (respectively homeomorphism-germs) r such that $f \circ r = f$. We say f is a *critical simplification* if $J(f) = I(S(f))$ and f is one-to-one on an open, dense subset of $S(f)$ (in particular, critical normalizations are critical simplifications). It is obvious that $\mathrm{Iso}_R(f)$ and $\mathrm{Iso}_R{}^0(f)$ are trivial if $n < p$ (since $S(f)$ is the entire domain). We will prove that these are almost trivial if $n = p$. However, if $n > p$, these groups are infinite dimensional. The following theorem is proved in [10].

THEOREM 2.2: *Suppose $f:(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a critical simplification.* $\mathrm{Iso}_R(f)$ *contains a maximal compact subgroup G_f; all compact subgroups of* $\mathrm{Iso}_R(f)$ *are conjugate by an element in* $\mathrm{Iso}_R(f)$ *to a subgroup of G_f. If the rank of $df(0)$ is $< p - 1$, then $G_f = \{id\}$. If the rank of $df(0)$ is $p - 1$, then $G_f \simeq O(i) \times O(r - i)$, where i is the index and r is the rank of $d^2f(0)$.*

Furthermore, the quotient $\mathrm{Iso}_R(f)/G_f$ is contractible in the sense of Jänich (see [15] and [10]).

For the rest of this section we will restrict our attention to the case $n = p$.

PROPOSITION 2.3: *Suppose $f:(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a critical simplification. Then* $\mathrm{Iso}_R{}^0(f)$ *has at most two elements; if it has two elements, then the complement of $S(f)$ has exactly two connected components, which are permuted by the nontrivial element of* $\mathrm{Iso}_R{}^0(f)$.

PROOF: By assumption, grad(det(df)) is nonzero and f is one-to-one on open, dense subsets of the germ $S(f)$. Since det(df) must therefore take on both positive and negative values, the complement X of $S(f)$ has at least two connected components.

Fix $h \in \mathrm{Iso}_R{}^0(f)$. Since $f|S(f)$ is generically one-to-one, h fixes $S(f)$. Furthermore, if h fixes a regular point of f, then it fixes an open neighborhood of that point, and hence fixes the entire connected component of X of that point. Now the fold points of f are dense in $S(f)$, and it is easy to see that, at a fold point, h either fixes a neighborhood of that point or permutes the two neighboring components of X. Thus, if h preserves all components of X, then h is the identity on a neighborhood of the fold set of f, hence is the identity on each component of X, and hence is the identity.

Suppose h is not the identity. Then there exist distinct components C and $h(C)$ of X. Let Y be the union of the closures of all other components of X, let Z be the union of the closures of C and $h(C)$, and let $V = Y \cap Z$. Then $Y \backslash V$ and $Z \backslash V$ are open and closed in the complement of V. However, any fold points in Z must have C and $h(C)$ as neighboring components of X, so must lie in the interior of Z. Thus V contains no fold points, and hence is of codimension 2. Consequently, the complement of V is connected. Thus $Y \backslash V$ is empty. So C and $h(C)$ are the only components of X.

Suppose h and k are two nontrivial elements of $\mathrm{Iso}_R{}^0(f)$. Then $h^{-1}k$ preserves the two components of X and hence is the identity. □

Since $\mathrm{Iso}_R(f)$ is contained in $\mathrm{Iso}_R{}^0(f)$, it is in particular compact, hence equals its maximal compact subgroup. Thus from Theorem 2.2 we deduce:

COROLLARY 2.4: *Suppose $f:(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a critical simplification. If f is a fold, then* $\mathrm{Iso}_R(f)$ *has two elements; otherwise,* $\mathrm{Iso}_R(f)$ *is trivial.*

One must note that a rank $n-1$ germ from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^n, 0)$ with nonzero quadratic differential must be a fold.

Putting Corollary 2.4 together with Theorem 2.1, we get:

COROLLARY 2.5: *Suppose f, $g:(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ are critical normalizations and $f(S(f)) = g(S(g))$. If f is a fold, we also assume f and g have the same image. Then f and g are right equivalent. The right equivalence is unique unless f and g are folds, in which case there are two right equivalences.*

Suppose $f:(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a critical simplification. If $\mathrm{Iso}_R{}^0(f)$ has two elements, but f is not a fold germ, then we say f is a *pseudofold*.

Notice that if the cusp locus of f contains the origin in its closure, then f is not a pseudofold at the origin (for $h \in \mathrm{Iso}_R{}^0(f)$ must be the identity in a neighborhood of the cusp set, and hence must preserve components of the complement of $S(f)$). This shows that the Thom-Mather topologically stable maps have no pseudofolds.

EXAMPLE 2.6: The maps $f_t(x, y) = (x, y^{2t} + xy)$, $t > 1$, have pseudofold points at the origin. These maps are one-to-one on their singular sets and two-to-one over non-critical values. They send a disk about the origin to the exterior of a higher order cusp.

EXAMPLE 2.7: Consider the germ f at 0 of the map

$$F(x_1, \ldots, x_{n-1}, y) = (x_1, \ldots, x_{n-1}, y^4 + (x_1^2 + \ldots + x_{n-1}^2)y^2).$$

This germ is stable in a deleted neighborhood of the origin. It has been shown in [8] that such a germ is finitely C^k-A-determined for all k, $0 \leq k < \infty$. There are arbitrarily high order perturbations of f which are finitely C^∞-A-determined, since being finitely C^∞-A-determined is a general property among the K-simple map-germs (see [6]). Thus, there is a finitely C^∞-A-determined g (which is therefore a critical simplification, and even a critical normalization if $n > 2$) which is C^k equivalent to f. The C^k isotropy group of g must have order 2 since the C^∞ isotropy group of f is of order 2. Thus such a g is a pseudofold.

PROPOSITION 2.8: *Suppose f, $g: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ are critical normalizations, but not pseudofolds. If $f = g \circ h$, where h is a homeomorphism-germ, then h is a C^∞ diffeomorphism-germ.*

PROOF: Since f and g are critical normalizations, they are not local homeomorphisms at any singular point. Thus $h(S(f)) = S(g)$. Consequently, $f(S(f)) = g(S(g))$; also f and g have the same image. By Theorem 2.1, there is a C^∞ diffeomorphism-germ r such that $f \circ r = g$. Thus $r \circ h$ is in $\mathrm{Iso}_R{}^0(f) = \mathrm{Iso}_R(f)$. Thus h is a C^∞ diffeomorphism-germ. □

3. Global right equivalence

In this section, we give extensions of the local right equivalence results of Sections 1 and 2 to the global case. We consider C^∞ maps $f: N^n \to P^p$.

The case $n < p$ is easily dealt with. In this case f is called a C^∞ *normalization* (of $f(N)$) if, for each $y \in f(N)$, $S = f^{-1}(y)$ is finite and the germ $f_S: N_S \to f(N)_y$ is a C^∞ normalization, as defined for germs in Section 1. A priori, $f(N)_y$ need not equal $f_S(N_S)$; however, if it does not, f_S can't be a C^∞ normalization of $f(N)_y$. Thus, a useful reformulation of our definition is: for each $y \in f(N)$, $S = f^{-1}(y)$ is finite, the germ $f_S: N_S \to f_S(N_S)$ is a C^∞ normalization, and $f: N \to f(N)$ is proper (which is weaker than requiring that $f: N \to P$ be proper).

THEOREM 3.1: *Suppose* $f_i: N_i^n \to P^p$, $n < p$, *are* C^∞ *normalizations of* $f_1(N_1) = f_2(N_2)$. *Then* f_1 *and* f_2 *are right equivalent.* (*In particular,* N_1 *and* N_2 *are diffeomorphic.*)

PROOF: By Theorem 1.11, there exists for each $y \in f_1(N_1)$ a diffeomorphism-germ $h_y: (N_1, f_1^{-1}(y)) \to (N_2, f_2^{-1}(y))$ such that $f_2 \circ h_y = f_1$. Since these diffeomorphism-germs are unique, they must be compatible. Hence they piece together to give $h: N_1 \to N_2$ such that $f_2 \circ h = f_1$. □

The case $n = p$ is more complicated. The principal reason is that right equivalences between two fold germs, or between two covering space map-germs (at a finite set), are not unique, and hence do not necessarily piece together to form a global equivalence.

For the moment, we consider the case of general n and p. While the definitions and results of Section 2 were stated for germs at 0 in Euclidean space, they extend immediately to include germs at a finite set in a manifold. A C^∞ map $f: N \to P$ is called a *critical normalization* if, for each $y \in f(S(f))$, $S = f^{-1}(y) \cap S(f)$ is finite, the germ $f_S: N_S \to P_y$ is a critical normalization, and $f|S(f): S(f) \to f(S(f))$ is proper. When $n < p$, the critical set $S(f) = N$, so the notions of critical normalization and C^∞ normalization coincide.

All stable mappings are critical normalizations, as are those mappings of finite A-codimension in dimensions $p > 2$ which have no isolated critical points (we claim that a finite A-codimension map f must satisfy: $f|S(f): S(f) \to f(S(f))$ is proper).

From now on, we will restrict our attention to the case $n = p$. Suppose f and g are two critical normalizations with $f(S(f)) = g(S(g))$. The same argument as for Theorem 3.1 implies that $f|S(f)$ and $g|S(g)$ are right equivalent, that is there is a diffeomorphism r from a neighborhood of $S(f)$ to a neighborhood of $S(g)$ such that $r(S(f)) = S(g)$ and, letting $h = r|S(f)$, $f|S(f) = g \circ h$.

Let $F(f)$ denote the set of fold points of f and let $C(f)$ be $S(f)\backslash F(f)$. We can apply Corollary 2.5 to show that r can be chosen in a neighborhood of $C(f)$ so that $f = g \circ r$ on this neighborhood.

Let V be a connected component of $S(f)$ containing at least one point of $C(f)$, and let $V^1 = h(V)$. Let $W = f(V) = g(V^1)$. Choose some x in $F(f) \cap V$. There is a continuous curve $\Gamma \subset V$ connecting x to some point y in $C(f) \cap V$, with $\Gamma \cap C(f) = \{y\}$. Since $f = g \circ r$ in a neighborhood of y, there is a fold point z in Γ at which $f = g \circ r$; thus f near z and g near $h(z)$ fold in the same direction (that is, there are neighborhoods U_1 of z and U_2 of $h(z)$ such that $f(U_1)$ and $g(U_2)$ lie on the same side of $f(U_1 \cap S(f)) = g(U_2 \cap S(g))$). Since folding in the same direction and folding in the opposite direction are both open properties, it follows that f and g must fold in the same direction at x, as well.

Now let V be a component of $S(f)$ consisting entirely of fold points, and let $V^1 = h(V)$. We say f and g satisfy the *fold condition* if they fold in the same direction at one, and hence every, pair of corresponding points on V and V^1, for each such V. Assume that f and g satisfy the fold condition. Then, at each x in $F(f)$, there are exactly two diffeomorphism-germs R_x such that $f_x = g_{h(x)} \circ R_x$ (where f_x denotes the germ of f at x). It is not necessarily possible to piece together the R_x's to get a right equivalence defined in a neighborhood of $S(f)$.

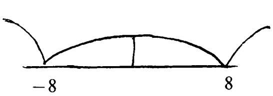
EXAMPLE 3.2: Let M be the annulus $[0,1] \times (-1,1)$ with $(0,y)$ and $(1,y)$ identified, for all $y \in (-1,1)$. Let N be the Möbius strip $[0,1] \times (-1,1)$ with $(0,y)$ and $(1,-y)$ identified, for all $y \in (-1,1)$. Let $f: M \to M$ and $g: N \to M$ be the maps induced by $(x,y) \to (x,y^2)$. Both f and g are C^∞ stable mappings, hence critical normalizations. They have the same image of their critical sets and they satisfy the fold condition, but they are not right-equivalent in any neighborhood of their critical sets.

In order to get f and g right equivalent in a neighborhood of their fold sets, we need that their fold sets have equivalent normal bundles. There is a simple cohomological condition for this. Let $w_1(f)$ denote the first Stiefel-Whitney class of the normal bundle to $F(f)$ in N_1. We say that *f and g have the same normal structure at folds* if $w_1(f) = h^* w_1(g)$, where h is the diffeomorphism from $S(f)$ to $S(g)$ such that $f|S(f) = g \circ h$.

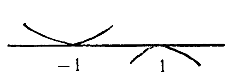
Let us now assume that $w_1(f) = h^* w_1(g)$. Let Γ be a simple closed curve in $F(f)$, and let i be the inclusion map of Γ into $F(f)$. Then the restriction to Γ of the normal bundle to $F(f)$ is orientable, and so trivial, if, and only if, $i^* w_1(f) = 0$. (See for example [14] for a detailed discussion of the properties of Stiefel-Whitney classes.) Thus the restriction to Γ of the normal bundle to $F(f)$ is trivial if, and only if, the restriction to $h(\Gamma)$ of the normal bundle to $F(g)$ is trivial. Pick a point $x \in \Gamma$ and sections v_1 and v_2 into the normal bundles along Γ and $h(\Gamma)$, respectively, which are nonvanishing at each point and continuous except possibly at x and $h(x)$. The map $v_1(y) \to v_2(h(y))$ determines one of the two diffeomorphism-germs R_y with the property $f_y = g_{h(y)} \circ R_y$, for each $y \in \Gamma$. Furthermore, the R_y's agree all along Γ, except possibly at $y = x$. The left and right hand limits of $v_1(y)$ at x will lie on the same side of the fold set if, and only if, the normal bundle is trivial along Γ if, and only if, the left and right hand limits of $v_2(y)$ at $h(x)$ lie on the same side of the fold set. Thus the R_y's agree at x, as well. Using this, we see that f and g are right equivalent in a neighborhood of the fold set, and at each connected component of the fold set there are exactly two germs of right equivalences.

Let F be a connected component of $F(f)$, and let C be a connected component of $C(f)$ such that C touches the closure of F. The unique right equivalence between f and g defined in a neighborhood of C is compatible with one of the two right equivalences defined in a neighborhood of F. We say that C determines this right equivalence at F. Another component of $C(f)$ might determine the other right equivalence at F, in which case f and g are not right equivalent in any neighborhood of their critical sets.

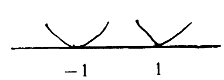
EXAMPLE 3.3: The function $a(x) = (3x^5 - 10x^3 + 15x,\ x^4 - 2x^2 + 1)$ is an immersion except at $x = \pm 1$. The image curve has simple cusps at $(\pm 8, 0)$ (see the illustration below). The vector $b(x) = (15(x^2 - 1), 4x)$ is a non-zero tangent vector to this image curve at $a(x)$. Let $f(x, y) = a(x) + yb(x)$. This f is a stable map with critical set the x-axis and cusps at $x = \pm 1$. Since f has only fold singularities along $A = \{0 \leq x \leq \frac{1}{2},\ y = 0\}$, there is an orientation-reversing diffeomorphism h defined on a neighborhood U of the positive x-axis such that $f \circ h = f$ on some open neighborhood of A. Let g be defined to be $f \circ h$ on U and f on some sufficiently small neighborhood of the negative x-axis. This is well-defined since $f = f \circ h$ on the intersection of the two neighborhoods. g is a stable map with critical set $S = S(f)$ and $g|S = f|S$. But $f^{-1}f(S) - S$ has one connected component on each side of S, whereas $g^{-1}g(S) - S$ has both its components on the same side of S. Though the germs of f and g are right equivalent at each point of the x-axis, f and g are not right equivalent an any neighborhood of the x-axis. The right equivalence at one cusp is orientation preserving while that at the other cusp is orientation reversing. These cusps therefore determine different right equivalences at $(-1, 1) \times \{0\}$.



$f(S) = g(S)$



$f^{-1}f(S)$ near S



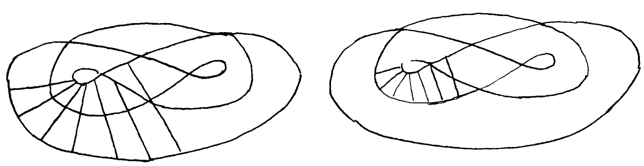
$g^{-1}g(S)$

We have proved the following.

THEOREM 3.4: *Suppose $f,\ g: N^n \to P^n$ are critical normalizations with $f(S(f)) = g(S(g))$. Suppose f and g satisfy the fold condition and have the same normal structure at folds. Finally, suppose that for each component F of the fold set of f, the components of $C(f)$ touching the closure of F all determine the same right equivalence at F. Then f and g are right equivalent in a neighborhood of their critical sets.*

The next question is: if f and g are right equivalent in neighborhoods of their critical sets, then can the right equivalences be extended globally? Let U be the domain of the given right equivalence r, and let M_1 be a closed submanifold-with-boundary of N_1 such that $M_1 \subset N_1 - S(f)$ and $M_1 \cup U = N_1$ (and so $\partial M_1 \subset U$). Assume $r(\partial M_1)$ is the boundary of a closed submanifold $M_2 \subset N_2 - S(g)$. Then the restrictions of f to the boundary and to the interior of M_1 are immersions, as are the corresponding restrictions of g. The question of global right equivalence of f and g reduces to the question: can the right equivalence $r|\partial M_1$ between $f|\partial M_1$ and $g|\partial M_2$ be extended to a right equivalence between $f|M_1$ and $g|M_2$? The following example shows that this is not always possible, even if M_1 and M_2 are diffeomorphic.

EXAMPLE 3.5: (Also see [23] and [3] for this example.) There are two non right equivalent immersions i and j of the disk D into the plane as in the illustration below (one should superimpose the two pictures, that is, i and j have the same image).



i j

Let π_N be the projection of the northern hemisphere H_N onto D and let π_S be the projection of the southern hemisphere H_S onto D. We define f by $f|H_N = i \circ \pi_N$, $f|H_S = i \circ \pi_S$, $g|H_N = i \circ \pi_N$, $g|H_S = j \circ \pi_S$. Then $S(f) = S(g) =$ the equator, $f(S(f)) = g(S(g))$, and f and g have only fold singularities. It is easy to verify that f and g satisfy the fold condition, and since the normal bundle to the equator is trivial, f and g have the same normal structure at folds. Thus there is a right equivalence between f and g which is defined in some neighborhood of $S(f)$. Choosing this equivalence so that it preserves hemispheres, we see that it extends over the northern hemisphere. The problem of extending it over the southern hemisphere is precisely the problem of finding a right equivalence between i and j.

Thus one wants to classify up to right equivalence immersions of a manifold with boundary (into another manifold of the same dimension) which extend a given immersion of the boundary. This has been done for immersions of D^2 into $\mathbb{R}^2$ by Blank in [3], and for more general 2-manifolds by several people (see [11]). While the problem in higher dimensions is probably very hard, it can be solved for some special cases.

An alternate approach to the problem of extending local right equivalences to global ones was developed by Wilson in [23] for stable maps between 2-manifolds. In this alternate approach, one stratifies the target using $f(S(f))$, and one pulls back this stratification by f and g. Then f and g restricted to source strata are covering spaces over target strata. One refines the target stratification so that all the strata are simply connected, inducing refinements of the source stratifications. Now the restrictions of f and g are diffeomorphisms between strata. This reduces the problem of finding a homeomorphism h such that $f = g \circ h$ to that of finding a map of stratifications H such that H is one-to-one, $H(\operatorname{st} S) = \operatorname{st}(H(S))$ (where $\operatorname{st} S$ is the set of strata whose closure contains S) and $f(S) = g(H(S))$ for all strata S. The problem then is to show that if $f = g \circ h$, h must be a diffeomorphism. The next theorem shows that this is often true.

THEOREM 3.6: *Suppose f and g are critical normalizations without pseudofold points. If $f = g \circ h$ for some homeomorphism h, then h must be a C^∞ diffeomorphism.*

This is an immediate consequence of Proposition 2.8.

Note that Thom–Mather topologically stable mappings have cusp set dense in the set of non-fold singular points, and hence have no pseudo-fold points.

In a related question. Damon in [5] has given an example of topologically stable maps which are topologically right-left equivalent but not C^∞ right-left equivalent. In Damon's example, $f(S(f))$ and $g(S(g))$ are not C^∞ diffeomorphic. Theorem 3.1 and 3.6 show that, for $n \leq p$, C^0 right-C^∞ left equivalence implies C^∞ right-left equivalence.

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