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MEAN-VALUE THEOREMS AND ERGODICITY OF CERTAIN GEODESIC RANDOM WALKS

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Abstract

We give some geometric conditions which guarantee that all the invariant functions of the spherical mean operator with certain radius on a Riemannian manifold are necessarily constant. A geometric model of a Markov process, so-called geodesic random walks, whose transition operator is the spherical mean, plays a fundamental role in our argument.

1. Introduction

Let M be a connected complete Riemannian manifold without boundary. Throughout we assume $\dim M \geq 2$. The spherical mean (operator) with radius r (≥ 0) on M is the operator L_r defined by

$$(L_r f)(x) = \int_{S_x M} f(\exp rv)\, dS_x(v),$$

where dS_x is the normalized uniform density on the unit sphere $S_xM = \{v \in T_xM; \|v\| = 1\}$. If $M = \mathbf{R}^n$ (with the standard metric), L_r is the classical spherical mean, and invariant functions of L_r are just harmonic functions. To be exact, a locally integrable function f on $\mathbf{R}^n$ is harmonic if and only if $L_r f = f$ for sufficiently small $r < \varepsilon$. A direct generalization of this classical mean-value theorem is the following.

THEOREM A: *There exists a family of self-adjoint elliptic operators* $\{P_k\}_{k=1,2,\cdots}$ *with* $P_1 = \Delta$ *such that, if* $L_r f = f$ *for sufficiently small* $r < \varepsilon$,

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then $P_k f \doteq 0$ for all k, and conversely if M is a real analytic Riemannian manifold, and if $P_k f = 0$ for all k, then $L_r f = f$ for $r \geq 0$.

This has been essentially proven in [8]. In fact this is almost equivalent to the formal expansion of L_r with respect to r;

$$L_r \sim I + \frac{\Delta}{2n} r^2 + \sum_{k=2}^{\infty} P_k r^{2k},$$

which, in the classical case, reduces to the Pizzetti's formula

$$L_r \sim \Gamma(\tfrac{1}{2}n) \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k!\Gamma\left(\frac{n}{2}+k\right)} \Delta^k.$$

We should point out that this kind of infinitesimal properties of L_r is usefull in characterizing Riemannian manifolds in terms of mean-value properties. But our discussion will not enter into this direction because the global character of L_r is our concern.

Suppose now M is compact. Since harmonic functions on such a M are constant, any integrable function f such that $L_r f = f$ for sufficiently small $r < \varepsilon$ is necessarily constant. An interesting thing is that, as we have showed in the previous paper [8], this is true even for a function with $L_r f = f$ for a fixed $r > 0$. For instance, if the restriction of the exponential mapping $\exp_x$ to the sphere rS_xM of radius r is an immersion for every point x in M, then the number 1 is a simple eigenvalue of the operator $L_r : L^2(M) \to L^2(M)$. For brevity, we call L_r *ergodic* if one can conclude the simplicity of the eigenvalue 1, which, as is known, is equivalent to the ergodicity of the Markov process on M whose transition operator is L_r. The primary purpose of this paper is to give a somewhat relaxed criterion of ergodicity, which, in some sense, resembles the criterion in the case of finite Markov chains.

THEOREM B: *Let M be a compact Riemannian manifold. If there exists a point x in M such that almost all points can be joined to x by r-geodesic chains of finite length, then L_r is ergodic. In particular, if any two points in M can be joined by an r-geodesic chain, then L_r is ergodic.*

Here r-geodesic chains of length k, k being a natural number, are continuous mappings $c: [0, k] \to M$ such that all restrictions $c|[i-1, i]$ $(i = 1, 2, \ldots, k)$ are geodisic curves with the same length r. Two points

x and y are said to be joined by an r-geodesic chain $c:[0,k] \to M$ if $c(0) = x$ and $c(k) = y$.

Our proof of Theorem B is quite elementary and supercedes the previous one [8] which relies heavily on regularity of Fourier integral operators and can be applied to only the limited case.

In connection with the above theorem, a natural question arises here. What kind of geometric condition guarantees that any two points are joined by r-geodesic chains? As was shown in [8], this is the case if $\exp_x : rS_xM \to M$ is an immersion for every point x. We will see in §3 that this condition is relaxed in the following way.

THEOREM C: *Let M be a complete Riemannian manifold. Suppose that for any point x in M there exist a natural number k and a vector $v \in krS_xM$ such that* $\exp_x : krS_xM \to M$ *is an immersion in a neighborhood of v. Then any two points in M can be joined by an r-geodesic chain of even length.*

As is illustrated by the example $M = S^n(1)$, $r = \pi$ or 2π, our assumption for the exponential mapping can not be omitted. On the other hand, if M is non compact, then the assumption in Theorem C is always satisfied, since one can find a geodesic ray through a point. Hence, if the fundamental group of a compact M is infinite, then one concludes that every two points are joined by r-geodesic chains. Together with Theorem B, one has

THEOREM D: *If $\pi_1(M)$ is infinite, then L_r is ergodic for any $r > 0$.*

In the last part of our discussion, we will see that two dimensional manifolds for which ergodicity of L_r is not satisfied have very remarkable properties.

REMARK: There are several references which are concerned with different kind of mean-value operators ([3] [4] [5] [7] [10]).

2. Proof of Theorem B

In view of ergodic theory of Markov processes, it is enough to prove that, for every pair of Borel sets A and B in M with positive measure, there exists a natural number k such that

$$\int_A L_r^k \chi_B dx > 0$$

(see [11]). We set

$$S_x^k M = S_x M \times \ldots \times S_x M, \text{ the } k\text{-ple product,}$$

here k is possibly infinite. We let $S^k M$ be the fiber bundle on M with fiber $S_x^k M$. The product probability measure on $S_x^k M$ and the canonical measure on M give rise to a fiber product measure P_k on $S^k M$. We identify $S_x^k M$ with the set of all r-geodesic chains of length k issued from x, by using parallel translations. This identification allows us to define a mapping

$$\pi_l : S^k M \to M \times M \qquad (0 \leq l \leq k)$$

by $\pi_l(c) = (c(0), c(l))$. The assumption in Theorem B is then equivalent to the union $\bigcup_{k=1}^{\infty} \pi_k(S^k M)$ having full measure in $M \times M$. As was shown in [8], the process $\tilde{\omega}_k : S^\infty M \to M$ defined by $\tilde{\omega}_k(c) = c(k)$ is a Markov process with the transition operator L_r, hence we have

LEMMA 1:

$$P_k(\pi_k^{-1}(A \times B) = \int_A L_r^k \chi_B dx$$

Therefore what we have to prove reduces to the following general lemma.

LEMMA 2: *Let* $\{\varphi_k : X_k \to Y, k = 1, 2, \ldots\}$ *be a family of smooth mappings of smooth paracompact manifolds such that the union* $\cup\, \varphi_k(X_k)$ *has full measure in Y. Then for any Borel subset A in Y with positive measure, there exists some k such that* $\varphi_k^{-1}(A)$ *has positive measure.*

PROOF: Let K_k be the set of critical value of φ_k, which, by the Sard's theorem (see [6]), has measure zero. The countable union $\cup\, K_k$ has also measure zero. One can choose a point y in $\cup\, \varphi_k(X_k) \backslash \cup\, K_k$ such that any open neighborhood of y and A have intersection with positive measure. Let $x_k \in X_k$ with $\varphi_k(x_k) = y$. Since φ_k is a submersion in a neighborhood of x_k, the inverse image $\varphi_k^{-1}(A)$ has positive measure, as desired.

Instead of L_r, consider the iterated operator L_r^2, which is also regarded as a transition operator of cerain Markov process. Applying a similar argument to L_r^2, we observe that 1 is a simple eigenvalue of L_r^2 whose eigen-functions are constant, provided that there exists a point x in M such that almost all points are joined to x by an r-geodesic chain of *even* length. In particular, we have

THEOREM E: *-1 is not an eigenvalue of L_r provided that there exists a point x in M such that the set of points joined to x by r-geodesic chains of even length has full measure in M.*

3. Geometry of geodesic chains

If an r-geodesic chain c corresponds to $(v_1, \ldots, v_k) \in S^k M$, we call c the chain associated with $(v_1, \ldots, v_k)$, and put $\tilde{\omega}_k(v_1, \ldots, v_k) = c(k)$. Let h and k be positive integers. Define a mapping $\tilde{\omega}_{h,k} : S_x M \times S_x M \to M$ by setting

$$\tilde{\omega}_{h,k}(u, v) = \exp_{\exp(hru)} P_{hru}(krv)),$$

where $P_{hru} : T_x M \to T_{\exp(hru)} M$ is the parallel translation along the geodesic curve: $t \mapsto \exp(thru)$ $(0 \le t \le 1)$. Then the diagram

$$\begin{array}{ccc} S_x M \times S_x M & \xrightarrow{\tilde{\omega}_{h,k}} & \\ \downarrow & & M \\ S^{h+k} M & \xrightarrow{\tilde{\omega}_{h+k}} & \end{array}$$

is commutative, where the vertical arrow is given by

$$(u, v) \mapsto (\underbrace{u, \ldots, u}_{h}, \underbrace{v, \ldots, v}_{k}).$$

From the assumption in Theorem C, one may choose vectors u and v in $S_x M$ such that

$$\exp : hrS_x M \to M$$

$$\exp : krS_{\exp(hru)} M \to M$$

are immersion around the points hru and $krP_{hru}(v)$ respectively. Note that one may choose such vectors with $u \neq \pm v$.

LEMMA 3: *$\tilde{\omega}_{h,k}: S_xM \times S_xM \to M$ is a submersion around the point (u, v) provided that $u \neq \pm v$.*

PROOF: From the Gauss' lemma it follows that

$$d\tilde{\omega}_{h,k}(0 \oplus T_vS_xM) = \text{the orthogonal complement of } \varphi_{kr}(P_{hru}(v)) \text{ in } T_{\tilde{\omega}_{h,k}(u,v)},$$

where $\varphi_t: SM \to SM$ is the geodesic flow. Given a $X \in T_uS_xM$, there is a Jacobi field J_x along the curve

$$t \mapsto c(t) = \exp(tP_{hru}(krv))$$

such that

$$J_X(1) = d\tilde{\omega}_{h,k}(X \oplus 0)$$
$$J_X(0) = (d_{hrv}(\exp_x))(hrX)$$
$$(\nabla_{P_{hru}(v)} J_X(0), P_{hru}(v)) = 0.$$

In fact, J_X is given as the infinitesimal variation of c associated with the variation

$$c_s(t) = \exp(tP_{hru(s)}(krv)), \qquad -\varepsilon < s < \varepsilon,$$

where $s \mapsto u(s)$ is a curve in S_xM with $u(0) = u$, $du(0)/ds = X$. We show that there exists some vector X in T_uS_xM such that $(J_X(1), \varphi_{kr}(P_{hru}(v))) \neq 0$, which certainly implies the assertion. Suppose it is not the case. Since

$$\frac{d^2}{dt^2}(J_X(t), \dot{c}(t)) = 0$$
$$(\nabla_{\dot{c}} J_X(0), \dot{c}(0)) = k^2r^2(\nabla_{P_{hru}(v)} J_X(0), P_{hru}(v)) = 0,$$

we find that

$$0 = (J_X(1), \varphi_{kr}(P_{hru}(v))) \equiv \frac{1}{kr}(J_X(t), \dot{c}(t)) \equiv \frac{1}{kr}(J_X(0), \dot{c}(0)) = (J_X(0), P_{hru}(v)).$$

Using again the Gauss' lemma, we have

$\{J_X(0); X \in T_u S_x M\}$ = the orthogonal complement of $\varphi_{hr}(u)$ in $T_{\exp(hru)}M$,

from which it follows that $P_{hru}(v) = \pm\varphi_{hr}(u) = \pm P_{hru}(u)$, or equivalently $u = \pm v$, contradicting our choice of u and v.

PROOF OF THEOREM C: Take $(u, v) \in S_x M \times S_x M$ as above. For brevity we set

$$y = \tilde{\omega}_{h,k}(u, v)$$

$$v^* = -\varphi_{kr}(P_{hru}(v))$$

$$u^* = -P_{krv}(\varphi_{hr}(u)).$$

It is easy to see that the associated chain to the $k + h - ple$ vectors

$$(\underbrace{v^*, \ldots, v^*}_{k}, \underbrace{u^*, \ldots, u^*}_{h}) \in S_y^{k+h}M$$

is just the chain obtained by traversing the chain associated to $(u, \ldots, u, v, \ldots, v)$ in the opposite direction. Since, in general, exp: $hrS_xM \to M$ is an immersion around the point hru if and only if x and $\exp(hru)$ is not conjugate along the geodesic : $t \mapsto \exp(hrtu)$ $(0 \leq t \leq 1)$, we observe that

$$\exp: hrS_{\exp(hru)}M \to M$$

$$\exp: krS_yM \to M$$

are immersions around the points $-hr\varphi_{hr}(u)(= hrP_{krv^*}(u^*))$ and krv^* respectively. Since $v^* \neq \pm u^*$ if and only if $u \neq \pm v$, we may apply the above lemma to the mapping $\tilde{\omega}_{k,h}: S_yM \times S_yM \to M$, that is, $\tilde{\omega}_{k,h}$ is a submersion around the point (v^*, u^*). From the commutative diagram $(*)$, it follows that $\tilde{\omega}_{k+h}: S_y^{k+h}M \to M$ is a submersion around the point $(v^*, \ldots, v^*, u^*, \ldots, u^*)$, so that the image of $\tilde{\omega}_{k+h}$ contains an open neighborhood U of x. Connecting the chain associated to $(u, \ldots, u, v, \ldots, v)$ with the chains issued from y associated to the $(k + h)$-ple vectors of the form $(v_1, \ldots, v_1, u_1, \ldots, u_1)(u_1, v_1 \in S_yM)$, we obtain r-geodesic chains of length $2(h + k)$ whose end points fill up U. In other words, any point in U can be joined to x by an r-geodesic chain of length $2(h + k)$. Note that the relation given by setting $x \sim y$ iff x and y are joined by an r-geodesic chain of even length is an equivalence relation. What we have proved is that each equivalence class is open. Since M is connected, this completes the proof.

REMARK: Under the assumption of Theorem C, we may further prove that, in the case M is compact, there is a positive integer k_0 such that any two points can be joined by an r-geodesic chain of length k_0.

Suppose M is not compact. For each point x in M, one may find a geodesic ray $c:[0,\infty)\to M$ with $c(0)=x$ (see [2]). The point x is not conjugate to $c(r)$ along c for any $r>0$. Therefore the assumption in Theorem C is always satisfied in this case. We should note the argument in §2 is valid to complete manifolds with finite volume, since the total space $S^\infty M$ has also finite volume and one can apply the ergodic theory. Thus we obtain the following which is the contrast to compact cases.

THEOREM F: *If M is a complete non compact Riemannian manifold with finite volume, then L_r is ergodic, and -1 is not eigenvalue for any $r>0$.*

We now apply Theorem D to the case of surfaces. Since compact 2-dimensional manifolds with finite $\pi_1(M)$ are S^2 or $P^2(R)$, we have

THEOREM G: *If M is a 2-dimensional compact manifold, not diffeomorphic to S^2 nor $P^2(R)$, then L_r is ergodic for $r>0$. A metric on S^2 or $P^2(R)$ for which L_r is not ergodic must be a Y_l^m-metric ($l=2r$) in the sense of A. L. Besse* [1]. *Namely, if M is not ergodic, then there must be a point m in M such that all the geodesic issued from m come back to m at length $2r$.*

It remains only to prove the last part. From Theorem B and C, it follows that, if L_r is not ergodic, we may find a point m in M such that the rank of $\exp_m|rS_mM$ is zero, that is, $\exp_m(rS_mM)=n\in M$. Thus it suffices to show

LEMMA 4: *Let M be a complete Riemannian manifold such that there are points m, n in M with $\exp(rS_mM)=n$. Then all the geodesics issued from m come back to m at length $2r$, and $\exp(rS_nM)=m$.*

PROOF: Since $\varphi_r(S_mM)\subset S_nM$, the restriction $\varphi_r|S_mM$ is necessarily a diffeomorphism of S_mM onto S_nM, thus for any $u\in S_mM$, there exists a vector $v\in S_mM$ such that $\varphi_r(v)=-\varphi_r(u)$. Since $-\varphi_r(u)=\varphi_{-r}(-u)$, we get $\varphi_{2r}u=\varphi_r(-\varphi_rv)=-v$, which implies $\exp(2rS_mM)=m$.

REMARK: If M is a Y_l^m-manifold for each point $m\in M$, then $L_l=Id$.

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