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CARSTEN SCHÜTT

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ON THE VOLUME OF UNIT BALLS IN BANACH SPACES

Carsten Schütt

Abstract

We estimate the volume ratio of $l_n^p \otimes_\pi l_n^r$, $1 \leq p, r \leq \infty$, unitary operator ideals and symmetric spaces. We also study the structure of the n-dimensional James space.

We consider the volume of unit balls in finite dimensional Banach spaces and an invariant of such spaces, the volume ratio.

We start with giving estimations for the volume of unit balls. In particular, for spaces with 1-unconditional bases we get simple formulas. These formulas are extended in a natural way to spaces without 1-unconditional bases. Then, we estimate the volume ratio of 1-symmetric spaces and of $l_n^p \otimes_\pi l_n^r$, $1 \leq p, r \leq \infty$, a problem posed in [11]. We get in an easy way estimations for the volume ratio of the n-dimensional James space J_n. We also show that the Banach–Mazur distance of J_n and l_n^2 is at most of the order $\log n$ and that the ké-constant of J_n [8] is uniformly bounded.

Other aspects concerning volumes of unit balls are considered in [12].

0. Preliminaries

In this paper we estimate volumes of unit balls of the space $\mathbb{R}^n$ provided with various norms. The measure is the usual Lebesgue measure. Since we always consider the $\mathbb{R}^n$ it is also clear what we understand by the natural identity between two spaces.

We denote the unit ball of a Banach space E by B_E and the unit sphere in l_n^2 by S^{n-1}. The volume ratio of a space E is given by

$$\mathrm{vr}(E) = \inf_{\varepsilon \subset B_E} \left(\frac{\mathrm{vol}(B_E)}{\mathrm{vol}(\varepsilon)} \right)^{\frac{1}{n}}$$

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where ε is an ellipsoid. By $\varkappa_n$ we denote the volumes of the unit balls in l_n^2. The Banach–Mazur distance of two Banach spaces E and F is defined by

$$d(E,F) = \inf\{\|I\|\,\|I^{-1}\| \mid I \in L(E,F),\ I \text{ is isomorphism}\}$$

One has

$$\mathrm{vr}(E) \le \mathrm{vr}(F)\,d(E,F)$$

By C_E we understand the unitary operator ideal with the norm induced by the symmetric space E. If $E = l_n^p$ we write C_p.

The ε-tensor product is the tensor product with the smallest tensor norm and the π-tensor product that with the biggest tensor norm. We say that $\{e_i\}_{i=1}^n$ is a C-unconditional basis of E if

$$\|\sum_{i=1}^n a_i e_i\| \le C\,\|\sum_{i=1}^n \varepsilon_i a_i e_i\|$$

for all $a_i \in \mathbb{R}$, $\varepsilon_i = \pm 1$, $i = 1,\ldots,n$, and that it is C-symmetric if

$$\|\sum_{i=1}^n a_i e_i\| \le C\,\|\sum_{i=1}^n \varepsilon_i a_{\pi(i)} e_i\|$$

for all $a_i \in \mathbb{R}$, $\varepsilon_i = \pm 1$, $i = 1,\ldots,n$ and all permutations π of $\{1,\ldots,n\}$.

The dual basis is denoted by $\{e_i^*\}_{i=1}^n$. If the indices are too awkward we write $e(i)$ instead of e_i.

The n-dimensional James space is $\mathbb{R}^n$ with the norm

$$\|x\| = \sup_{p_1 < \ldots < p_m} (|\xi_{p_1} - \xi_{p_2}|^2 + |\xi_{p_2} - \xi_{p_3}|^2 + \ldots$$
$$\ldots + |\xi_{p_{m-1}} - \xi_{p_m}|^2 + |\xi_{p_m}|^2)^{\frac{1}{2}} \tag{0.1}$$

where the sup is taken over all strictly increasing sequences.

The volume of the Euclidean unit ball is

$$\varkappa_n = \frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)}$$

1. Basic estimates for volumes of unit balls

We introduce here estimates for the volumes of unit balls. We start with the case of E having an unconditional basis. It is possible to estimate the volume in terms of certain vectors. The existence of these vectors were proved in [4], [7].

LEMMA 1.1: *Let $\{e_i\}_{i=1}^n$ be an 1-unconditional basis of E. Then there is a sequence $\{s_i\}_{i=1}^n$ of real numbers such that*

$$\|\sum_{i=1}^n s_i e_i\| \, \|\sum_{i=1}^n s_i^{-1} e_i^*\| = n.$$

LEMMA 1.2: *Let $\{e_i\}_{i=1}^n$ be an 1-unconditional basis of E. For every sequence s_i, t_i, $i = 1, \ldots, n$ such that $s_i t_i = 1/n$ and*

$$\|\sum_{i=1}^n s_i e_i\| = \|\sum_{i=1}^n t_i e_i^*\| = 1$$

we have

$$(2e)^{-n} \operatorname{vol}(B_E) \leq \prod_{i=1}^n s_i \leq 2^{-n} \operatorname{vol}(B_E)$$

REMARK: If $\{e_i\}_{i=1}^n$ is a 1-symmetric basis s_i can be taken as $\|\sum_{k=1}^n e_k\|^{-1}$ for all i.

PROOF: First we prove the right hand inequality. Obviously B_E contains

$$\operatorname{conv}\{\sum_{i=1}^n \varepsilon_i s_i e_i \,|\, \varepsilon_i = \pm 1, \, i = 1, \ldots, n\}$$

Therefore we get

$$2^n \prod_{i=1}^n s_i \leq \operatorname{vol}(B_E).$$

On the other hand B_E is contained in

$$\operatorname{conv}\{\varepsilon_i t_i^{-1} e_i \,|\, \varepsilon_i = \pm 1, \, i = 1, \ldots, n\}$$

since $\|\sum_{i=1}^{n} \pm t_i e_i^*\| = 1$. Thus

$$\mathrm{vol}(B_E) \le (\prod_{i=1}^{n} t_i^{-1})\,\mathrm{vol}(B_{l_n^1})$$

$$= 2^n \frac{1}{n!} \prod_{i=1}^{n} t_i^{-1} = 2^n \frac{n^n}{n!} \prod_{i=1}^{n} s_i \le (2e)^n \prod_{i=1}^{n} s_i \qquad \square$$

In the case that we have to deal with spaces that don't have "nice" unconditional bases we will need a different formula.

LEMMA 1.3: *Let μ be the Haar measure on S^{n-1}. Then we have for every unit ball B in $\mathbb{R}^n$.*

$$\mathrm{vol}(B) = \varkappa_n \int_{S^{n-1}} \|x\|^{-n}\, d\mu$$

where $\|\ \|$ is the norm with respect to B.

The proof is elementary.

LEMMA 1.4: *Let T_i, $i = 1, \dots, N$ be isometries of the n-dimensional Euclidean space and $\|\ \|$ any norm on $\mathbb{R}^n$ with unit ball B. Moreover, let*

$$\|x\|_T = N^{-1} \sum_{i=1}^{N} \|T_i(x)\|$$

with unit ball B^T. Then we have

$$\mathrm{vol}(B^T) \le \mathrm{vol}(B).$$

PROOF: By Lemma 1.3 and T_i^{-1} being isometries in l_n^2 we have

$$\mathrm{vol}(B) = \varkappa_n \int_{S^{n-1}} \|T_i(x)\|^{-n}\, d\mu$$

Therefore we get

$$\mathrm{vol}(B)^{1/n} = \varkappa_n^{1/n} N^{-1} \sum_{i=1}^{N} \left(\int_{S^{n-1}} \|T_i(x)\|^{-n}\, d\mu \right)^{1/n}$$

By triangle inequality we get

$$\mathrm{vol}(B)^{1/n} \ge \varkappa_n^{1/n} \left(\int_{S^{n-1}} \left| N^{-1} \sum_{i=1}^{N} \|T_i(x)\|^{-1} \right|^n d\mu \right)^{1/n}$$

Using now the inequality $N^2 \le (\sum_{i=1}^N a_i)(\sum_{i=1}^N a_i^{-1})$ we get

$$\mathrm{vol}(B) \ge \varkappa_n \int_{S^{n-1}} \left| N^{-1} \sum_{i=1}^{N} \|T_i(x)\| \right|^{-n} d\mu$$

Now we apply Lemma 1.3 again. □

LEMMA 1.5: *Let B be the unit ball of E and $\{e_i\}_{i=1}^n$ a normalized basis of E. Suppose $\{s_i\}_{i=1}^n$ is a sequence of real numbers such that*

$$\operatorname*{Av}_{\pm} \|\sum_{i=1}^{n} \pm s_i e_i\| \le 1$$

Then we have

$$2^n \prod_{i=1}^{n} s_i \le \mathrm{vol}(B)$$

PROOF: We apply Lemma 1.4 and choose as isometries diagonal operators.

$$T_\varepsilon((a_i)_{i=1}^n) = (\varepsilon_i a_i)_{i=1}^n \qquad \varepsilon_i = \pm 1$$

So we have by Lemma 1.4 for the norm

$$\|x\|_{\mathrm{unc}} = \operatorname*{Av}_{\pm} \|\sum_{i=1}^{n} \pm \xi_i e_i\|$$

with the unit ball B^{unc} that

$$\mathrm{vol}(B^{\mathrm{unc}}) \le \mathrm{vol}(B)$$

Now we use the same argument as in the proof of Lemma 1.2. □

We shall also need a result due to Santaló [9].

LEMMA 1.6: *Let B be an unit ball in $\mathbb{R}^n$. Then we have*

$$\mathrm{vol}(B)\,\mathrm{vol}(B^*) \le \varkappa_n^2.$$

2. Volume ratio for symmetric spaces and unitary operator ideals

PROPOSITION 2.1: *Let* $\{e_i\}_{i=1}^n$ *be a* 1-*symmetric basis of* E *and* $\mathrm{id} \in L(l_n^2, E)$ *be the natural identity* $\mathrm{id}((a_i)_{i=1}^n) = \sum_{i=1}^n a_i e_i$. *Then we have*

$$c^{-1}\,\mathrm{vr}(E) \le \sqrt{n}\,\|\mathrm{id}\|\,\|\sum_{i=1}^n e_i\|^{-1} \le c\,\mathrm{vr}(E)$$

where $c > 0$ *is an universal number.*

The right hand side inequality can also be found in [11].

PROOF: By Lemma 1.2 we get

$$\frac{1}{2e}\mathrm{vol}(B_E)^{1/n} \le \|\sum_{i=1}^n e_i\|^{-1} \le \frac{1}{2}\mathrm{vol}(B_E)^{1/n}$$

since we can choose $s_i = \|\sum_{i=1}^n e_i\|^{-1}$, $i = 1,\dots,n$. By John's theorem [5] the ellipsoid of maximal volume must be unique. Considering the symmetries of the unit ball we conclude that the ellipsoid of maximal volume must be a multiple of the unit ball in l_n^2. The factor is given by $\|\mathrm{id}\|^{-1}$. □

PROPOSITION 2.2: *Let* C_E *be the unitary operator ideal generated by the symmetric space* E. *Then we have*

$$c^{-1}\,\mathrm{vr}(E) \le \mathrm{vr}(C_E) \le c\,\mathrm{vr}(E)$$

where $c > 0$ *is an absolute number.*

The right hand inequality was proved in [11]. The next lemma can be found in [2].

LEMMA 2.3: *Let* $\|\ \|_{p,q}$ *denote the norm in* $l_n^p \otimes_\varepsilon l_n^q$. *Suppose that* $1 \le q \le p$ *and*

$$\beta(p,q) = \begin{cases} -\dfrac{1}{2} + \dfrac{1}{p} + \dfrac{1}{q} & \text{if } p \le 2 \\[2ex] \dfrac{1}{q} & \text{if } p \ge 2 \end{cases}$$

Then there are numbers $C_{p,q} > 0$ such that

$$C_{p,q}^{-1} n^{\beta(p,q)} \leq \operatorname*{Av}_{\pm} \| \sum_{i,j=1}^{n} \pm e_i \otimes e_j \| \leq C_{p,q} n^{\beta(p,q)}$$

LEMMA 2.4: *There is an absolute number $c > 0$ such that*

(i) $\left(\dfrac{1}{c\sqrt{n}}\right)^{n^2} \leq \operatorname{vol}(B_{c_\infty}) \leq \left(\dfrac{c}{\sqrt{n}}\right)^{n^2}$

(ii) $(c^{-1} n^{-3/2})^{n^2} \leq \operatorname{vol}(B_{c_1}) \leq (c n^{-3/2})^{n^2}$

PROOF: The right hand inequality of (i) follows from

$$\frac{1}{\sqrt{n}} B_{c_\infty} \subset B_{c_2}.$$

The left hand inequality in (ii) follows from $B_{c_2} \subset \sqrt{n}\, B_{c_1}$. The left hand inequality in (i) follows from Lemma 1.5 and Lemma 2.3. The right hand inequality in (ii) follows from (i) and Lemma 1.6. □

PROOF OF PROPOSITION 2.2: The proof is essentially a repetition of the proof of Proposition 2.1. The ellipsoid of maximal volume ε contained in B_{c_E} is for reasons of symmetry a multiple of the unit ball of the Hilbert–Schmidt operators C_2.

The factor is $\|\mathrm{Id}\|^{-1}$ where $\mathrm{Id} \in L(C_2, C_E)$ is the natural identity. In fact, we have $\|\mathrm{Id}\| = \|\mathrm{id}\|$ where $\mathrm{id} \in L(l_n^2, E)$ is the natural identity. Therefore we have

$$\left(c^{-1} \|\mathrm{id}\|^{-1} \frac{1}{n}\right)^{n^2} \leq \operatorname{vol}(\varepsilon) \leq \left(c \|\mathrm{id}\|^{-1} \frac{1}{n}\right)^{n^2} \tag{2.1}$$

And for the volume of B_{C_E} we get

$$\left(c^{-1} \|\sum_{i=1}^{n} e_i\|^{-1} \frac{1}{\sqrt{n}}\right)^{n^2} \leq \operatorname{vol}(B_{C_E}) \leq \left(c \|\sum_{i=1}^{n} e_i\|^{-1} \frac{1}{\sqrt{n}}\right)^{n^2} \tag{2.2}$$

Indeed, we have

$$\|\sum_{i=1}^{n} e_i\|^{-1} B_{C_\infty} \leq B_{C_E} \leq n \|\sum_{i=1}^{n} e_i\|^{-1} B_{C_1}$$

We apply Lemma 2.4 and get (2.2). Now, the proposition follows from (2.1), (2.2) and Proposition 2.1. □

3. Volume ratio for tensors of l_n^p and l_n^r

We get here a complete description of the volume ratio for π-tensor products of l_n^p-spaces. It turns out that the volume ratio of the tensor product is small provided the volume ratio of the factors is already small. Moreover, we even obtain small volume ratio in certain tensor products of factors with large volume ratio.

THEOREM 3.1: *Up to a constant* $\mathrm{vr}(l_n^r \otimes_\pi l_n^s)$ *is equal to*

$$\begin{array}{ll} 1 & \text{for } 1 \le r, s \le 2 \\ n^{\frac{1}{2}-\frac{1}{r}} & \text{for } 2 \le r \le s' \\ n^{\frac{1}{s}-\frac{1}{2}} & \text{for } 2 \le s' \le r \\ 1 & \text{for } 2 \le r, s \le \infty \text{ and } \frac{1}{2} \le \frac{1}{r} + \frac{1}{s} \\ n^{\frac{1}{2}-\frac{1}{r}-\frac{1}{s}} & \text{for } \frac{1}{r} + \frac{1}{s} \le \frac{1}{2} \end{array}$$

The case $r = 2$ *and* $1 \le s \le 2$ *can be found in* [11].

For the proof of Theorem 3.1 we require several lemmas.

LEMMA 3.2: *Let* $B_{p,q}$ *be the unit ball in* $l_n^p \otimes_\varepsilon l_n^q$ *and* $B_{p,q}^*$ *its dual unit ball. Then there are numbers* $C_{p,q} > 0$ *such that*
(i) $C_{p,q}^{-1} n^{-\beta(p,q)} \le \mathrm{vol}(B_{p,q})^{1/n^2} \le C_{p,q} n^{-\beta(p,q)}$
(ii) $C_{p,q}^{-1} n^{\beta(p,q)-2} \le \mathrm{vol}(B_{p,q}^*)^{1/n^2} \le C_{p,q} n^{\beta(p,q)-2}$
where $\beta(p,q)$ *are as given in Lemma* 2.3.

PROOF: The left hand inequality of (i) is an immediate consequence of Lemma 1.5 and 2.3. The right hand inequality of (ii) follows from the left hand inequality of (i) and Lemma 1.6. The right hand inequality of (i) follows from the left hand inequality of (ii) and Lemma 1.6. The left hand inequality of (ii) is deduced from

$$\max_{\pm} \Big\| \sum_{i,j=1}^{n} \pm e_i \otimes e_j \Big\| \le (1+\sqrt{2}) \begin{cases} n^{\frac{1}{p'}+\frac{1}{q'}+\frac{1}{2}} & \text{if } 1 \le p, q \le 2 \\ n^{1+\frac{1}{q'}} & \text{if } 1 \le q \le 2 \le p \le \infty \\ & \text{or } 2 \le q \le p \le \infty \end{cases} \tag{3.1}$$

where the norm is the one in $l_n^{p'} \otimes_\pi l_n^{q'}$. Indeed, the left hand inequality of

(ii) follows from (3.1) since we find a cube of a sufficient magnitude in the unit ball.

We verify (3.1). Clearly,

$$\max_{\pm} \Big\| \sum_{i,j=1}^{n} \pm e_i \otimes e_j \Big\| \le \mathrm{ubc}\{(e_i \otimes e_j)_{i,j=1}^n\} \Big\| \sum_{i,j=1}^{n} e_i \otimes e_j \Big\|$$
$$= \mathrm{ubc}\{(e_i \otimes e_j)_{i,j=1}^n\} n^{\frac{1}{p'}+\frac{1}{q'}}$$

Now we apply Corollary 4 and Proposition 7 of [10]. □

So far we have estimated the volume of the unit balls in $l_n^r \otimes_\pi l_n^s$. Now we provide the necessary estimates to see what the volume of the ellipsoid of maximal volume is.

First of all we need an estimate that is due to Hardy and Littlewood [3].

LEMMA 3.3: *Suppose*

$$\frac{1}{\mu} = \frac{1}{2p} + \frac{1}{2q} - \frac{1}{4}$$

Then we have for p and q with $\frac{3}{2} \le \frac{1}{p} + \frac{1}{q}$

$$\Big(\sum_{ij} |a_{ij}|^{\mu}\Big)^{1/\mu} \le c \|A\|_{p,q}$$

for some absolute number $c > 0$.

LEMMA 3.4: *Let* $\mathrm{id} \in L(l_n^p \otimes_\varepsilon l_n^q, l_{n^2}^2)$ *be the natural identity. Then we have for*

(i) $\frac{1}{p} + \frac{1}{q} \ge \frac{3}{2}$

$$1 \le \|\mathrm{id}\| \le c$$

(ii) $1 \le q \le 2, q' \le p$

$$\sqrt{n} = \|\mathrm{id}\|$$

(iii) $1 \le q \le 2,\ p \le q',\ \frac{1}{p} + \frac{1}{q} \le \frac{3}{2}$

$$n^{\frac{3}{2}-\frac{1}{p}-\frac{1}{q}} \le \|\mathrm{id}\| \le c n^{\frac{3}{2}-\frac{1}{p}-\frac{1}{q}}$$

(iv) $2 \leq q \leq p$

$$c_{p,q}^{-1} n^{\frac{1}{q'}} \leq \|\mathrm{id}\| \leq n^{\frac{1}{q'}}$$

PROOF: The left hand inequality in (i) is obvious. The right hand inequality follows from Lemma 3.3. The left hand side inequalities in (ii) and (iii) follow by considering the matrix

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

The right hand inequality of (ii) follows from

$$\|A\|_{p,q} \geq \max_{j=1,\ldots,n} \left(\sum_{i=1}^{n} |a_{ij}|^q \right)^{\frac{1}{q}} \geq \frac{1}{\sqrt{n}} \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$$

The right hand inequality of (iii) follows from (i). Since $1/p + 1/q \leq 3/2$ there are $\tilde{p}$ and $\tilde{q}$ with $\tilde{p} \leq p$, $\tilde{q} \leq q$ and $1/\tilde{p} + 1/\tilde{q} = 3/2$. Thus we get by (i)

$$\Big(\sum_{ij} |a_{ij}|^2\Big)^{\frac{1}{2}} \leq c\|A\|_{\tilde{p},\tilde{q}} \leq c n^{3/2 - 1/p - 1/q} \|A\|_{p,q}$$

The left hand inequality of (iv) follows from Lemma 2.3. The right hand side inequality is simple to prove

$$\|A\|_{p,q} \geq \max_{j=1,\ldots,n} \left(\sum_{i=1}^{n} |a_{ij}|^q \right)^{\frac{1}{q}} \geq n^{-\frac{1}{2}} n^{-\frac{1}{2}+\frac{1}{q}} \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \quad \square$$

Now, Theorem 3.1 follows from Lemma 3.2 (ii) and 3.4.

4. The n-dimensional James space

We study here the structure of the n-dimensional James space J_n. It was proved in [6] that the unconditional basis constant of J_n tends to infinity with n.

We prove that the distance $d(l_n^2, J_n)$ is at most of order $\log n$. Therefore the same holds for the volume ratio of J_n and the dual J_n^*. But, in fact, we prove that the volume ratio of J_n and J_n^* are uniformly bounded.

The examples studied before usually give that $\mathrm{vr}(E)\,\mathrm{vr}(E^*)$ is of the order of $d(E, l_n^2)$. The arguments for estimating $\mathrm{vr}(J_n)$ and $\mathrm{vr}(J_n^*)$ are simple and quite different from those used before. We also get that $\mathrm{k\acute{e}}(J_n)$ is uniformly bounded. This invariant was studied in [8].

PROPOSITION 4.1: *There is an absolute number C such that*

$$d(J_n, l_n^2) \le C \log n$$

PROPOSITION 4.2: *There is an absolute number C such that*

$$\mathrm{vr}(J_n) \le C \text{ and } \mathrm{vr}(J_n^*) \le C$$

PROPOSITION 4.3: *There is an absolute number C such that*

$$\mathrm{K\acute{e}}(J_n) \le C$$

PROOF OF PROPOSITION 4.2: Let $\{e_i\}_{i=1}^n$ denote the usual unit vector basis in J_n and

$$f_j = \sum_{i=1}^{j} e_i \qquad j = 1, \dots, n$$

We have for all $x = (\xi_i)_{i=1}^n$

$$\|x\|_\infty \le \Big\|\sum_{i=1}^{n} \xi_i e_i\Big\|_J \le 2\|x\|_2 \tag{4.1}$$

and

$$\|x\|_2 \le \Big\|\sum_{j=1}^{n} \xi_j f_j\Big\|_J \le \sqrt{2}\,\|x\|_1 \tag{4.2}$$

(4.1) is obvious. The right hand inequality of (4.2) follows by triangle inequality and the left hand inequality by choosing the sequence $p_i = i$, $i = 1, \dots, n$.

Now we observe that the map A^{-1} with $A(e_j) = \sum_{i=1}^{j} e_i$ maps the unit ball of J_n with respect to the basis $\{f_j\}_{j=1}^n$, say B_f, onto the unit ball with respect to $\{e_i\}_{i=1}^n$, say B_e. Since $\det(A) = 1$ we have

$$\mathrm{vol}(B_e) = \mathrm{vol}(B_f)$$

By (4.1) and (4.2) we conclude that for some $c > 0$ we have

$$\left(\frac{1}{c\sqrt{n}}\right)^n \le \mathrm{vol}(B_e) = \mathrm{vol}(B_f) \le \left(\frac{c}{\sqrt{n}}\right)^n$$

By this and (4.1) we get $\mathrm{vr}(J_n) \le C'$. The arguments for J_n^* are by duality the same. □

Now we are going to prove Proposition 4.1. Proposition 4.3 will be a consequence of the construction in the proof of Proposition 4.1. We shall give essentially a proof for the case when the dimension of J_n is a power of 2, $n = 2^m$. Thus we define

$$f_i^k = \sum_{j=1}^{2^{k-1}} e(2^k i - 2^k + j),\ f_1^{m+1} = \sum_{j=1}^{2^m} e(j) \tag{4.3}$$

where $k = 1, \dots, m$, $i = 1, \dots, n2^{-k}$ and $\{e(l)\}_{l=1}^n$ denotes the usual unit vector basis in J_n. This basis induces an isomorphism between J_n and l_n^2 and will give Proposition 4.1.

Lemma 4.4: *Let $\{f_i^k\}_{i,k}$ be as defined in* (4.3). *Then we have*

$$\left(\sum_{i=1}^{n2^{-k}} |a_i|^2\right)^{\frac{1}{2}} \le \left\|\sum_{i=1}^{n2^{-k}} a_i f_i^k\right\| \le \sqrt{2}\left(\sum_{i=1}^{n2^{-k}} |a_i|^2\right)^{\frac{1}{2}}$$

for all $k = 1, \dots, m$.

Lemma 4.5: *Let $\{f_i^k\}_{i,k}$ be as defined in* (4.3) *and let $P_l \in L(J_n, J_n)$ with*

$$P_l\Big(\sum_k \sum_i a_i^k f_i^k\Big) = \sum_i a_i^l f_i^l$$

for $l = 1, \dots, m+1$. Then we have

$$\|P_l\| \le \sqrt{2}$$

Clearly Proposition 4.1 follows from Lemma 4.4 and 4.5. The proof of Lemma 4.4 is immediate. For the proof of Lemma 4.5 we need the following sublemma.

Sublemma 4.6: *Let $\{f_i^k\}_{i,k}$ be as defined in* (4.3). *We have for all $(a_i)_{i=1}^n$*

(i) $$\left\langle \sum_{i=1}^{n2^{-k}} a_i f_i^k, e^*(2^k i_0 - 2^{k-1}) \right\rangle = a_{i_0}$$

for $k = 1, \dots, m$, and $i_0 = 1, \dots, n2^{-k}$

$$\left\langle \sum_{i=1}^{n2^{-k}} a_i f_i^k, e^*(2^k i_0) \right\rangle = 0$$

for $k = 1, \ldots, m$, *and* $i_0 = 1, \ldots, n2^{-k}$

(ii) $$\left\langle \sum_{i=1}^{n2^{-k}} a_i f_i^k, e^*(2^l i_0 - 2^{l-1}) \right\rangle = 0$$

$$\left\langle \sum_{i=1}^{n2^{-k}} a_i f_i^k, e^*(2^l i_0) \right\rangle = 0$$

for $l > k$, $i_0 = 1, \ldots, n2^{-l}$, $l = 1, \ldots, m$

(iii) $$\left\langle \sum_{i=1}^{n2^{-k}} a_i f_i^k, e^*(2^l i_0 - 2^{l-1}) \right\rangle = \left\langle \sum_{i=1}^{n2^{-k}} a_i f_i^k, e^*(2^l i_0) \right\rangle$$

for $l < k$ *and* $i_0 = 1, \ldots, n2^{-k}$.

PROOF: (i) is immediately clear since the blocks f_i^k, $i = 1, \ldots, n2^{-k}$, are disjoint and $e(2^k i_0 - 2^{k-1})$ appears just in the block $f_{i_0}^k$.

Concerning the second equality in (i) we just have to observe that $e(2^k i_0)$ does not appear in any of the blocks f_i^k, $i = 1, \ldots, n2^{-k}$.

We prove now (ii). Assume the first expression is not zero. Then there is an $i_1 \in \mathbb{N}$ such that

$$2^k i_1 - 2^k + 1 \leq 2^l i_0 - 2^{l-1} \leq 2^k i_1 - 2^{k-1}$$

or

$$i_1 - 1 + 2^{-k} \leq 2^{l-k} i_0 - 2^{l-1-k} \leq i_1 - \tfrac{1}{2}$$

Since $l > k$ we have that $2^{l-k} i_0 - 2^{l-1-k}$ is a natural number. But there is no natural number in the interval $[i_1 - 1 + 2^{-k}, i_1 - \frac{1}{2}]$. The same argument holds for the second equality.

Now we prove (iii). Clearly, it is enough to prove that

$$2^k i - 2^k + 1 \leq 2^l i_0 - 2^{l-1} \leq 2^k i - 2^{k-1} \tag{4.4}$$

holds for $i \in \mathbb{N}$ if and only if

$$2^k i - 2^k + 1 \leq 2^l i_0 \leq 2^k i - 2^{k-1} \tag{4.5}$$

holds for $i \in \mathbb{N}$. Indeed, if there is no $i \in \mathbb{N}$ fulfilling (4.4) or (4.5) then both are not fulfilled and therefore both expressions in (iii) are zero. If there is an $i \in \mathbb{N}$ fulfilling (4.4) or (4.5) then (4.4) and (4.5) are satisfied and consequently both expressions in (iii) are equal to the same a_i.

Obviously it is enough to prove that the right hand inequality of (4.4) implies that of (4.5) and the left hand inequality of (4.5) implies that of (4.4).

We start with the right hand inequalities. We have that

$$2^{l-k}i_0 - 2^{l-1-k} \leq i - \tfrac{1}{2} \tag{4.6}$$

If $2^{l-k}i_0 \leq i - \frac{1}{2}$ nothing is left to prove. If $2^{l-k}i_0 > i - \frac{1}{2}$ then $2^{l-k}i_0 - 2^{l-k} \geq i - \frac{1}{2}$ since $i_0, i \in \mathbb{N}$ and $l < k$. But this contradicts (4.6). Now we consider the left hand side inequalities. We have

$$i - 1 + 2^{-k} \leq 2^{l-k}i_0$$

Clearly, we must have $i - 1 < 2^{l-k}i_0$. But this implies

$$i - 1 + 2^{l-k} \leq 2^{l-k}i_0$$

since $i, i_0 \in \mathbb{N}$ and $l < k$. □

Proof of Lemma 4.5: In view of Lemma 4.4 it is enough to prove

$$\Big(\sum_i |a_i^l|^2\Big)^{\frac{1}{2}} \leq \Big\|\sum_k \sum_i a_i^k f_i^k\Big\|$$

for all $l = 1, \ldots, m+1$. According to Sublemma 4.6 we choose a sequence $p_1 < \ldots < p_\alpha$, $\alpha \in \mathbb{N}$. If $l \in \{1, \ldots, m\}$ we put

$$\begin{aligned} p_{2i-1} &= 2^l i - 2^{l-1} \\ p_{2i} &= 2^l i \end{aligned} \qquad i = 1, \ldots, n2^{-l}$$

If $l = m + 1$ we put simply $p_1 = n$. Applying this sequence to the definition of the norm (0.1) and using Sublemma 4.6 gives the estimation. □

Proof of Proposition 4.3: As pointed out in [8] we have to find a basis $\{x_i\}_{i=1}^n$ such that

$$\Big\|\sum_{i=1}^n \pm x_i\Big\| \, \Big\|\sum_{i=1}^n \pm x_i^*\Big\| \leq cn$$

for all changes of signs. In fact, we have for the basis (4.3).

$$\tfrac{1}{2}\sqrt{n} \leq \Big\|\sum_k \sum_i \pm f_i^k\Big\| \leq (2 + \sqrt{2})\sqrt{n} \tag{4.7}$$

The left hand inequality follows from Lemma 4.4 and 4.5.

$$\sqrt{2}\|\sum_k\sum_i \pm f_i^k\| \geq \|\sum_{i=1}^{n/2} \pm f_i^1\| \geq \sqrt{\frac{n}{2}}$$

The right hand inequality follows from Lemma 4.4.

$$\|\sum_k\sum_i \pm f_i^k\| \leq \sum_k \|\sum_i \pm f_i^k\| \leq \sqrt{2}\sum_{k=1}^{m+1}\sqrt{n2^{-k}} \leq (2+\sqrt{2})\sqrt{n}$$

For the dual basis $\{f_i^{k*}\}_{i,k}$ we get analogous estimates to (4.7). Indeed, by dualization Lemma 4.4 and 4.5 are also valid for the dual basis.

Thus we get the result for dimensions n that are powers of 2. The case of other dimensions is treated in an analogous way. □

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Johannes Kepler Universität
Institut für Mathematik
Linz, Austria