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ON ISOSPECTRAL DEFORMATIONS OF RIEMANNIAN METRICS. II

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1. Introduction

Let M be an $n(\geq 2)$ dimensional compact oriented C^∞ manifold without boundary. Let g be a C^∞ Riemannian metric on M, and $\mathrm{Spec}(M, g)$ denote the set of eigenvalues of the Laplace-Beltrami operator $\Delta_g = -g^{jk}\nabla_j\nabla_k$ acting on real C^∞ functions on M. A 1-parameter C^∞ deformation $g(t)$ $(-\varepsilon < t < \varepsilon)$ of a Riemannian metric on M is called an *isospectral deformation* of $g(0)$ if $\mathrm{Spec}(M, g(t)) = \mathrm{Spec}(M, g(0))$ holds for every t. We call $g(t)$ to be *trivial* if there is a 1-parameter family $\eta(t)$ of diffeomorphisms of M such that $g(t) = \eta(t)^*g(0)$. We have considered in [1], [2] the following problem (given in [3, p. 233]).

PROBLEM A: *Is there a non-trivial isospectral deformation of a Riemannian metric?*

So far, we have few results concerning this problem except for special cases [1] ~ [6]. Among others the following is known.

THEOREM: *There are no non-trivial isospectral deformations of g, if*

(1) (M, g) is $(1/n)$-pinched, that is, for each $x \in M$, there exists a positive number A(depending on x) such that $-1 - (1/n) < K/A < -1 + (1/n)$, K being the sectional curvature associated with any two dimensional subspace of T_xM, or

(2) (M, g) is of non-negative constant curvature.

The case (1) was proved by Guillemin and Kazhdan [4], [5], and (2) is due to Kuwabara [2] for flat case and to Tanno [6] for the case of positive constant curvature. Moreover, for the case (2), a stronger result

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was shown as follows. Let $\mathscr{R}$ be the manifold of C^∞ Riemannian metrics on M with C^∞ topology. *If (M, g) is flat or a standard sphere, there is a neighborhood U of g in $\mathscr{R}$ such that if* $\mathrm{Spec}(M, g) = \mathrm{Spec}(M, g')$ *and $g' \in U$ then (M, g') is isometric with (M, g).*

In the previous paper [1], [2] we studied the problem by considering the variations of Minakshisundaram's coefficients under the deformation of the metric. We try in this paper a different approach to the problem based on Lax's idea which plays a fundamental role in theory of nonlinear waves [7]. We consider the isospectral deformations confined to Lax's sense which are called L-isospectral deformations, and set up the following problem.

PROBLEM B: *Is there a non-trivial L-isospectral deformation of a matric?*

We see that there are no non-trivial L-isospectral deformations under suitable conditions.

In §2 we introduce the notion of L-isospectral deformations. In §3 we consider the non-existence of L-isospectral deformations and give a sufficient condition for it. It is shown in §4 that this condition is related to the non-existence of first integrals of the geodesic flow, and we give some results concerning the non-existence of L-isospectral deformations.

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2. L-isospectral deformations

Let $g(t)$ be a C^∞ isospectral deformation of $g = g(0)$, that is,

$$\Delta_{g(t)}\phi_j(t) \equiv \Delta_t\phi_j(t) = \lambda_j\phi_j(t), \tag{2.1}$$

and $\{\phi_j(t)\}_{j=0}^{\infty}$ is the system of real eigenfunctions orthonormal with respect to the inner product $(\,,)_t$ defined from the metric $g(t)$, namely, $(\phi, \psi)_t = \int \phi\psi\, dV(g(t))$, $\quad dV(g(t)) = \sqrt{\det g(t)}\, dx^1 \ldots dx^n$. Moreover by Browder's theorem [8], we can choose $\phi_j(t)$ to be of C^∞ class with respect to t.

First, we give the following lemma.

LEMMA 2.1: *Let $g(t)$ be a C^∞ isospectral deformation of g, and $\mu = dV(g)$. Then, there is a C^∞ isospectral deformation $\tilde{g}(t)$ of g such that $\tilde{g}(t) = \eta(t)^*g(t)$ for a 1-parameter family $\eta(t)$ of diffeomorphisms of M, and $dV(\tilde{g}(t)) = \mu$.*

Proof: It is well known that $\mathrm{vol}(M, g(t))$ is left invariant under the isospectral deformation $g(t)$ (cf. [3, p. 216]). Hence, the lemma is immediately obtained by the following lemma due to Moser [9].

Lemma (Moser): *Let $\mu(t)$ be a C^∞ deformation of n-form on M which is non-degenerate and $\int_M \mu(t) = \int_M \mu(0)$ for each t. Then, there is a C^∞ family $\eta(t)$ of diffeomorphisms of M such that $\eta(t)^*\mu(t) = \mu(0)$.*

By Lemma 2.1, we consider hereafter only volume-element preserving deformations, for which the *infinitesimal deformation* (*i-deformation*, for short) $h(t) = dg(t)/dt$ satisfies (cf. [10])

$$\mathrm{Tr}_{g(t)} h(t) = h_{jk}(t) g^{jk}(t) = 0.$$

We denote the set of all square integrable real functions on M by $L^2(M)$, the inner product being $(,) = (,)_t = (,)_0$, and the space of distributions on M by $\mathscr{E}'(M)$. For an isospectral deformation $g(t)$, we introduce a linear operator $B_t : L^2(M) \to \mathscr{E}'(M)$ for each t as follows. Suppose an element ϕ of $L^2(M)$ is expressed as $\sum_{j=0}^{\infty} a_j(t)\phi_j(t)$, $a_j(t) \in \mathbf{R}$. Then for $\psi \in C^\infty(M)$, we define

$$\langle B_t\phi, \psi\rangle = \sum_{j=0}^{\infty} a_j(t)(\phi_j'(t), \psi),$$

where $\phi_j'(t) \equiv d\phi_j(t)/dt$ and the domain $D(B_t)$ of the operator B_t is the set of all ϕ $(\in L^2(M))$ for which the right hand side of the above has a real finite value. Note that $B_t\phi_j(t) = \phi_j'(t) \in C^\infty(M)$ holds good.

Now, differentiate (2.1) with respect to t, and we have

$$\Delta_t'\phi_j(t) + \Delta_t B_t \phi_j(t) - \lambda_j B_t \phi_j(t) = 0,$$

hence,

$$(\Delta_t' + \Delta_t B_t - B_t \Delta_t)\phi_j(t) = 0.$$

Therefore, we get the following equation of operators on $D(B_t) \cap C^\infty(M)$;

$$\Delta_t' + [\Delta_t, B_t] = 0. \tag{2.2}$$

Thus we have

Proposition 2.2: *If $g(t)$ is an isospectral deformation, there is a linear operator B_t satisfying* (2.2), *where*

$$\Delta_t' = h^{jk}\nabla_j\nabla_k + (\nabla_k h^{jk})\nabla_j = \nabla_j(h^{jk}\nabla_k), \tag{2.3}$$

∇ being the covariant differentiation defined by $g(t)$.

PROOF: (2.3) is immediately derived from variational formulas of Riemannian structure [10]. Q.E.D.

REMARK: The operator B_t depends on the choice of the orthonormal basis of eigenfunctions $\{\phi_j(t)\}$.

The equation (2.2) may be called Lax's equation, which is originally studied concerning Korteweg-de Vries (KdV) equation (see Lax [7]):

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0.$$

For the Schrödinger operator $L_t = (d^2/dx^2) + (1/6)u(x, t)$, consider a third order differential operator

$$B_t = -4\frac{d^3}{dx^3} - u\frac{d}{dx} - \frac{1}{2}\frac{du}{dx}.$$

Then the equation $L'_t + [L_t, B_t] = 0$ is equivalent to the KdV equation and $\mathrm{Spec}(L_t)$ is left invariant when u changes with t subject to the KdV equation. Moreover, for higher odd order differential operators B_t we get a series of higher order KdV equations, and $\mathrm{Spec}(L_t)$ is invariant if u changes according to them.

On the basis of the above discussion, we introduce the following definition.

DEFINITION: Let $g(t)$ be an isospectral deformation. If B_t is a differential operator for each t, we call $g(t)$ an *isospectral deformation in Lax's sense*, or *L-isospectral deformation.* If B_t is a k-th order differential operator for each t, we call $g(t)$ an *L_k-isospectral deformation.* Note that $D(B_t) = L^2(M)$ for the L-isospectral deformation.

LEMMA 2.3: *Let $g(t)$ be an L_k-isospectral deformation. Then, the k-th differential operator B_t is skew-symmetric, that is,*

$$B_t + B_t^* = 0, \tag{2.4}$$

where B_t^* is the formal adjoint of B_t with respect to $(\,,)$.

PROOF: By differentiating $(\phi_j(t), \phi_k(t)) = \delta_{jk}$ with respect to t, we have

$$(B_t\phi_j(t), \phi_k(t)) + (\phi_j(t), B_t\phi_k(t)) = 0,$$

and (2.4) because the above holds for all ϕ_j's. Q.E.D.

As a converse of Proposition 2.2, we have the following.

PROPOSITION 2.4: *Suppose there are a volume-element preserving C^∞ deformation $g(t)$ of a metric and a skew-symmetric k-th order differential operator B_t smoothly depending on t, which satisfy eq.* (2.2). *Assume that there exists a* 1-*parameter family of linear operators $T_t: C^\infty(M) \to C^\infty(M)$, $-\varepsilon < t < \varepsilon$, whose infinitesimal generator is B_t, that is, $T_t = \exp(\int_0^t B_s ds)$ and $T_0 = Identity$. Then the deformation $g(t)$ $(-\varepsilon < t < \varepsilon)$ is an isospectral deformation of* $g(0)$.

PROOF: Let $\{\psi_j\}$ be a set of orthonormal eigenfunctions associated with $\mathrm{Spec}(M, g(0)) = \{\lambda_j\}$, and set $\phi_j(t) = T_t\psi_j$. Then $\{\phi_j(t)\}_{j=0}^\infty$ forms an orthonormal basis of $L^2(M)$ for each t. In fact,

$$\frac{d}{dt}(\phi_j(t), \phi_k(t)) = (B_t\phi_j(t), \phi_k(t)) + (\phi_j(t), B_t\phi_k(t)) = 0,$$

hence $(\phi_j(t), \phi_k(t)) = (\psi_j, \psi_k) = \delta_{jk}$ holds. Set

$$\Delta_t\phi_j(t) = \sum_{k=0}^{\infty} a_j^k(t)\phi_k(t),$$
$$a_j^k(t) = (\Delta_t\phi_j(t), \phi_k(t)), \; a_j^k(0) = \lambda_j\delta_j^k.$$

The coefficients $a_j^k(t)$ are C^∞ functions and

$$\frac{d}{dt}a_j^k(t) = (\Delta_t'\phi_j(t) + \Delta_t B_t\phi_j(t), \phi_k(t)) + (\Delta_t\phi_j(t), B_t\phi_k(t)) =$$
$$= ((\Delta_t' + [\Delta_t, B_t])\phi_j(t), \phi_k(t)) = 0.$$

Therefore $a_j^k(t) = \lambda_j\delta_j^k$ and accordingly $\mathrm{Spec}(M, g(t)) = \{\lambda_j\}$.

Q.E.D.

A fundamental example of L-isospectral deformation is a trivial deformation, that is,

LEMMA 2.5: *A trivial deformation is an L_1-isospectral deformation.*

PROOF: Let $g(t) = \eta(t)^*g(0)$ for a 1-parameter family $\eta(t)$ of volume preserving diffeomorphisms of M. Then, we have for each eigenfunction,

$$\phi_j(x, s) = \phi_j(\eta(s-t)x, t) = \eta(s-t)^*\phi_j(x, t).$$

Therefore, we get $\phi_j'(t) = X_t\phi_j(t)$, where $X_t = d\eta(t)/dt$ is a vector field satisfying $\nabla_j X_t^j = 0$ (cf. [11]). Thus $B_t = X_t$ is a first order differential operator and satisfies (2.2) and (2.4). Q.E.D.

3. Non-existence of L-isospectral deformations

Let $g(t)$ be a C^∞ deformation with $g(0) = g$. We consider the equation (2.2) at $t = 0$ (the suffix 0 being omitted). A k-th order differential operator B on (M, g) is expressed as

$$B = a_{(k)}^{i_1 \dots i_k} \nabla_{i_1} \dots \nabla_{i_k} + a_{(k-1)}^{j_1 \dots j_{k-1}} \nabla_{j_1} \dots \nabla_{j_{k-1}} + \dots + a_{(0)}, \tag{3.1}$$

where $a_{(m)}^{i_1 \dots i_m}$ are components of a contravariant symmetric m-tensor. For this operator B, we have

$$B^* = (-1)^k a_{(k)}^{i_1 \dots i_k} \nabla_{i_1} \dots \nabla_{i_k} + \text{(lower order terms)}.$$

Therefore, k is odd because B is skew-symmetric (Lemma 2.3). Thus we have only to consider odd order differential operators B.

First, we deal with L_1-isospectral deformations, and have the following which is the converse of Lemma 2.5.

PROPOSITION 3.1: *There are no non-trivial L_1-isospectral deformations.*

PROOF: Let B is a first order skew-symmetric differential operator, namely, $B = a^i \nabla_i + (1/2)(\nabla_i a^i)$. Then, we have from (2.2),

$$(h^{jk} - 2\nabla^j a^k)\nabla_j \nabla_k + \{\nabla^k h_k^j - \nabla_k \nabla^k a^j - \nabla^j \nabla_i a^i - a^k R_k^j\}\nabla_j + \tfrac{1}{2}\Delta(\nabla_i a^i) = 0,$$

where R_{jk} is the Ricci curvature tensor of (M, g). Therefore, we get $h^{jk} = \nabla^j a^k + \nabla^k a^j$, that is, $h(=(dg/dt)(0))$ is a trivial i-deformation (see [1]). Thus, if $g(t)$ is an L_1-isospectral deformation, then $h(t)$ is trivial with respect to $g(t)$ for each t. Hence the proposition is obtained by the following lemma.

LEMMA (Koiso [12, Lemma 2.9]): *If $h(t) = dg(t)/dt$ is trivial for each t, then $g(t)$ is a trivial deformation.*

Next, we consider L_k-isospectral deformations for $k(\text{odd}) \geq 3$. Substituting the differential operator B given by (3.1) into eq. (2.2), we get a necessary and sufficient condition that the coefficients $a_{(m)}$ and h should be satisfied. The computation, however, is so complicated that we cannot write it explicitly.

As a necessary condition, we have the following.

PROPOSITION 3.2: *If $g(t)$ is an L_k-isospectral deformation for $k(odd) \geq 3$, then the highest order coefficients of B satisfy*

$$\nabla^\rho a_{(k)}^{j_1 \dots j_k} + \nabla^{j_1} a_{(k)}^{\rho j_2 \dots j_k} + \dots + \nabla^{j_k} a_{(k)}^{\rho j_1 \dots j_{k-1}} = 0. \tag{3.2}$$

PROOF: By straightforward calculations, eq. (2.2) leads to

$$(\nabla^{\rho} a_{(k)}^{j_1 \dots j_k}) \nabla_{\rho} \nabla_{j_1} \dots \nabla_{j_k} + (\text{lower order terms}) = 0.$$

Thus we get (3.2). Q.E.D.

Let S_k be the space of all C^∞ contravariant symmetric k-tensor fields on M endowed with C^∞ topology. For a C^∞ Riemannian metric g, we define $\hat{\nabla}_g^k : S_k \to S_{k+1}$ by

$$(\hat{\nabla}_g^k a)^{i_1 \dots i_{k+1}} = \nabla^{i_1} a^{i_2 \dots i_{k+1}} + \nabla^{i_2} a^{i_1 i_3 \dots i_{k+1}} + \dots + \nabla^{i_{k+1}} a^{i_1 \dots i_k},$$

where ∇ is the covariant differentiation defined by g. Let $\mathscr{R}$ be the manifold of all C^∞ Riemannian metrics with C^∞ topology, and

$$\mathscr{N}_k = \{g \in \mathscr{R};\ (\hat{\nabla}_g^k)^{-1}(0) = \{0\}\}.$$

LEMMA 3.3:
(1) *$\mathscr{N}_k$ is an open subset of $\mathscr{R}$.*
(2) $\mathscr{R} \supset \mathscr{N}_1 \supset \mathscr{N}_3 \supset \dots \supset \mathscr{N}_{2m-1} \supset \mathscr{N}_{2m+1} \supset \dots.$

PROOF: (1) Define $\hat{\nabla}^k : \mathscr{R} \times (S_k \backslash \{0\}) \to S_{k+1}$ by $\hat{\nabla}^k(g, a) = \hat{\nabla}_g^k a$. Then we have $\mathscr{N}_k = \mathscr{R} \backslash \pi(\ker(\hat{\nabla}^k))$, where $\pi : \mathscr{R} \times (S_k \backslash \{0\}) \to \mathscr{R}$ is the projection. It is easy to see that $\hat{\nabla}^k$ is continuous and π is an open mapping. Hence $\mathscr{N}_k$ is open in $\mathscr{R}$.

(2) We show $(\mathscr{R} \backslash \mathscr{N}_{2m-1}) \subset (\mathscr{R} \backslash \mathscr{N}_{2m+1})$. Let $g \in (\mathscr{R} \backslash \mathscr{N}_{2m-1})$ and $\hat{\nabla}_g^{2m-1} a = 0$. Then, obviously, $\hat{\nabla}_g^{2m+1}(a \mathbin{\hat{\otimes}} g^{-1}) = 0$ holds, where $a \mathbin{\hat{\otimes}} g^{-1}$ denotes the symmetrization of $a \otimes g^{-1}$. Q.E.D.

We have the following proposition by virtue of Proposition 3.2.

PROPOSITION 3.4: *If the metric g belongs to $\mathscr{N}_k$, $k(odd) \geq 3$, then there are no non-trivial L_k-isospectral deformations of g.*

PROOF: Assume B is the k-th order differential operator satisfying (2.2). If $g \in \mathscr{N}_k$, then it follows from Proposition 3.2 and Lemma 3.3, (2) that the operator B reduces to be of first order. Since the set $\mathscr{N}_k$ is open, the isospectral deformation must be trivial by virtue of Proposition 3.1. Q.E.D.

REMARK: We conjecture that for each positive odd integer k, the set $\mathscr{N}_k$ is dense in $\mathscr{R}$. It is known that the statement is valid for the case of $k = 1$ (cf. Ebin [13, Proposition 8.3]).

Set $\mathscr{N}_\infty = \bigcap_{k:\mathrm{odd}} \mathscr{N}_k$. Noting that $\mathscr{N}_\infty$ is not necessarily open, we get the following.

PROPOSITION 3.5: *If the metric g belongs to $\mathscr{N}_\infty$, there are no non-trivial L-isospectral i-deformations of g.*

4. Relation with first integrals of geodesic flows

Consider the cotangent bundle T^*M with the natural symplectic structure. Let (x^i, p_i) be the local coordinate system of T^*M naturally induced from the coordinates (x^i) of M. For a Riemannian metric g on M, define a function H_g on T^*M by

$$H_g = \tfrac{1}{2}g^{jk}p_jp_k.$$

The Hamiltonian flow on T^*M defined by H_g is called the geodesic flow, and the image of its integral curves projected on M are geodesics of (M, g).

Let P_k (k: positive integer) be the set of all polynomial functions on T^*M which are homogeneous of degree k in (p_i). We define a one-one correspondence $\Phi: S_k \to P_k$ by

$$\Phi(a) = \frac{1}{k}\, a^{i_1\ldots i_k}p_{i_1}\cdots p_{i_k}.$$

Then, we have the following (cf. [5, Proposition 3.1]).

LEMMA 4.1: *For each positive integer k, the equation $\hat{\nabla}_g^k a = 0$ is equivalent to*

$$\{\Phi(a), H_g\} = 0.$$

Here $\{\,,\}$ is the Poisson bracket defined from the symplectic structure of T^*M.

PROOF: For $\Phi(a) = (1/k)a^{i_1\ldots i_k}p_{i_1}\cdots p_{i_k}$, we have

$$\begin{aligned}\{\Phi(a), H_g\} &= \frac{1}{k}\frac{\partial a^{i_1\ldots i_k}}{\partial x^j}\, p_{i_1}\cdots p_{i_k}g^{jm}p_m - \\ &- \tfrac{1}{2}a^{ji_1\ldots i_k}p_{i_1}\cdots p_{i_k}\frac{\partial g^{km}}{\partial x^j}\, p_kp_m = \\ &= \frac{1}{k}(\nabla^m a^{i_1\ldots i_k})p_mp_{i_1}\cdots p_{i_k}.\end{aligned}$$

Thus the lemma is proved. Q.E.D.

DEFINITION: A C^∞ function f on T^*M is called the *first integral* of the geodesic flow if $\{f, H_g\} = 0$, and f is not constant on any open set of any level surface of H_g. Moreover, if f belongs to P_k, we call f the *first integral of degree k.*

From the above lemma, we have for odd k,

$$\mathcal{N}_k = \{g \in \mathcal{R};\ \text{the geodesic flow has no first integral of degree } k\}.$$

We have the following theorem from Propositions 3.4 and 3.5.

THEOREM 4.2: *There are no non-trivial L-isospectral i-deformations (resp. L_k-isospectral deformations for odd integer $k \geq 3$) of g, if the geodesic flow defined by g has no first integrals (resp. first integrals of degree k).*

By Anosov [14] the geodesic flow defined by the metric of negative curvature is ergodic and has no first integrals. Thus we have

COROLLARY 4.3: *If (M, g) is of negative sectional curvature, there are no non-trivial L-isospectral deformations of g.*

REMARK: In [4] Guillemin and Kazhdan showed that if (M, g) is of negative sectional curvature and $g(t)$ is an isospectral deformation of g, then there is a C^1 function f on T^*M such that

$$H_g' + \{H_g, f\} = 0, \tag{4.1}$$

where $H_g' = (1/2)h^{jk}p_j p_k$. Moreover if (M, g) is $(1/n)$-pinched, it is shown that the function f satisfying (4.1) belongs to P_1 and accordingly $h = (dg/dt)(0)$ is trivial. We note that the equation (2.2) may be regarded as a quantum version of eq. (4.1).

Finally, we consider the case where the metric does not belong to $\mathcal{N}_k$, and have the following theorem.

THEOREM 4.4: *Let k be a positive odd integer, and assume that every first integral of odd degree $\leq k$ of the geodesic flow defined by the metric g is expressed as a linear combination of the products of the first integrals of degree one and H_g. Then there are no non-trivial L_k-isospectral i-deformations of g.*

PROOF: We prove the theorem by induction on k. For the case $k = 1$, the statement reduces to Proposition 3.1. For general odd k, suppose h

is an L_k-isospectral i-deformation of g, and

$$\Delta' + [\Delta, B] = 0,$$

where

$$\Delta' = \nabla_j(h^{jk}\nabla_k),$$
$$B = a^{i_1 \cdots i_k}\nabla_{i_1}\cdots\nabla_{i_k} + \text{(lower order terms)}.$$

By Proposition 3.2, Lemma 4.1, and the assumption of the theorem, we have

$$a = \sum_{k=2r+s} g^{-1}\overbrace{\hat{\otimes}\cdots\hat{\otimes}}^{r} g^{-1}\hat{\otimes}\,\xi_1\,\hat{\otimes}\cdots\hat{\otimes}\,\xi_s,$$

where $\xi_1,\ldots,\xi_s$ are the Killing vectors on (M, g). Set $\Omega_k = \xi_k^j\nabla_j$, $k = 1,\ldots,s$, and

$$B_1 = \sum_{k=2r+s} (\Delta^r\Omega_1\ldots\Omega_s)$$

corresponding to a, where ( ) denotes the symmetrization. We see easily that B_1 is a skew-symmetric k-th differential operator, and $[\Delta, B_1] = 0$. Moreover, we have $B = B_1 + B_2$, where B_2 is a skew-symmetric $(k-2)$-th differential operator, and

$$\Delta' + [\Delta, B_2] = 0$$

holds good. Thus h is an L_{k-2}-isospectral i-deformation of g. Therefore h is trivial by the assumption of induction. Q.E.D.

We conjecture that the assumption of the theorem is satisfied for every Riemannian symmetric spaces.

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