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HIGHER JACOBIANS AND CYCLES ON ABELIAN VARIETIES

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1. Introduction

A long outstanding problem in the transcendental theory of algebraic varieties is to find a cohomological criterion for a cycle on an algebraic variety to be itself algebraic (see Hodge [3], Chapter 4). Criteria are available in certain cases; but they are far from complete.

Because of the sensitivity of abelian varieties to moduli, they offer an attractive terrain upon which to test the so-called "Hodge conjecture" concerning the connexion between algebraic cycles and real harmonic forms of type (p, p) with integral periods. For sufficiently general abelian varieties, as expressed by suitable independence of the periods in a normalized period matrix, the Hodge conjecture is known to be true (Mattuck [7], Comessatti [2]).

Here we show that the problem can be related to A. Weil's higher Jacobian varieties. For odd p, the p-th higher Jacobian $\hat{A}$ of an abelian variety A turns out to admit a homomorphism π onto A. We compute explicitly the kernel K of π and the resulting splitting of $\hat{A}$. We then show that to any real harmonic form φ of type (p, p) with integral periods on A we can associate a real $(1, 1)$-form with integral periods on $\hat{A}$. In this way it is possible to define an algebraic (divisorial) correspondence between A and the kernel K, from which we are able to obtain various algebraic cycles on A associated with the form φ. However, their connexion with φ is rather obscure, because of the highly transcendental nature of our construction, and we have not yet succeeded in making the computations needed to clarify it.

Some of our results, especially concerning π, were obtained ca. 1960.

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2. Ingredients

The materials required for a polarized abelian variety over the complex field $\mathbf{C}$ are as follows (cf. Weil [11], [12]): a real vector space V of even dimension, say $2n$; a lattice $L \subset V$, i.e. a discrete subgroup of rank $2n$; an endomorphism J of V such that $J^2 = -Id$; an alternating bilinear form $E: V \times V \to \mathbf{R}$ (real field) such that $E(u, v)$ is integral for all vectors u, v in L and, finally, such that the bilinear form $Q(x, y) = E(x, Jy)$ on V is symmetric and positive definite.

V is then made into a complex vector space by defining $i \cdot x = Jx$ $(i = \sqrt{-1})$; and $H(x, y) = Q(x, y) - i \cdot E(x, y)$ is an Hermitian metric on V. It induces a Hodge metric on the quotient, $A = V/L$, which thereby becomes a polarized abelian variety. H is called a *Riemann form* for A (but terminology is rather variable).

Observe that $E(Jx, Jy) = Q(Jx, y) = Q(y, Jx) = E(x, y)$.

3. Subspaces

In the sequel we shall require only a special case of the following elementary result.

Let $e_1, \ldots, e_{2n}$ be a base for L, and let U be the subspace of V spanned by linearly independent vectors $u_1, \ldots, u_m$, with say $u_\alpha = u_\alpha^j e_j$ (summation convention here and elsewhere).

PROPOSITION 1: *$U \cap L$ has rank m if and only if there is a number $c \neq 0$ such that $c \cdot D$ is integral for every $m \times m$ minor D of the matrix $u = (u_\alpha^j)$.*

PROOF: If $U \cap L$ has rank m, then there is a non-singular matrix $a = (a_\beta^\alpha)$ such that

$$u_\alpha^j a_\beta^\alpha = \text{integer for all } j, \beta.$$

If u' is an $m \times m$ submatrix of u, then $u'a$ is integral, and so $(\det u')(\det a)$ is an integer.

For the converse, let $c \cdot \det u' \in \mathbf{Z}$ for all such u'. Write u, with possibly permuted rows, as

$$u = \begin{pmatrix} u' \\ u'' \end{pmatrix},$$

where u'' is of size $2n \times m$. Assuming $\det u' \neq 0$, let v be the inverse of u'.

If u^k denotes the k-th row of u, then $u^k = p_1 u^1 + \ldots + p_m u^m$ for certain numbers p_α. Now replace row j of u' by u^k. The determinant is then $p_j \cdot D$, where $D = \det u'$. By assumption, c times this determinant is integral, i.e.

$$cDp_j = t_j \in \mathbf{Z}.$$

We have

$$u^k vcD = t_j u^j v = (t_1, \ldots, t_m),$$

and so

$$uvcD = \begin{pmatrix} cD \cdot I_m \\ T \end{pmatrix}, \quad T \text{ integral.}$$

Finally, cD is integral, and therefore so is ua, where $a = cDv$. Q.E.D.

If U satisfies the conditions of the proposition and is moreover stable under the operator J, then $U/U \cap L$ has an induced structure of abelian subvariety of A.

4. Higher Jacobians

Let p be an odd integer $< 2n$ (later we shall want $1 < p \leq n$). Form the exterior products

$$\hat{V} = \wedge^p V, \; \hat{J} = \wedge^p J,$$

and let $\hat{L}$ denote the lattice generated by the elements

$$e_I = e_{i_1} \wedge \ldots \wedge e_{i_p},$$

where again e_i is a base for L. I is the multi-index $I = (i_1, \ldots, i_p)$ with $1 \leq i_1 < \ldots < i_p \leq 2n$. Next, writing $\hat{x} = x_1 \wedge \ldots \wedge x_p$, $\hat{y} = y_1 \wedge \ldots \wedge y_p$, we set

$$\hat{E}(\hat{x}, \hat{y}) = \det(E(x_i, y_j))_{i,j=1,\ldots,p}$$

and we have

$$\begin{aligned} \hat{Q}(\hat{x}, \hat{y}) = \hat{E}(\hat{x}, \hat{J}\hat{y}) &= \det(E(x_i, Jy_j))_{i,j} \\ &= \det(E(y_j, Jx_i)) \\ &= \hat{Q}(\hat{y}, \hat{x}). \end{aligned}$$

If $u_1, \ldots, u_{2n}$ is an orthonormal basis for V relative to the form Q, then

$$\hat{Q}(u_I, u_J) = \det(Q(u_{i_\mu}, u_{j_\nu}))_{\mu, \nu = 1, \ldots, p}$$
$$= \begin{cases} 0 & \text{if } I \neq J \\ 1 & \text{if } I = J. \end{cases}$$

(Here and below we use J both for the complex operator in V and as a multi-index. No confusion can possibly result.)

From §2 and the foregoing we have thus obtained a new polarized abelian variety

$$\hat{A} = \hat{V}/\hat{L},$$

which can be identified with Weil's p-th Jacobian of A (cf. [10]).

5. The homomorphism π

Again let $e_1, \ldots, e_{2n}$ be a base for the lattice L, and introduce real coordinates x in V or A by $x \to x^i e_i$. From the Riemann form E we obtain the differential 2-form

$$\omega = \tfrac{1}{2} a_{ij} dx^i \wedge dx^j, \quad a_{ij} = E(e_i, e_j).$$

Write $q = \dfrac{p+1}{2}$ and build the $2q$-form

$$\omega_q = 2^{-q} a_{i_1 j_1} \cdots a_{i_q j_q} dx^{i_1} \wedge dx^{j_1} \wedge \ldots \wedge dx^{i_q} \wedge dx^{j_q}$$
$$= \frac{1}{(2q)!} a_{i_1 \ldots i_{p+1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{p+1}},$$

where

$$a_{i_1 \ldots i_{p+1}} = 2^{-q} \delta^{j_1 \ldots j_{p+1}}_{i_1 \ldots i_{p+1}} a_{j_1 j_2} \cdots a_{j_p j^{p+1}}.$$

Let A^{ij} denote the cofactor of a_{ji}. Thus

$$A^{ij} a_{jk} = (\det a) \cdot \delta^i_k,$$

where $a = (a_{ij})$. Set

$$a^j_I = a^j_{i_1 \ldots i_p} = A^{kj} a_{i_1 \ldots i_p k},$$

and define $\pi : \hat{V} \to V$ by

$$e_I \to a_I^j e_j.$$

Here and below, $I = (i_1, \ldots, i_p)$ with $i_1 < \ldots < i_p$, etc.

Since the a_I^j are integers, this map carries $\hat{L}$ into L and so induces a homomorphism, still called π, of $\hat{A}$ to A.

It is easy to verify the following fact:

PROPOSITION 2: *If $\pi' : \hat{V} \to V$ is the homomorphism defined in an analogous manner starting from any basis $v_1, \ldots, v_{2n}$ of V, then $\pi' = (\det c)^2 \pi$, where $v_i = c_i^j e_j$ and $c = (c_i^j)$.*

In particular, if $\{v_i\}$ is a new base for L, then the matrix c is unimodular, and so then $\pi' = \pi$.

PROPOSITION 3: *π is surjective.*

PROOF: Let $\{e_i'\}$ be a sympletic basis for V:

$$E(e_i', e_j') = \begin{cases} 0 & \text{if } j \neq n + i,\ i = 1, \ldots, n \\ 1 & \text{if } j = n + i. \end{cases}$$

By Prop. 2 we have only to show that the homomorphism π' constructed from $\{e_i'\}$ is surjective. We introduce real coordinates (y^i) by $y \to y^i e_i'$. Then

$$\omega = \sum_{i=1}^{n} dy^i \wedge dy^{i^*},$$

where we write

$$i^* = i + n \text{ for } i = 1, \ldots, n.$$

It will also prove convenient to write

$$i^* = i - n \text{ for } i = n + 1, \ldots, 2n,$$

and

$$I^* = (i_1^*, \ldots, i_p^*)$$

for a multi-index $I = (i_1, \ldots, i_p)$.

We have

$$\omega^q = (-1)^q \sum_{i_k \leq n} dy^{i_1} \wedge \ldots \wedge dy^{i_q} \wedge dy^{i_1^*} \wedge \ldots \wedge dy^{i_q^*},$$

where again $q = \frac{1}{2}(p+1)$; and so

$$\omega^q = \frac{1}{(p+1)!} b_{i_1 \ldots i_p} dy^{i_1} \wedge \ldots \wedge dy^{i_{p+1}},$$

$b_{i_1 \ldots i_p}$ having the value ± 1 if $\{i_1, \ldots, i_{p+1}\} = \{i_1^*, \ldots, i_{p+1}^*\}$ and if there are no repeated indices. Otherwise $b_{i_1 \ldots i_p} = 0$. In the index equality we mean that the *set* $\{i_1, \ldots, i_{p+1}\}$ is invariant under the involution $i \to i^*$. If now B^{jk} is the cofactor of $b_{kj} = E(e'_k, e'_j)$, we have

$$b_I^k = B^{jk} b_{i_1 \ldots i_p j} = \pm b_{i_1 \ldots i_p k^*}.$$

Accordingly the homomorphism π' is given by

$$e'_I = e'_{i_1} \wedge \ldots \wedge e'_{i_p} \to b_I^j e'_j.$$

Fix k and take

$$I = (i_1, i_1^*, \ldots, i_q, i_q^*, k).$$

For this I, $b_I^j = 0$ unless $j = k$. It is then clear that e'_k occurs in the image of π', hence also of π. Q.E.D.

Proposition 4: *π is* **C**-*linear.*

Proof: Again let $\{e_i\}$ be a base for L and write

$$Je_i = h_i^j e_j, \quad E(e_i, e_j) = a_{ij}.$$

Then

$$Q(e_i, e_j) = E(e_i, Je_j) = h_j^k a_{ik}.$$

Next,

$$\begin{aligned}\pi(\hat{J}e_I) &= (Je_{i_1} \wedge \ldots \wedge Je_{i_p}) \\ &= (h_{i_1}^{j_1} \ldots h_{i_p}^{j_p} e_{j_1} \wedge \ldots \wedge e_{j_p}) \\ &= h_{i_1}^{j_1} \ldots h_{i_p}^{j} a_{j_1 \ldots j_p}^{i} e_i;\end{aligned}$$

and

$$J(\pi e_I) = a^m_{i_1 \dots i_p} h^i_m e_i.$$

We must show that

$$h^{j_1}_{i_1} \dots h^{j_p}_{i_p} a^i_{j_1 \dots j_p} = a^m_{i_1 \dots i_p} h^i_m.$$

Multiply both sides by a_{ki}. Since $A^{ji} a_{ki} = -(\det a)\delta^j_k$, the left side is

$$-h^{j_1}_{i_1} \dots h^{j_p}_{i_p} (\det a) a_{j_1 \dots j_p k}.$$

The right side is

$$a^m_{i_1 \dots i_p} h^i_m a_{ki} = a^m_{i_1 \dots i_p} h^i_k a_{mi} = (\det a) a_{i_1 \dots i_p i} h^i_k.$$

Multiply both members by $h^k_{i_{p+1}}$: We must show that

$$h^{j_1}_{i_1} \dots h^{j_{p+1}}_{i_{p+1}} a_{j_1 \dots j_{p+1}} = a_{i_1 \dots i_{p+1}}.$$

Observe that

$$a_{km} h^k_i h^m_j = E(Je_i, Je_j) = a_{ij}.$$

Now the left side of the previous equation can be written

$$2^{-q} h^{j_1}_{i_1} \dots h^{j_{p+1}}_{i_{p+1}} \delta^{m_1 \dots m_{p+1}}_{j_1 \dots j_{p+1}} a_{m_1 m_2} \dots a_{m_p m_{p+1}}$$
$$= 2^{-q} \sum_s (\operatorname{sign} s) h^{m_1}_{i_{s1}} \dots h^{m_{p+1}}_{i_{s(p+1)}} a_{m_1 m_2} \dots a_{m_p m_{p+1}},$$

where s runs through the permutations of $\{1, \dots, p+1\}$. By the remark above, this reduces to

$$2^{-q} \sum_s (\operatorname{sign} s) a_{i_{s1} i_{s2}} \dots a_{i_{sp} i_{s(p+1)}} = a_{i_1 \dots i_{p+1}}. \qquad \text{Q.E.D.}$$

6. A special base

We now fix a *quasi-symplectic* base e_i in L for the form E. That is, for $i = 1, \dots, n$,

$$E(e_i, e_j) = \begin{cases} d_j & \text{for } j = i^* \\ 0 & \text{for } j \neq i^*, \end{cases}$$

where the d_i are integers such that $d_1|d_2|\dots|d_p$; viz., they are the elementary divisors of E (cf. Siegel [9], p. 65). For indices $n+1,\dots,2n$ we write $d_{i+n} = -d_i$; i.e. $d_{i_*} = -d_i$.

As before, we introduce coordinates x^i in V by $x \to x^i e_i$, and similarly x^I in $\hat{V}$ by $x^I e_I$. That being so, we can write

$$\omega = \tfrac{1}{2} a_{ij} dx^i \wedge dx^j, \quad a_{ij} = E(e_i, e_j),$$

where $a_{ij} = 0$ if $j \neq i^*$, and $a_{ii^*} = d_i$. Similarly, on $\hat{V}$ we have

$$\hat{\omega} = \tfrac{1}{2} \hat{a}_{IJ} dx^I \wedge dx^J, \quad \hat{a}_{IJ} = \hat{E}(e_I, e_J).$$

Here $\hat{a}_{IJ} = 0$ if $J \neq I^*$, and $\hat{a}_{II^*} = d_I = d_{i_1} \dots d_{i_p}$. Indeed, the products $e_I = e_{i_1} \wedge \dots \wedge e_{i_p}$ yield a quasi-symplectic base for $\hat{L}$:

$$\hat{E}(e_I, e_J) = \begin{cases} \gamma(I) = (-1)^{p(l+1)} d_I & \text{if } J = I^* \\ 0 & \text{if } J \neq I^* \end{cases}$$

where $l =$ number of $i \leq n$ in I.

7. The kernel of π

We continue with the notation of §3, but henceforth referred to the quasi-symplectic base $\{e_i\}$ of the preceding paragraph. Then $\pi(e_I) = a_I^k e_k$, and it follows from the proof of Prop. 3 that

> $a_I^k = 0$ *unless* a_{Ik^*} *has a set* Ik^* *of*
> $p + 1$ *distinct indices, invariant under* $*$.

Call an index I "good" ($I \in G$) if I contains $\dfrac{p-1}{2}$ distinct pairs i, i^* and one index, denoted by $\rho(I)$, different from all of those. Otherwise write $I \in B$ ("bad"). If $I \in B$, then $a_I^k = 0$ for all k.

For $k = 1,\dots,2n$ we now let $\lambda(k)$ denote an index $J = (j_1,\dots,j_p)$ in G with $\rho(J) = k$ and with maximum absolute value of

$$\prod_{j \neq k} d_j.$$

We can clearly do that in such a way that $\lambda(k^*) = \lambda(k)^*$. Finally, we let G_0 be the set of good indices I such that $I \neq \lambda(\rho(I))$. That being fixed we have

PROPOSITION 5: *$W = \operatorname{Ker} \pi : \hat{V} \to V$ is spanned by the elements*

$$\begin{cases} \text{(i) } e_I \text{ with } I \in B, \\ \text{(ii) } e_I - m_I e_{\lambda\rho(I)}, \quad I \in G_0, \end{cases}$$

where m_I is a product of certain d_j.

PROOF: For bad I, $\pi(e_I) = 0$, as already pointed out. For $I \in G_0$, $\pi(e_I) = a_I^j e_j$ and $\pi(e_{\lambda\rho(I)}) = a^j_{\lambda\rho(I)}$. Let $\rho(I) = k$. Then as remarked above, $\pi(e_I) = a_I^k e_k$ (no sum on k); and $\pi(e_{\lambda\rho(I)}) = a^k_{\lambda\rho(I)} e_k$. From the definitions and the fact that $d_1 | d_2 | \ldots | d_p$ it follows easily that the coefficients here are both products of d_i's, and that the latter is larger than a_I^k in absolute value, hence is divisible by a_I^k.

The elements (i), (ii) are obviously linearly independent. $\hat{V}$ has dimension $\binom{2n}{p}$, and the number of elements (i), (ii) is clearly $\binom{2n}{p} - 2n$. Then Prop. 5 follows from the fact that π is surjective. Q.E.D.

Let K denote the kernel of $\pi : \hat{A} \to A$. Then K is an abelian subvariety of $\hat{A}$, hence has the form

$$K = W/W \cap \hat{L}.$$

The elements (i), (ii) are in $\hat{L}$ and consequently generate a sub-lattice L' of $\hat{L}$ of finite index in $W \cap \hat{L}$. Then the abelian variety

$$K' = W/L'$$

is a finite covering of K.

8. Splitting of $\hat{A}$

W being as above the kernel of $\pi : \hat{V} \to V$, let U denote the orthogonal complement of $\hat{W}$ with respect to the form $\hat{E}$ of §4.

If $\hat{x} \in U$, so that $\hat{E}(\hat{x}, W) = 0$, then

$$\hat{E}(\hat{J}\hat{x}, W) = \hat{E}(\hat{J}\hat{x}, \hat{J}W) = \hat{E}(\hat{x}, W) = 0.$$

Therefore U is a complex subspace of $\hat{V}$. Further, if $\hat{x} \in U \cap W$, then $\hat{J}\hat{x} \in W$, and so $\hat{E}(\hat{x}, \hat{J}\hat{x}) = 0$, i.e. $\hat{Q}(\hat{x}, \hat{x}) = 0$, whence $\hat{x} = 0$. Thus $\hat{V} = U + W$ (direct sum). This splitting leads to a splitting of $\hat{A}$ in accordance with the "complete reducibility theorem" of Poincaré (cf. Weil [12], Chapitre VI, no. 11, Théorème 6). Here we shall make this explicit (cf. §2, Prop. 1).

For $k = 1, \ldots, 2n$ set

$$\bar{e}_k = e_{\lambda(k)} + \sum_{J \in S_k} c_k^J e_J,$$

where

$$S_k = \{I \in G_0, \rho(I) = k\},$$

and where

$$c_k^J = m_{J^*} d_{\lambda(k)} / d_{J^*}$$

(see Prop. 5 for notations). Observe that $c_k^{\lambda(k)^*} = 1$.

Proposition 6: *The elements $\bar{e}_k$ form a basis for U.*

Proof: For $I \in B$ it is clear that

$$\hat{E}(\bar{e}_k, e_I) = 0.$$

Next take $I \in G$ with $\rho(I) = k$, $I \neq \lambda(k)$. That is, $I \in S_k$. Then it is plain that $I \neq \lambda(k)^*$. We have (cf. Prop. 5)

$$\begin{aligned}\hat{E}(\bar{e}_k, e_I - m_I e_{\lambda(k)}) &= \hat{E}(e_{\lambda(k)}, e_I) - m_I \hat{E}(e_{\lambda(k)}, e_{\lambda(k)}) \\ &\quad + \sum_{J \in S_k} c_k^J \hat{E}(e_J, e_I) - m_I \sum_{J \in S_k} c_k^J \hat{E}(e_J, e_{\lambda(k)}).\end{aligned}$$

The first two terms vanish; the last two reduce to

$$\begin{aligned}&c_k^{I^*} \hat{E}(e_{I^*}, e_I) - m_I c_k^{\lambda(k)^*} \hat{E}(e_{\lambda(k)^*}, e_{\lambda(k)}) \\ &\quad = -d_I c_k^{I^*} + m_I d_{\lambda(k)} c_k^{\lambda(k)^*} \\ &\quad = -d_I m_I d_{\lambda(k)} / d_I + m_I d_{\lambda(k)} = 0.\end{aligned}$$

Hence the $\bar{e}_k$ are orthogonal to the kernel of π. They are obviously linearly independent, which establishes the proposition. Q.E.D.

It is evident that the elements $\bar{e}_k$ are in $\hat{L}$. They therefore generate a sub-lattice L_0 of rank $2n$ contained in U. Thus $U \cap \hat{L}$ has rank $2n$, and accordingly we have an abelian subvariety

$$A' = U / U \cap \hat{L}$$

in $\hat{A}$ and a finite covering

$$B = U/L_0$$

of A'. Via π, A' is a finite covering of A, and therefore so is B.

Write

$$\bar{e}_k = e_{\lambda(k)} + \sum_{J \in S_k} c_k^J(e_J - m_J e_{\lambda(k)}) + \sum_{J \in S_k} c_k^J m_J e_{\lambda(k)}.$$

Then

$$\bar{e}_k = h_k e_{\lambda(k)} \,(\mathrm{mod}\ W),$$

where h_k is an integer $\neq 0$, and

$$\pi(\bar{e}_k) = h_k \pi(e_{\lambda(k)}) = h_k a^j_{\lambda(k)} e_j.$$

But $a^j_{\lambda(k)} = 0$ unless $j = k^*$, and so we have

$$\pi(\bar{e}_k) = p_k e_k,$$

the p_k being non-zero integers.

9. Forms of type (p, p)

Let $v_1, \ldots, v_n$ be a complex basis in V. We introduce complex coordinates $z^\alpha = \xi^\alpha + i\eta^\alpha$ by $z \to z^\alpha v_\alpha$. And we now assume that the odd integer p is $\leq n$. A harmonic (p, p)-form φ on A has an expression

$$\varphi = \sum \varphi_{A\bar{B}} dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \ldots \wedge d\bar{z}^{\beta_p},$$

where the coefficients are *constants*. We assume that φ is real: $\varphi_{B\bar{A}} = -\bar{\varphi}_{A\bar{B}}$. We assume further that all of the periods of φ are integral.

We now define a bilinear form Φ on $\hat{V}$ by

$$\Phi(e_I, e_J) = \int_{T_{IJ}} \varphi,$$

where T_{IJ} is the oriented torus generated by $e_{i_1}, \ldots, e_{i_p}, e_{j_1}, \ldots, e_{j_p}$, the e_i as in §7. It is well known that the cycles T_{IJ} with $I \cap J = \varnothing$ generate the integral homology of A in dimension $2p$ (cf. Lefschetz [6], Chapitre 6, No. 6).

PROPOSITION 7: *If $\varphi \neq 0$, then the bilinear form Φ on $\hat{V}$ is not the zero form. It is alternating and has integral values on $\hat{L}$. Moreover, it is of type* (1, 1).

PROOF: The first assertion follows from the preceding remark; Φ is alternating because JI is an odd permutation of IJ; it is integral on $\hat{L} \times \hat{L}$ because of our assumption that φ has integral periods. We come now to the last point.

Set

$$v_\alpha = q^j_\alpha e_j, \quad e_j = r^\beta_j v_\beta.$$

Then

$$x^j r^\beta_j = z^\beta,$$

where (x^j) is the real coordinate system on V defined by the basis $\{e_i\}$ (see §6). For uniformity of notation, let $J = (i_{p+1}, \ldots, i_{2p})$. Then

$$\int_{T_{IJ}} \varphi = \sum \varphi_{A\bar{B}} \cdot \int_{(0 \le x^i \le 1)} \frac{\partial(z^{\alpha_1}, \ldots, z^{\alpha_p}, \bar{z}^{\beta_1}, \ldots, \bar{z}^{\beta_p})}{\partial(x^{i_1}, \ldots, x^{i_{2p}})} dx^{i_1} \ldots dx^{i_{2p}}$$

$$= \sum \varphi_{A\bar{B}} \begin{vmatrix} r^1_{i_1} \ldots r^p_{i_1} \; \bar{r}^1_{i_1} \ldots \bar{r}^p_{i_1} \\ \cdots\cdots\cdots \\ r^1_{i_{2p}} \ldots r^p_{i_{2p}} \bar{r}^1_{i_{2p}} \ldots \bar{r}^p_{i_{2p}} \end{vmatrix}.$$

Then

$$\Phi(e_I, \hat{J} e_J) = h^{k_1}_{i_1} \ldots h^{k_p}_{i_p} \Phi(e_I, e_{k_1} \wedge \ldots \wedge e_{k_p}).$$

But

$$h^k_j r^\alpha_k = i r^\alpha_j$$

(cf. §5), and so

$$\Phi(e_I, \hat{J} e_J) = i^p \sum \varphi_{A\bar{B}} \cdot \left| \begin{array}{c|c} r^\alpha_i & -\bar{r}^\beta_i \\ \hline r^\alpha_j & \bar{r}^\beta_j \end{array} \right| = \Phi(e_J, \hat{J} e_I).$$

Q.E.D.

Φ defines a harmonic differential form $\hat{\varphi}$ of type (1, 1) on $\hat{A}$ whose periods are the quantities $\Phi_{IJ} = \Phi(e_I, e_J)$. In terms of the real coordinates x^I on $\hat{A}$ (or $\hat{V}$),

$$\hat{\varphi} = \tfrac{1}{2} \Phi_{IJ} dx^I \wedge dx^J.$$

It should be noted that we have not attempted to associate complex coordinates in $\hat{V}$ with the z^α in V.

PROPOSITION 8: *If the form φ is positive, then so is $\hat{\varphi}$.*

PROOF: φ is positive if the hermitian form

$$i^p \sum \varphi_{A\bar{B}} U^A \overline{U^B}$$

is positive, which we now suppose. Then

$$\Phi(\hat{x}, \hat{J}\hat{x}) = x_1^{i_1} \dots x_p^{i_p} x_1^{j_1} \dots x_p^{j_p} \Phi(e_I, \hat{J}e_J).$$

Here

$$\hat{x} = x_1 \wedge \dots \wedge x_p,$$

and

$$x_i = x_i^j e_j = x_i^j r_j^\alpha v_\alpha = z^\alpha v_\alpha.$$

From the expression exhibited above for $\Phi(e_I, \hat{J}e_J)$ we obtain

$$\begin{aligned}\Phi(\hat{x}, \hat{J}\hat{x}) &= i^p \sum \varphi_{A\bar{B}} \cdot \left| \begin{array}{c|c} z_i^\alpha & -\bar{z}_i^\beta \\ \hline z_i^\alpha & \bar{z}_i^\beta \end{array} \right| \\ &= 2^p i^p \sum \varphi_{A\bar{B}} \det(z_i^\alpha) \det(\bar{z}_i^\beta),\end{aligned}$$

which is positive, for the decomposable vector $\hat{x}$. If we replace $\hat{x}$ by $\hat{x} + \hat{y}$, writing $y_i = y_i^j e_j = w^\alpha v_\alpha$, then in the determinant above z_i^α will be replaced by $z_i^\alpha + w_i^\alpha$, and the same conclusion obtains. The assertion follows easily for an arbitrary element of $\hat{V}$ different from zero. Q.E.D.

Of course in general φ will not be positive; but for a suitable integer b the form $\varphi + b\omega^p$ will be positive.

10. Algebraic cycles

The construction of the preceding paragraph provides us with an isomorphic mapping, call it f, from the group F_p of real harmonic (p, p)-forms φ with integral periods on A into the group of real $(1, 1)$-forms $\hat{\varphi}$ with integral periods on $\hat{A}$. We recall that any algebraic cycle of (real) dimension $2n - 2p$ on A (more precisely, its homology class) is dual to such a (p, p)-form φ, and so the group of real homology classes G_p of such cycles is embedded in F_p, by duality.

On the other hand, $\hat{\varphi} = f(\varphi)$ is dual to a class of $(2N-2)$-cycles on $\hat{A}$ (N = complex dimension of $\hat{A}$), and the class contains an algebraic cycle (divisor). If D is such a divisor, then D establishes an algebraic correspondence between A and the kernel K of π (cf. [1, Chapter 6], [4, Book III, Chapter XI] and §7 above). Let Z denote an algebraic cycle on K of real dimension $2r$, and put $\hat{Z} = \pi^{-1}(Z)$. Then the divisor D can be so translated in $\hat{A}$ that the intersection $D \cdot \hat{Z}$ is defined. Its projection in A is then an algebraic cycle Z_A of dimension $2r - 2$ (which we assume of course to be $\leq 2n$), or else is zero. In this way we obtain algebraic cycles of various dimensions in A.

As we are concerned here with cycle classes with respect to real homology, the above operations can be taken in the homological sense. Thus, if $\hat{\omega}$ is the fundamental Kähler $(1, 1)$-form on $\hat{A}$ and ω_K is the induced form on K, the class dual to $\omega_K^{N'-r}$ ($N' = \dim K = N - n$) contains an algebraic $2r$-cycle Z (intersection of divisors on K). Then $\hat{Z}$ is dual as cycle to some multiple $k\hat{\omega}^{N'-r}$. The harmonic form on A dual to the projection Z_A is then obtained by integration of this latter form over the fibres of $\pi : \hat{A} \to A$.

In particular, for suitable r we get a homomorphism f^* from F_p to G_p.

Despite the very explicit nature of the calculations, or perhaps because of it, i.e. using the results of §§7, 8, it seems quite difficult to clarify the connexion between $f^*(\varphi)$ and φ. If however it could be shown that $\operatorname{Ker} f^* = 0$, then from simple rank considerations we could infer that our (p, p)-form φ is dual to a class c such that some multiple kc contains an algebraic cycle.

11. Remarks

In the foregoing p was always taken to be an odd integer. Weil's higher Jacobians for even p are defined by means of duality of abelian varieties. But with reference to forms of type (p, p), for even p it would seem more to the point to replace φ by $\varphi \wedge \omega$ or by $\Lambda\varphi$ (Λ as in [12]).

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