

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 47, n° 1 (1982), p. 85-93

<[http://www.numdam.org/item?id=CM\\_1982\\_\\_47\\_1\\_85\\_0](http://www.numdam.org/item?id=CM_1982__47_1_85_0)>

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## ABELIAN CLOSURE IN SOLUBLE LIE RINGS

P. McInerney

In this paper we examine the question of what properties are inherited by soluble Lie rings from their abelian subrings.

More formally, suppose  $\mathcal{X}$  is a class of Lie rings which is closed with respect to taking abelian subrings. We will say that  $\mathcal{X}$  is *abelian-closed* if given any soluble Lie ring  $L$  all of whose abelian subrings are in  $\mathcal{X}$  then  $L$  is also in  $\mathcal{X}$ .

Similar investigations have been made for groups by Mal'cev [6], Schmidt [7] and Čarin [2].

### §1. Notation and Initial Observations

Most of the terminology we will use is already fairly standard in the theory of Lie algebras (see Jacobson [5] or Amayo and Stewart [1]). However some notions special to Lie rings, and due for the most part to the existence of torsion elements, need explanation.

The collection of all torsion elements of a Lie ring  $L$  is a characteristic ideal and is denoted by  $T(L)$  (a characteristic ideal is invariant under any derivation of  $L$ ).

If  $p$  is a prime then  $L_p$  denotes the collection of all  $x \in L$  such that  $p^k x = 0$  for some positive integer  $k$ .  $L_p$  is also a characteristic ideal, called the  $p$ -component of  $L$  and  $T(L)$  is a direct sum of the various  $L_p$  as  $p$  ranges over all primes (cf. Fuchs [3] Ch XII).

Likewise the divisible subgroup of  $L$  is a characteristic ideal denoted by  $D(L)$ .

The *torsion free rank*  $r_0(L)$ , the  $p$ -rank  $r_p(L)$  and the *total rank*  $r(L)$  are just the corresponding ranks for the underlying abelian group of  $L$  (Fuchs [4] pg. 85 ff).

We will use the following notation for the classes of Lie rings we consider:

$\mathcal{F}$	finite Lie rings
Max	Lie rings satisfying the ascending chain condition on subrings
Min	Lie rings satisfying the descending chain conditions on subrings
$\mathcal{G}$	finitely generated Lie rings
$\mathcal{G}_Z$	Lie rings of finite type i.e. having a finitely generated underlying abelian group
$\mathcal{C}$	one generator (i.e. cyclic) Lie rings
$\mathcal{A}$	abelian Lie rings
$\mathcal{A}_0$	abelian Lie rings with $r_0(L) < \infty$
$\mathcal{A}_1$	abelian Lie rings with $r_0(L) < \infty$ and $r_p(L) < \infty$ for all primes $p$ .
$\mathcal{A}_2$	abelian Lie rings with $r(L) < \infty$
$\mathcal{A}_3$	abelian Lie rings with $r_0(L) < \infty$ and $T(L) \in \mathcal{F}$

Note that  $\mathcal{C} < \mathcal{A}$  and also  $\mathcal{A}_3 < \mathcal{A}_2 < \mathcal{A}_1 < \mathcal{A}_0 < \mathcal{A}$  since these relations are already true for abelian groups.  $\mathcal{A}$ ,  $\mathcal{A}_0$ ,  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  and  $\mathcal{A}_3$  are all closed under taking subrings but only  $\mathcal{A}$ ,  $\mathcal{A}_0$  and  $\mathcal{A}_1$  are closed under quotients.

If  $\mathcal{X}$  is a class of Lie rings then we write  $E\mathcal{X}$  for the class of Lie rings with a finite ideal series each of whose factors is in  $\mathcal{X}$ . We shall be interested in the classes  $E\mathcal{C}$  (polycyclic Lie rings),  $E\mathcal{A}$  (soluble Lie rings) and  $E\mathcal{A}_i$ ,  $i = 0, \dots, 3$ . All these classes are closed under taking subrings.

Note that we will often consider abelian Lie rings simply as abelian groups, carrying over properties and terminology where convenient. Fuchs [4] can always be used as a reference. Similarly, properties of the underlying abelian group of a Lie ring  $L$  are often useful in determining the properties of  $L$ . We will frequently use the fact that the derivation ring,  $\text{Der}(L)$ , of  $L$  is a Lie subring of the endomorphism ring of  $L$  considered as an abelian group and supplied with the usual commutator product.

## §2. Abelian Closure and Chain Conditions

First we will consider the minimal condition on subrings. If  $L \in \text{Min}$  then it contains a unique minimal ideal of finite index. If  $L$  is also soluble then we have:

**LEMMA 1:** *Let  $L \in \mathcal{EA} \cap \text{Min}$ , then  $L$  is a finite extension of a central, divisible, abelian ring (and consequently is countable).*

**PROOF:**  $L$  has an invariant abelian series each of whose factors satisfies the descending chain condition on subrings and hence each factor is a torsion abelian group. Hence  $L$  is a torsion ring.

Let  $N$  be the unique minimal ideal of finite index. Then  $N$  has no proper subrings of finite index. For suppose  $P$  were such a subring, then for some integer  $m$

$$mL \leq P$$

Now  $mL$  is always a (characteristic) ideal of  $L$  and so we can form  $N/mL$  and by construction this has finite exponent.

By solubility there is a characteristic abelian series.

$$mL = L_0 < L_1 < \cdots < L_n = N$$

Then  $L_n/L_{n-1}$  is abelian of finite exponent and satisfies the minimal condition for subrings. Hence it is finite. But  $L_{n-1}$  is a characteristic ideal in  $N$  and so is an ideal in  $L$  (if  $I$  is an ideal of  $L$  and  $J$  is a characteristic ideal of  $I$  then  $J$  is always an ideal of  $L$ ). But this is a contradiction, hence  $N$  has no proper subrings of finite index.

If  $m > 0$  and  $mN < N$  then as above there exists a characteristic abelian series from  $mN$  to  $N$ . Looking at the top factor again gives a contradiction and so  $mN = N$  for all  $m$  and so  $N$  is divisible.

Finally, in a torsion Lie ring  $D(L)$  is contained in the centre  $Z(L)$ . Indeed suppose  $y \in L$  and  $ny = 0$  for some integer  $n > 0$ . Let  $x \in D(L)$ . By divisibility we can find  $z \in D(L)$  such that  $x = nz$ . Then

$$[x, y] = [nz, y] = [z, ny] = 0$$

and so  $x \in Z(L)$ .

Note that the initial conditions in this result can be weakened to allow  $L$  to have only the minimal condition on subideals. But then the result in any case forces  $L$  to satisfy the minimal condition on subrings.

**LEMMA 2:** *Let  $L \in \mathcal{EA} \cap \text{Min}$  and let  $\Gamma \leq \text{Der}(L)$  be a torsion Lie ring of derivations of  $L$ . Then  $\Gamma \in \mathcal{F}$*

**PROOF:** By Fuchs [3] p. 207 the endomorphism ring of a divisible

abelian group is torsion free. Hence any Lie ring of derivations of a divisible Lie ring is torsion free.

By Lemma 1,  $D = D(L)$  is torsion abelian, and  $L/D \in \mathcal{F}$ . Now  $\Gamma$  induces a Lie ring of derivations on the characteristic ideal  $D$ , and since  $\Gamma$  is torsion this action must be trivial by the above.

Now consider a derivation  $d: L \rightarrow L$  inducing zero on  $L/D$  and killing  $D$ . Then  $d$  is fully determined by its action on a finite set  $X$  (of coset representatives for  $L/D$ ), and it sends  $X$  to a subgroup  $Y$  of  $D$ , whose exponent is bounded (by the exponent of  $L/D$ ). Hence  $Y$  is finite since  $D$  is divisible. Consequently, since  $d$  sends a finite set  $X$  to a finite set  $Y$ , only finitely many such  $d$  are possible. Further, since  $L/D \in \mathcal{F}$ , only finitely many derivations  $L/D \rightarrow L/D$  are possible. These two facts taken together mean  $\Gamma \in \mathcal{F}$ .

**THEOREM 3:** *Min is abelian-closed.*

**PROOF:** Suppose  $L$  is soluble with each of its abelian subrings in Min. Let  $L$  have derived length  $d$ . We will use induction on  $d$ .

Let  $N$  be an ideal in  $L$  maximal with respect to  $N^{(d-1)} = 0$  and  $N \geq L^{(1)}$  ( $L^{(n)}$  denotes the  $n$ th term in the derived series of  $L$ ). By the induction hypothesis  $N \in \text{Min}$ .

Now consider

$$C = C_L(N) = \{y \in L \mid [N, y] = 0\}$$

By the maximality of  $N$  we have  $C \leq N$ . The cyclic subrings generated by each element of  $L$  are abelian and so are in Min. Hence  $L$  is a torsion ring.

Since  $N$  is an ideal of  $L$ , so is  $C$  and we can consider  $L/C$  to be a (torsion) Lie ring of derivations of  $N$ . Hence by lemma 2

$$L/C \in \mathcal{F} \leq \text{Min}$$

The result now follows since Min is closed under extensions.

We now consider what happens with the maximal condition. Once again we need information about derivations.

**LEMMA 4:** *Let  $L$  be a Lie ring and  $\Gamma \leq \text{Der}(L)$ . (i) If  $L \in \mathcal{G} \cap \mathcal{A}$  then  $\Gamma \in \mathcal{G}_z < \mathcal{G}$ . (ii) If  $L \in \mathcal{G}_z$  then  $\Gamma \in \mathcal{G}_z < \mathcal{G}$ . (iii) If  $L \in \text{E}\mathcal{C}$  then  $\Gamma \in \mathcal{G}_z < \mathcal{G}$ . Further if  $\Gamma \in \text{E}\mathcal{A}$  then  $\Gamma \in \text{E}\mathcal{C}$ .*

**PROOF:** (i) This follows from the fact that if  $L$  is a finitely

generated abelian group then its ring of endomorphisms is finitely generated as an abelian group (Fuchs [3] p. 212 ff). (ii) By case (i). (iii) Since  $\mathcal{G}_z \cap E\mathcal{A} = E\mathcal{C}$ .

**THEOREM 5:** *If  $L \in E\mathcal{A}$  and each of its abelian subideals is finitely generated then  $L \in E\mathcal{C}$ .*

**PROOF:** This is clearly true when  $L \in \mathcal{A}$  so assume  $L \notin \mathcal{A}$ . Let  $N$  be the last nontrivial term of the derived series of  $L$ . By hypothesis  $N \in \mathcal{G}$ .

Let  $H/N$  be an abelian subideal of  $L/N$  and let  $C = C_H(N)$ . Now  $C \geq N$  and so  $H/C \in \mathcal{A}$  and hence by Lemma 4(i),  $H/C \in \mathcal{G}$ .

Now  $C^2 = [C, C] \leq N$  and so  $[C^2, C] = 0$  and  $C$  is nilpotent (of length 2). Let  $M$  be a maximal abelian ideal of  $C$ .  $M$  is a subideal of  $L$  and so  $M \in \mathcal{G}$ . By the maximality of  $M$  we have

$$M = C_C(M)$$

Hence since  $C^2 \leq Z(C) \leq M$  we have  $C/M \in \mathcal{A}$ . By lemma 4(i) again we have  $C/M \in \mathcal{G}$ . Hence  $H \in \mathcal{G}$ .

Thus  $H/N \in \mathcal{G}$  and  $L/N$  satisfies the initial hypotheses of the theorem. By induction on the derived length the result now follows.

We can now restate this result in a number of forms:

**COROLLARY:** (i)  $E\mathcal{C}$  is abelian-closed. (ii) If  $L \in E\mathcal{A}$  is such that all its abelian subrings are finitely-generated then  $L \in \mathcal{G}$ . (Note that the terminology of abelian-closure cannot be used here since  $\mathcal{G}$  is not closed with respect to taking abelian subrings.) (iii)  $\text{Max}$  is abelian-closed. (iv)  $\mathcal{F}$  is abelian-closed.

**PROOF:** (i) and (ii) follow from  $E\mathcal{C} < \mathcal{G}$ . (iii) Follows since  $E\mathcal{C} < \text{Max}$ . (iv) Let  $L \in E\mathcal{A}$  with all its abelian subrings finite. Now  $\mathcal{F} < \mathcal{G} \cap \text{Min}$ . Hence by Theorem 3 and Theorem 5

$$L \in E\mathcal{C} \cap \text{Min}$$

Now if  $L \in E\mathcal{C}$  then  $D(L) = 0$  and so by Lemma 1 we have  $L \in \mathcal{F}$ .

### §3. Abelian Closure and Rank Conditions

**LEMMA 7:** *Let  $L$  be a torsion Lie ring, and suppose (by slight abuse of notation) that the underlying abelian group of  $L$  is in  $\mathcal{A}_1$ . Then every finite set of elements of  $L$  lies in a finite characteristic ideal of  $L$ .*

PROOF: Since  $L$  is torsion it is the direct sum of its  $p$ -components  $L_p$ , and the underlying abelian group of each  $L_p$  is a (group) direct sum of finitely many cyclic groups of order  $p^k$  for various  $k$  and finitely many Prüfer  $\mathcal{C}_p$  groups.

Let  $x_1, \dots, x_n \in L$  with each  $x_i$  of order  $m_i$  say. Let  $m = m_1 m_2 \dots m_n$ . Clearly  $m$  involves only finitely many primes and so

$$L[m] = \{x \in L \mid mx = 0\}$$

is finite. But this is a characteristic ideal since it is a fully invariant subgroup of  $L$  considered as an abelian group.

LEMMA 8: Suppose  $L$  is a Lie ring and  $H$  is an ideal of  $L$  with  $H \in \mathcal{E}\mathcal{A}_1$  and  $L/H \in \mathcal{A} \setminus \mathcal{A}_0$ . Then  $L$  contains a free abelian subring of countable rank.

PROOF: Case (i)  $H = 0$ . This follows immediately from the definition of  $\mathcal{A}_0$ .

In view of this case, since  $L/H$  will always contain a free abelian subring of countable rank we may assume without loss of generality that  $L/H$  is in fact such a ring.

Case (ii)  $H \in \mathcal{A}_1$  and  $H$  is torsion free.

Let  $A$  be a maximal abelian subring of  $L$  with  $A \geq H$ . Let  $r_0(H) = n$  say. Suppose  $r_0(A) = m (\geq n)$ . By the maximality of  $A$  we have  $A = C_L(A)$  and since  $L/H \in \mathcal{A}$  we have that  $A$  is an ideal of  $L$ . Hence  $L/A$  may be considered as a subring of  $\text{Der}(A)$ .

Now we can consider  $A$  as being embedded in  $V = Q \otimes_{\mathbb{Z}} A$  a vector space over the rationals of dimension  $m$ . Now  $V$  has dimension  $m$ , hence the endomorphism ring of  $V$  has dimension  $m^2$  (being isomorphic to the ring of  $m \times m$  matrices over  $Q$ ). Hence  $\text{Der}(A)$  has rank  $\leq m^2$  and consequently so too does  $L/A$ . This means  $L/A \in \mathcal{A}_1$  and in particular  $r_0(L) < \infty$ , which is a contradiction.

Case (iii)  $H \in \mathcal{A}$  and  $H$  is torsion. Since we are assuming that  $L/H$  is free abelian of countable rank suppose

$$L/H \cong \bigoplus_{i \in \mathbb{Z}} \langle x_i + H \rangle$$

We will construct a sequence of elements  $y_1, y_2, \dots$  such that  $y_i = k_i x_i$  for nonzero integers  $k_i$  and

$$[y_i, y_j] = 0 \quad \text{for all } i \text{ and } j.$$

Since  $L/H \in \mathcal{A}$  we have  $[x, y] \in \mathcal{A}$  for all  $x, y \in L$ . Take  $y_1 = x_1$  and suppose that  $y_1, \dots, y_n$  have been constructed. By Lemma 7  $[y_i, x_{n+1}]$  for  $i = 1, \dots, n$  all lie in a finite characteristic subring  $F \leq H$ . Since  $H$  is an ideal of  $L$  and  $F$  is characteristic,  $F$  is also an ideal of  $L$ . Suppose  $|F| = m$ , then for all  $i = 1, \dots, n$

$$[y_i, mx_{n+1}] = m[y_i, x_{n+1}] = 0$$

Put  $y_{n+1} = mx_{n+1}$  and then  $y_{n+1}$  is as required.

Now let  $A$  be the subring generated by  $y_1, y_2, \dots$ . The natural homomorphism

$$\xi: L \rightarrow L/H$$

maps  $A$  onto  $(A + H)/H$ . Now  $L/H$  is torsion free so  $nx_i \notin H$  for all  $n$ , and since the  $x_i + H$  generate  $L/H$ ,  $\xi$  restricted to  $A$  is injective. Hence  $A$  is free abelian of countable rank.

*Case (iv)  $H \in \mathcal{A}$ .*

Let  $T = T(L)$  and use induction on  $r_0(H/T) = n$ .

If  $n = 0$  then  $T = H$  and case (ii) applies. If  $n > 0$  choose an ideal  $K$  of  $L$  with  $T \leq K \leq H$  and  $K$  of maximal rank subject to

$$r_0(K/T) < r_0(H/T)$$

We may assume  $L/K$  is torsion free (for otherwise we can just factor out the torsion ideal). Case (ii) now applies to show that  $L/K$  has a subring  $A/K$  which is free abelian of countable rank.

The induction hypothesis can now be applied to show that  $A$  has a free abelian subring of countable rank.

*Case (v) The general case.*

We now use induction on the derived length  $d$  of  $H$ .

If  $d = 1$  use case (iv). Suppose  $d > 1$ . Then by induction  $L/H^{(d-1)}$  has a free abelian subring of countable rank. But  $H^{(d-1)} \in \mathcal{A}$  so a further application of case (iv) finishes the argument.

**LEMMA 9:** *If  $L \in \mathcal{EA}$  and  $L$  is torsion free then  $L$  has a finite characteristic series with factors which are torsion free abelian.*

**PROOF:** The proof will be by induction on the derived length  $d$  of  $L$ .

The result is clear when  $d = 1$  so suppose  $d > 1$ . Then  $L/L^{(d-1)} \in \mathcal{EA}$  and has derived length  $d - 1$ . Put

$$T/L^{(d-1)} = T(L/L^{(d-1)})$$

Now  $T$  is a characteristic ideal of  $L$ ,  $L/T$  is torsion free and has derived length  $\leq d - 1$ , hence by induction  $L/T$  has a finite series of the required type.

Let  $C = C_T(L^{(d-1)})$ . Then  $C$  is a characteristic ideal of  $L$  and as usual  $T/C$  may be considered a subring of  $\text{Der}(L^{(d-1)})$ . Now  $L^{(d-1)}$  is torsion free so considered as an abelian group its endomorphism ring is torsion free. Hence  $T/C$  is torsion free. But  $T/C$  is a quotient of  $T/L^{(d-1)}$  which is a torsion ring. Hence  $T = C$ .

So  $[T, L^{(d-1)}] = 0$  and  $L^{(d-1)} \leq Z(T)$ .

Now if  $L$  is a torsion free Lie ring then  $L/Z(L)$  is also torsion free. Indeed let  $x \in L$  be such that  $nx \in Z(L)$  for some integer  $n \neq 0$ . Then for any  $y \in L$

$$0 = [nx, y] = n[x, y]$$

Hence  $[x, y] = 0$  since  $L$  is torsion free and  $x \in Z(L)$ .

Now  $T/Z(T)$  is a quotient of  $T/L^{(d-1)}$  and so is torsion, but by the above observation it is also torsion free. Hence  $T = Z(T)$  and  $T \in \mathcal{A}$ . The result now follows from the case  $n = 1$ .

Suppose  $\mathcal{X}$  is any class of torsion, abelian Lie rings. Define a class  $\bar{\mathcal{X}}$  by  $L \in \bar{\mathcal{X}}$  if and only if

- (i)  $L \in \mathcal{A}$
- (ii)  $T(L) \in \mathcal{X}$

and

$$(iii) \quad r_0(L/T(L)) < \infty$$

**THEOREM 10:** *Let  $\mathcal{X}$  be a class of torsion, abelian Lie rings such that: (a)  $\mathcal{F} \cap \mathcal{A} \leq \mathcal{X} \leq \mathcal{A}_1$ ; (b)  $\mathcal{X}$  is closed under the taking of subrings. Then if  $E\mathcal{X}$  is abelian-closed,  $E\bar{\mathcal{X}}$  is abelian-closed.*

**PROOF:** Let  $L \in E\mathcal{A}$  and suppose that all its abelian subrings lie in  $\bar{\mathcal{X}}$ . We will use induction on the derived length  $d$  of  $L$ .

If  $d = 1$  then  $L \in \bar{\mathcal{X}}$ . If  $d > 1$  then by induction we may assume

$$L^2 \in E\bar{\mathcal{X}} \leq E\mathcal{A}_1$$

If  $L/L^2 \notin \mathcal{A}_0$  then by Lemma 8  $L$  has a free abelian subring of countable rank which is a contradiction. Hence  $L/L^2 \in \mathcal{A}_0$  and so  $L \in \mathcal{A}_0$ .

Put  $T = T(L)$ . Then by Lemma 9  $L/T$  has a finite characteristic

series with torsion free, abelian factors. Hence  $L/T \in E\bar{\mathcal{X}}$  (clearly each of the factors is in  $\bar{\mathcal{X}}$ ).

By hypothesis  $T \in E\mathcal{X}$  and since  $\mathcal{X}$  and  $\bar{\mathcal{X}}$  coincide for torsion rings we have  $T \in E\bar{\mathcal{X}}$ . Hence  $L \in E\bar{\mathcal{X}}$  as required.

We now obtain the required results as corollaries to this theorem.

**COROLLARY 11:** (i)  $E\mathcal{A}_1$  is abelian-closed. (ii)  $E\mathcal{A}_2$  is abelian-closed. (iii)  $E\mathcal{A}_3$  is abelian-closed.

**PROOF:** (i) Take  $\mathcal{X}$  to be the class of torsion Lie rings in  $\mathcal{A}_1$ . Let  $L \in E\mathcal{A}$  be a torsion ring and suppose that all its abelian subrings are in  $\mathcal{A}_1$ .

Now  $L \in E\mathcal{A}_1$  if and only if

$$L_p \in \text{Min} \cap E\mathcal{A}$$

for all primes  $p$ . (Use induction on the derived length  $d$ . Then  $L_p^{(d-1)}$  is in  $\mathcal{A}_1$  and hence, since it is torsion, in  $\text{Min}$ .)

Since  $L_p$  is a direct factor of  $L$ , the abelian subrings of  $L_p$  are precisely the abelian subrings of  $L$  intersected with  $L_p$ . Hence Theorem 3 together with Theorem 10 gives the result.

(ii) Take  $\mathcal{X} = \mathcal{A} \cap \text{Min}$  and use Theorems 3 and 10.

(iii) Take  $\mathcal{X} = \mathcal{A} \cap \mathcal{F}$  and use Corollary 6(iv) and Theorem 10.

### Acknowledgement

I would like to thank the referee for correcting an incomplete proof of Lemma 2.

### REFERENCES

- [1] R.K. AMAYO and I.N. STEWART: Infinite-dimensional Lie algebras. Noordhoff, 1974.
- [2] V.S. ČARIN: On soluble groups of type  $A_4$ . *Mat. Sb.* 52 (1960) 895–914.
- [3] L. FUCHS: *Abelian Groups*. Pergamon Press, 1960.
- [4] L. FUCHS: *Infinite Abelian Groups Vol. 1*. Academic Press, 1970.
- [5] N. JACOBSON: *Lie algebras*. Interscience, 1962.
- [6] A.I. MAL'CEV: On certain classes of soluble groups. *Mat. Sb.* 28 (1951) 567–588. *A.M.S. Transl.* (2) 2 (1956) 1–21.
- [7] O.J. SCHMIDT: Infinite soluble groups. *Mat. Sb.* 17 (1945) 145–162.

(Oblatum 15-IX-1981 &  
2-II-1982)

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