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THE ZEROS OF CERTAIN POINCARÉ SERIES

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1. Introduction

K. Wohlfahrt [6] showed in 1964 that the only zeros of the Eisenstein series E_k for the modular group lie on transforms of the unit circle when $4 \leq k \leq 26$, and conjectured that this holds for all $k \geq 4$. The range of k was extended to $k \leq 34$ and $k = 38$ in [4], but in [2] F.K.C. Rankin and H.P.F. Swinnerton-Dyer proved Wohlfahrt's conjecture for all k by a simple argument. The purpose of the present paper is to show that similar properties hold for a wide class of meromorphic modular forms belonging to the modular group. In particular, it is shown that, if $G_k(z, m)(m \in \mathbb{Z})$ is the mth Poincaré series of weight k, then for $m \leq 1$ all its finite zeros in the standard fundamental region lie on the lower arc A, while for $m > 1$ at most $m - 1$ of these zeros do not lie on A; for $m = 0$ this reproduces the result of [2].

Throughout the paper I shall be concerned with meromorphic modular forms of even positive weight $k \geq 4$ on the upper half-plane $\mathsf{H} = \{z\colon \operatorname{Im} z > 0\}$ for the modular group

$$\Gamma(1) = \mathrm{SL}(2, \mathbb{Z}).$$

The vector space of all such forms is denoted by M_k. Thus, if $f \in M_k$, f has a Fourier series expansion of the form

$$f(z) = \sum_{n=-N}^{\infty} a_n\, e^{2\pi i n z}, \tag{1.1}$$

which is convergent when $\operatorname{Im} z$ is sufficiently large. The subspace of

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M_k consisting of forms $f \in M_k$ that are holomorphic on $\mathbb{H}$ is denoted by H_k; for such forms the series (1.1) converges for all $z \in \mathbb{H}$. The subspace of H_k consisting of cusp forms, for which we can take $N = -1$, is denoted by C_k.

If $f \in M_k$, then f has at most a finite number of poles in any fundamental region. From the work of Petersson [1] it follows that $f(z)$ can be represented as the sum of Poincaré series

$$f(z) = \frac{1}{2} \sum_{c,d} \frac{R(e^{2\pi i Tz})}{(cz+d)^k} =: G_k(z; R). \tag{1.2}$$

Here

$$Tz = \frac{az+b}{cz+d},$$

where

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1),$$

and the summation is over all pairs of coprime integers c, d; for each such pair we choose a single $T \in \Gamma(1)$ with $[c, d]$ as bottom row. Here $R(t)$ is a suitably chosen rational function of t. The series is absolutely and uniformly convergent on every compact subset of $\mathbb{H}$ free of poles of f.

To illustrate this result we take two special cases. In the first place, take

$$R(t) = t^m \quad (m \in \mathbb{Z})$$

and put

$$G_k(z; R) = G_k(z, m) \tag{1.3}$$

in this case. For $m > 0$, $G_k(z, m) \in C_k$ and C_k is spanned by those $G_k(z, m)$ for which

$$0 < m \le \frac{k}{12};$$

see [5], Theorem 6.2.1. For $m = 0$, $G_k(z, 0)$ is an Eisenstein series, being usually denoted by $E_k(z)$. For $m<0$, $G_k(z,m) \in H_k$ and has a pole of order m at ∞.

As a second example take

$$R(t) = (t-q)^{-n} \quad (n \in \mathbb{N}), \tag{1.4}$$

where $0 < |q| < 1$ and

$$q = e^{2\pi i w} \quad (w \in \mathbb{H}).$$

Then $G_k(z;R)$ has poles of order n at those points of H congruent to w modulo $\Gamma(1)$.

By taking R to be an appropriate linear combination of the rational functions described in the previous two paragraphs we see that any function $f \in M_k$ can be expressed in the form (1.2).

If $f \in M_k$, then $f^K \in M_k$, where

$$f^K(z) = \overline{f(-\bar{z})};$$

see §8.6 of [5]. Moreover, $f^K = f$ if and only if f has real Fourier coefficients. Such a form we call a *real modular form* and denote by $M_k^\dagger$ the subset of M_k consisting of such forms; similarly for $H_k^\dagger$ and $C_k^\dagger$. These are clearly vector spaces over the real field $\mathbb{R}$. Note that, if $f \in M_k$, then both

$$f + f^K \text{ and } i(f - f^K)$$

are in $M_k^\dagger$, so that there is, in a sense, no loss of generality in confining attention to real modular forms.

If $f \in M_k^\dagger$ and if f has a zero or pole at a point $z \in \mathbb{H}$, then it has another of the same order at $-\bar{z}$. Further, if the rational function R has real coefficients, then clearly $G_k(z;R) \in M_k^\dagger$; we call such a function R a *real rational function*. When representing real modular forms as Poincaré series $G_k(z;R)$ we shall restrict our attention to real rational functions R. Such a function has the property that

$$R(\bar{t}) = \overline{R(t)} \tag{1.5}$$

for all $t \in C$.

An arbitrary modular form may have a zero at any point of $\mathbb{H}$. However, we shall show that there is a wide class of real forms that have all their zeros on transforms of the arc

$$S = \left\{ z = e^{i\theta} : \frac{\pi}{3} \le \theta \le \frac{2\pi}{3} \right\}$$

This is already known to be true for the Eisenstein series E_k; see [2].

2. General results

We denote by F the standard fundamental region for $\Gamma(1)$. This is the subset of $\mathbb{C}$ consisting of all points $z \in \mathbb{H}$ for which either

$$|z| > 1, -\tfrac{1}{2} < \operatorname{Re} z < 0$$

or

$$|z| \ge 1, 0 \le \operatorname{Re} z \le \tfrac{1}{2},$$

and we regard ∞ as belonging to F. F is bounded on its lower side by the arc S, but only half this arc, namely

$$A = \left\{ z = e^{i\theta} : \frac{\pi}{3} \le \theta \le \frac{\pi}{2} \right\} \tag{2.1}$$

is contained in F.

If $f \in M_k$ and has N zeros and P poles in F, counted with appropriate multiplicities, then

$$N - P = \frac{k}{12}; \tag{2.2}$$

see [5], Theorem 4.1.4. Here zeros or poles at i are counted with weight $\frac{1}{2}$, while those at $\rho = e^{\pi i/3}$ are counted with weight $\frac{1}{3}$.

Let

$$L_i = \{z \in \mathbb{H}: z = iy,\ y > 1\}, \tag{2.3}$$

and

$$L_\rho = \{z \in \mathbb{H}: z = \tfrac{1}{2} + iy,\ y > \tfrac{1}{2}\sqrt{3}\}, \tag{2.4}$$

so that L_i, A and L_ρ form the boundary in $\mathbb{H}$ of the right-hand half of F.

For $k \ge 4$ we express k in the form

$$k = 12l + s, \tag{2.5}$$

where

$$l = \dim C_k \geq 0 \tag{2.6}$$

and

$$s = 4,\ 6,\ 8,\ 10,\ 0 \text{ or } 14. \tag{2.7}$$

If f is holomorphic at i and ρ, then, since

$$\frac{k}{12} = l + \frac{s}{12},$$

we see that we must have

$$\frac{s}{12} = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3} + \frac{1}{2}, 0 \text{ and } \frac{2}{3} + \frac{1}{2} \tag{2.8}$$

in the six cases, respectively. Accordingly the total weighted order of the zeros of f at i and ρ is at least $s/12$ in each case.

Let $G_k(z; R)$ be defined as in (1.2) and suppose that this function is holomorphic on the arc A. We wish to count the number of its zeros on A. For this purpose it is convenient to consider points on the larger arc S and put

$$F_k(\theta, R) = e^{ki\theta/2} G_k(e^{i\theta}, R), \tag{2.9}$$

where $\theta \in [(\pi/3), (2\pi/3)] = I$, say. If we pair the terms of the Poincaré series corresponding to c, d and d, c, and use (1.5), we see that $F_k(\theta, R)$ is real for $\theta \in I$.

Further,

$$F_k(\theta; R) = 2 \operatorname{Re} g_k(\theta; R) + F_k^*(\theta; R), \tag{2.10}$$

where

$$g_k(\theta; R) = e^{(1/2)ki\theta} R(e^{2\pi i e^{i\theta}}) \tag{2.11}$$

and $F_k^*(\theta; R)$ consists of those terms of the series defining F_k for which $c^2 + d^2 \geqslant 2$. Note that $g_k(\theta; R)$ arises from the terms with $c, d = \pm 1, 0$ and $0, \pm 1$.

As θ increases from $\pi/3$ to $2\pi/3$ the point

$$t = e^{2\pi i e^{i\theta}} = e^{-2\pi \sin\theta + 2\pi i \cos\theta}$$

describes in a clockwise direction a curve γ beginning at

$$-r_0 = -e^{-\pi\sqrt{3}},$$

which encircles the origin, passing through the point

$$r_1 = e^{-2\pi}$$

and returning to $-r_0$. The curve γ is pear-shaped and symmetric about the real axis. It has a cusp at $-r_0$, the two tangents there making angles of $\pm\pi/3$ with the positive real axis. The curve γ and its interior $\mathbb{D}_\gamma$ are entirely contained in the unit disc

$$\mathbb{D} = \{t \in \mathbb{C}: |t| < 1\}.$$

Moreover there is a one-to-one correspondence between points $t = e^{2\pi i z}$ in $\mathbb{D}_\gamma$ and points z of $\mathbb{F}$ for which $|z| > 1$.

We now assume that R has no zero or pole on γ and that it has N_γ zeros and P_γ poles in $\mathbb{D}_\gamma$, counted with the appropriate multiplicities. Then the variation in the argument of $e^{ik\theta/2}R(t)$ as t describes S, i.e. as θ goes from $\pi/3$ to $2\pi/3$ is clearly

$$2\pi\left\{P_\gamma - N_\gamma + \frac{k}{12}\right\},$$

by the Argument Principle.

Because of the symmetry of γ about the real axis, the variation in the argument of $e^{ik\theta/2}R(t)$ as t describes A, i.e. as θ goes from $\pi/3$ to $\pi/2$, is half this amount, namely

$$\pi\left(P_\gamma - N_\gamma + \frac{k}{12}\right)$$

Now

$$g_k(\pi/3; R) = e^{ik\pi/6}R(-r_0)$$

and

$$g_k(\pi/2; R) = e^{ik\pi/4}R(r_1).$$

Thus we may take

$$\arg g_k(\pi/3; R) = \pi\left(n_0 + \frac{k}{6}\right), \arg g_k(\pi/2; R) = \pi\left(n_1 + \frac{k}{4}\right),$$

where n_0 and n_1 are integers and

$$n_1 - n_0 = P_\gamma - N_\gamma. \tag{2.12}$$

Now suppose that $G_k(z; R)$ has N_R zeros and P_R poles in F, counted with appropriate multiplicities and weights. Then

$$N_R - P_R = \frac{k}{12} = l + \frac{s}{12}. \tag{2.13}$$

We are now ready to prove our main theorems. These apply to rational functions R with certain properties. We shall say that R has property P_k if (i) R is a real rational function, (ii) all the poles of R lie in D_γ, R has no zeros on γ, (iii) $l \geq N_\gamma - P_\gamma$, and (iv)

$$|F_k^*(\theta; R)| < 2|R(e^{2\pi i e^{i\theta}})| \tag{2.14}$$

for $\theta \in I_0 = [\pi/3, \pi/2]$. Note that (2.14) ensures that $G_k(z; R)$ does not vanish identically.

THEOREM 1: *Suppose that R has property P_k. Then the Poincaré series $G_k(z; R)$ has at least $N_R - N_\gamma$ zeros at points of A.*

PROOF: Note that N_γ is an integer, but N_R need not be. Further, by our assumptions, $P_R = P_\gamma$. It can be checked in each of the six cases that the interval $[n_0 + k/6, n_1 + k/4]$ contains exactly

$$n_1 - n_0 + l + 1$$

integers N. Note that $n_1 - n_0 + l \geq 0$ by (2.12) and condition (iii).

At the corresponding points $N\pi$, $g_k(\theta; R)$ takes alternately the values $\pm|g_k(\theta; R)|$, so that it follows by continuity from (2.10, 14) that $F_k(\theta; R)$ vanishes at least once in each of the $n_1 - n_0 + l$ subintervals between these points. Hence $G_k(z; R)$ has at least

$$n_1 - n_0 + l = P_\gamma - N_\gamma + l$$

zeros at interior points of A and therefore by (2.13), at least

$$P_\gamma - N_\gamma + l + \frac{s}{12} = N_R - N_\gamma$$

zeros on A.

As an immediate corollary we have

THEOREM 2. *Suppose that R has property P_k and that it does not vanish in $\mathbb{D}_\gamma$. Then all the zeros of $G_k(z; R)$ in $\mathbb{F}$ lie on A. They are all simple zeros except that, when $k \equiv 2$ (mod 6), there are of necessity double zeros at $\rho = e^{\pi i/3}$.*

PROOF: For $N_\gamma = 0$ and we see that in (2.8), $\frac{2}{3}$ occurs only for $k \equiv 2$ (mod 6).

THEOREM 3: *Suppose that R has property P_k and that it has exactly one zero in $\mathbb{D}_\gamma$, which is at the origin and is simple. Suppose also that R is bounded on $\mathbb{D} - \mathbb{D}_\gamma$. Then $G_k(z; R)$ has a simple zero at ∞. All its other zeros in $\mathbb{F}$ lie on A and are simple except that, when $k \equiv 2$ (mod 6), there are double zeros at ρ.*

For it is easy to see that $G_k(z; R)$ has a zero at ∞ whenever $R(0) = 0$.

3. Applications

Before the theorems of the previous section can be applied, it is necessary to put condition (2.14) of property P_k into a more usable form. For our present purposes fairly crude estimates suffice, although we shall require more refined approximations in §4.

For $c^2 + d^2 \geq 2$ and $z = e^{i\theta} \in A$,

$$\operatorname{Im} Tz = \frac{\sin\theta}{c^2 + d^2 + 2cd\cos\theta} = \psi_T(\theta), \tag{3.1}$$

say. Now it is easily checked, since $|cd| \geq 1$, that

$$c^2 + d^2 + 2cd\cos\theta \geq \frac{2}{\sqrt{3}}\sin\theta \quad (\pi/3 \leq \theta \leq \pi/2) \tag{3.2}$$

and hence

$$\psi_T(\theta) \leq \frac{\sqrt{3}}{2}. \tag{3.3}$$

Accordingly,

$$|e^{2\pi i T z}| \geq e^{-\pi\sqrt{3}} = r_0 \quad (c^2 + d^2 \geq 2).$$

Define

$$M_R = \sup\{|R(t)|: r_0 \leq |t| \leq 1\}.$$

Note that M is finite by condition (ii) of P_k since, at any pole t of $R, |t| < r_0$.

Accordingly we have

$$|F_k^*(\theta; R)| \leq M_R \sum |c\, e^{i\theta} + d|^{-k}, \tag{3.4}$$

where, in the summation we take

$$c > 0, c^2 + d^2 \geq 2, (c, d) = 1. \tag{3.5}$$

Now

$$(c^2 + d^2 + 2cd \cos\theta)^{-k/2} + (c^2 + d^2 - 2cd \cos\theta)^{-k/2}$$

has, for $\theta \in I_0$, a maximum value when $\theta = \pi/3$ of

$$(c^2 + cd + d^2)^{-k/2} + (c^2 - cd + d^2)^{-k/2}$$

and accordingly

$$|F_k^*(\theta; R)| \leq M_R \sum (c^2 + cd + d^2)^{-k/2}, \tag{3.6}$$

subject to the same conditions (3.5). The series on the right is, apart from the omission of the terms with $c^2 + d^2 = 1$, a well-known Epstein zeta-function and we therefore have

$$|F_k^*(\theta; R)| \leq 2M_R \alpha_k, \tag{3.7}$$

where

$$\alpha_k = \frac{3Z_3(k/2)\zeta(k/2)}{2\zeta(k)} - 1. \tag{3.8}$$

Here ζ is the Riemann zeta-function and, for $s > 1, Z_3(s)$ is the Dirichlet L-series

$$Z_3(s) = 1 - 2^{-s} + 4^{-s} - 5^{-s} + 7^{-s} - 8^{-s} + \cdots.$$

α_k is a decreasing function of k. We have

$$\alpha_4 \le 0.795,\ \alpha_6 \le 0.568,\ \alpha_8 \le 0.520,\ \alpha_{10} \le 0.507,\ \alpha_{12} \le 0.503,$$

while

$$\alpha_{24} \le 0.500003$$

and for large k

$$\alpha_k = \tfrac{1}{2} + \tfrac{1}{2} 3^{1-k/2} + O(7^{-k/2}).$$

Accordingly, condition (2.14) will be satisfied if

$$M_R \alpha_k < |R(e^{2\pi i e^{i\theta}}) \quad (\theta \in I_0). \tag{3.9}$$

We now make a number of applications of these results.

CASE 1: Take

$$R(t) = t^{-m}, \text{ where } m \in \mathbb{Z}, m \ge 0,$$

so that $M_R = e^{\pi m \sqrt{3}}$, while

$$|R(e^{2\pi i e^{i\theta}})| = e^{2\pi m \sin\theta} \ge e^{\pi m \sqrt{3}},$$

so that (3.9) is satisfied because $\alpha_k < 1$.

Since $P_\gamma = m$ and $N_\gamma = 0$ it is clear that property P_k holds. We deduce that the Poincaré series $G_k(z, m)$ has all its zeros in F on A and that they are all simple except as specified in Theorem 2. This includes the case $m = 0$ considered in [2].

CASE 2: Let

$$R(t) = \frac{g_n(t)}{f_m(t)},$$

where f_m and g_n are real polynomials with leading coefficients 1 and of degrees m and n, respectively, where $m \geq n$. They therefore possess a total of $m+n$ non-leading coefficients all of which are real. We assume that the zeros of f_m and g_n lie in $\mathbb{D}_\gamma$. Property P_k then holds.

We deduce from Theorem 1 that, provided that

$$\inf\{|R(t)|: t \in \gamma\} > \alpha_k \sup\{|R(t)|: r_0 \leq |t| \leq 1\}, \tag{3.10}$$

the Poincaré series $G_k(z; R)$ has at least $N_R - N_\gamma$ zeros on A. Now (3.10) is satisfied when $f_m(t) = t^m, g_n(t) = t^n$ by Case 1. Because of continuity and the compactness of the sets involved, there exists a neighbourhood U of the origin in $\mathbb{R}^{m+n}$ such that, if the non-leading coefficients of f_m and g_n lie in U, then $G_k(z; R)$ has at least $N_R - N_\gamma$ zeros on A.

In particular, if $n = 1$, $g_n(t) = t$ and $m \geq 1$, it follows from Theorem 3 that, on some neighbourhood V of the origin in $\mathbb{R}^m$ containing the non-leading coefficients of f_m, $G_k(z; R)$ has the properties stated in that theorem, provided that $f_n(0) \neq 0$.

CASE 3: We examine in greater detail the special case when

$$R(t) = (t-q)^{-m},$$

where $m \in \mathbb{N}$ and $q \in \mathbb{R} \cap \mathbb{D}_\gamma$. Accordingly

$$-r_0 < q < r_1.$$

Note that

$$r_0 = 4.3334 \times 10^{-3}, \; r_1 = 1.8674 \times 10^{-3}.$$

Then, for $t \in \gamma$,

$$|t-q| \leq \max(q + r_0, r_1 - q) = r_3 + |q + r_2|,$$

where

$$r_2 = \tfrac{1}{2}(r_0 - r_1), \; r_3 = \tfrac{1}{2}(r_0 + r_1).$$

Also, for $|t| \geq r_0$

$$|t-q| \geq r_0 - |q|.$$

Accordingly (3.10) is satisfied whenever

$$\frac{r_0 - |q|}{r_3 + |q + r_2|} > \beta(k, m) = \alpha_k^{1/m}, \tag{3.11}$$

and we have

$$\tfrac{1}{2} < \beta = \beta(k, m) < 1.$$

Condition (3.11) is easily seen to be equivalent to

$$-\frac{r_0 - \beta r_1}{1 + \beta} < q < \frac{1 - \beta}{1 + \beta} r_0.$$

Thus, when q lies in this interval, all the zeros of $G_k(z; R)$ lie on A, for all $k \geq 4$.

4. Application to cusp forms

In what follows we take

$$R(t) = t^m \quad (m \in \mathbb{N})$$

so that, by (1.3),

$$G_k(z; R) = G_k(z, m).$$

We assume that

$$k = 24 \text{ or } k \geq 28. \tag{4.1}$$

For $G_k(z, m)$ vanishes identically for $k = 4, 6, 8, 10, 14$, while, for $k = 12, 16, 18, 20, 22, 26$, the location of its zeros is known, since

$$G_k(z, m) = B_{k,m}\Delta(z)E_{k-12}(z),$$

where E_{k-12} is an Eisenstein series ($E_0 = 1$) and $B_{k,m}$ is a constant. It is known that the functions $G_k(z, m)$ $(0 < m \leq l)$ span C_k and therefore do not vanish identically.

It is necessary to assume in what follows that

$$0 < m \leq l - 1. \tag{4.2}$$

By (2.12) and (4.2),

$$n_1 - n_0 + l = l - m \geq 1,$$

so that the interval $[n_0 + k/6, n_1 + k/4]$ contains $l - m + 1 \geq 2$ integers and condition (iii) of property P_k holds.

To obtain the results we wish to prove we must examine the function $F_k^*(\theta; R)$ in greater detail than previously. We consider first the terms with

$$\pm (c, d) = (-1, 1) \text{ and } (1, 1).$$

These give contributions

$$\frac{(-1)^{m+(1/2)k} e^{-\pi m \cot (1/2)\theta}}{(2 \sin \frac{1}{2}\theta)^k} \quad \text{and} \quad \frac{(-1)^m e^{-\pi m \tan (1/2)\theta}}{(2 \cos \frac{1}{2}\theta)^k}.$$

Write

$$g_1(\theta) = 2 \sin \theta - \cot \tfrac{1}{2}\theta, \; g_2(\theta) = 2 \sin \theta - \tan \tfrac{1}{2}\theta$$

and put

$$G_1(\theta) = \frac{\exp\{\pi m g_1(\theta)\}}{(2 \sin \frac{1}{2}\theta)^k}, \; G_2(\theta) = \frac{\exp\{\pi m g_2(\theta)\}}{(2 \cos \frac{1}{2}\theta)^k}$$

for $\pi/3 \leq \theta \leq \pi/2$. Then

$$2G_1'(\theta) \sin^2 \tfrac{1}{2}\theta = G_1(\theta)[\pi m\{1 + 2 \cos \theta(1 - \cos \theta)\} - \tfrac{1}{2}k \sin \theta].$$

The expression in square brackets decreases as θ increases taking its maximum value of $\frac{3}{4}(2\pi m - k\sqrt{3})$ at $\theta = \frac{1}{3}\pi$. This value is negative, so that $G'(\theta) \leq 0$ and therefore

$$G_1(\theta) \leq G(\tfrac{1}{3}\pi) = 1 \quad (\theta \in I_0). \tag{4.3}$$

Also

$$G_2'(\theta) \cos^2 \tfrac{1}{2}\theta = G_2(\theta)[\pi m\{\cos \theta(1 + \cos \theta) - \tfrac{1}{2}\} + \tfrac{1}{4}k \sin \theta],$$

which is positive since $k \geq 12m$. Hence

$$G_2(\theta) \leq G_2(\tfrac{1}{2}\pi) = \frac{e^{\pi m}}{2^{k/2}}. \tag{4.4}$$

We have $c^2+d^2 \geq 5$ for the remaining values of c, d summed over in $F_k(\theta; R)$, and $\psi_T(\theta) \geq 0$; see (3.1). Hence, as in §3, an upper bound for the remaining terms is given by

$$\delta_k = \tfrac{1}{2} \sum_{c^2+d^2>5} (c^2+d^2+cd)^{-k/2}$$

$$= 3\left\{\frac{Z_3(\frac{1}{2}k)\zeta(\frac{1}{2}k)}{\zeta(k)} - 1 - 3^{-1-k/2}\right\}. \tag{4.5}$$

We have

$$\delta_{24} = 10^{-6} \times 3.764,$$

and, by using the approximations

$$Z_3(x) \leq 1 - 2^{-x} + 4^{-x},$$

$$\zeta(x) \leq 1 + 2^{-x} + 3^{-x} + \frac{3^{1-x}}{x-1},$$

$$\{\zeta(2x)\}^{-1} \leq 1 - 2^{-2x},$$

we find that

$$\delta_k \leq 3^{-k/2}\left(2 + \frac{18}{k-2}\right). \tag{4.6}$$

The only condition of property P_k that remains to be checked is (2.14), which now takes the form

$$e^{-2\pi m \sin\theta}\left\{1 + \frac{e^{\pi m}}{2^{k/2}}\right\} + \delta_k < 2e^{-2\pi m \sin\theta}.$$

For this to hold we require that

$$\frac{e^{\pi m}}{2^{k/2}} + \delta_k e^{2\pi m} < 1$$

which, by (4.6), since $12(m+1) \leq k$, reduces to

$$e^{-\pi}\left(\frac{1}{2}e^{\pi/6}\right)^{k/2} + e^{-2\pi}\left(\frac{1}{3}e^{\pi/3}\right)^{k/2}\left(2 + \frac{18}{k-2}\right) < 1.$$

For $k \geq 24$ the left-hand side is less than 0.00849 so that condition (2.11) is satisfied.

THEOREM 4: *Suppose that* $l = \dim C_k \geq 1$ *and that* $0 < m \leq l$. *Then* $G_k(z, m)$ *has at least* $\frac{1}{12}k - m$ *zeros on* A *and at least one at* ∞. *In particular, all the zeros of* $G_k(1, m)$ *in* $\mathbb{F}$ *are simple, except for a double zero at* $\rho = e^{\pi i/3}$, *when* $k \equiv 2 (\mathrm{mod}\, 6)$. *One of these simple zeros is at* ∞ *and the others lie on* A.

In view of the preceding analysis we need only remark that the theorem is trivial when $m = l$ since in that case there are at least $s/12$ zeros at i and ρ.

5. Cusp forms of weight 24

Theorem 4 gives an exact estimate of the number of zeros of $G_k(z, m)$ on A only when $m = 1$. For $m > 1$ only a lower bound is given. It would be of interest to have more precise information about the location of zeros when $m > 1$. In this section we examine the first such case, which arises when

$$k = 24, l = 2, m \geq 2.$$

The space C_{24} has a basis consisting of the newforms

$$f_j(z) = \sum_{n=1}^{\infty} \lambda_j(n) e^{2\pi i n z} \quad (z \in \mathbb{H}; j = 1, 2), \tag{5.1}$$

where the coefficients $\lambda_j(n)$ are the eigenvalues corresponding to Hecke's operator T_n. We have (see [3], (9.7))

$$f_j = \Delta[E_{12} + \{\mu + (-1)^{j-1}\nu\}\Delta] \quad (j = 1, 2), \tag{5.2}$$

where

$$\mu = \frac{324204}{691}, \nu = 12\sqrt{144169} = 12\eta,$$

and

$$\eta = \sqrt{144169} = 379.69593.$$

It follows from (5.1, 2) that

$$\lambda_j(2) = 540 + (-1)^{j-1}\nu$$

so that

$$\lambda_1 = \lambda_1(2) = 5096.3512,\ \lambda_2 = \lambda_2(2) = -4016.3512.$$

For later purposes we also require the values

$$\mu_1 = \lambda_1(3) = 169740 - 576\eta = -48964.855,$$

and

$$\mu_2 = \lambda_2(3) = 169740 + 576\eta = 388444.85.$$

In all these and later estimates the last digit may be in doubt.

Write

$$g_k(z, m) = m^{k-1} G_k(z, m) \quad (m \in \mathbb{N}).$$

Then

$$g_k(z, m) = g_k(z, 1)|T_m,$$

where T_m is Hecke's operator; see [3]. If we write

$$g_{24}(z, 1) = \xi_1 f_1(z) + \xi_2 f_2(z),$$

then

$$g_n(z) := g_{24}(z, n) = \xi_1 \lambda_1(n) f_1(z) + \xi_2 \lambda_2(n) f_2(z). \tag{5.3}$$

We are particularly interested in the location of the zeros of g_2, and therefore require to evaluate ξ_1 and ξ_2.

From [3] (p. 205) we know that ξ_1 and ξ_2 are positive and that (see [3], equation (7.4), with $q = 20$, $r = 4$, $k = 24$)

$$\xi_j = \frac{\zeta(20)\{\Lambda_{j1}\beta(20, 4; 1) + \Lambda_{j2}\beta(20, 4; 2)\}}{\alpha_4 \phi_{24}^{(j)}(23)\phi_{24}^{(j)}(20)}. \tag{5.4}$$

Here

$$\Lambda_{11} = \lambda_2/(\lambda_2 - \lambda_1),\ \Lambda_{12} = -1/(\lambda_2 - \lambda_1),$$

$$\Lambda_{21} = -\lambda_1/(\lambda_2 - \lambda_1),\ \Lambda_{22} = 1/(\lambda_2 - \lambda_1),$$

and α_n is the coefficient of $e^{2\pi iz}$ in the Fourier expansion of $E_n(z)$, and not the quantity defined in (3.8). Also

$$\beta(20, 4, 1) = \alpha_{20} + \alpha_4 - \alpha_{24} = 240.07005,$$
$$\beta(20, 4, 2) = 9\alpha_4 + \alpha_4\alpha_{20} + 524289\alpha_{20} - 8388609\alpha_{24}$$
$$= 37161.979.$$

Finally

$$\phi_{24}^{(j)}(s) = \sum_{n=1}^{\infty} \lambda_j(n)n^{-s} = \prod_p \{1 - \lambda_j(p)p^{-s} + p^{23-2s}\}^{-1},$$

where the product is carried out over all prime numbers p.

By using the values of $\lambda_j(2)$ and $\lambda_j(3)$ and Deligne's bounds

$$|\lambda_j(p)| \leq 2p^{23/2}$$

for $p > 3$ we find that

$$\phi_{24}^{(1)}(20) = 1.00486,\ \phi_{24}^{(2)}(20) = 0.99629,$$
$$\phi_{24}^{(1)}(23) = 1.000607,\ \phi_{24}^{(2)}(23) = 0.999525,$$

which leads to the values

$$\xi_1 = 0.45537,\ \xi_2 = 0.54471$$

and

$$\xi_1\lambda_1 + \xi_2\lambda_2 = 133. \tag{5.5}$$

From (5.3, 5) we see that g_2 has a simple zero at ∞. We now show that it does not vanish on A, but that its remaining zero in F lies on L_ρ, the right-hand boundary of F, and is simple. For this purpose we put

$$g_n(z) = \Delta(z)h_n(z),$$

where $h_n \in H_{12}$ and

$$h_n = \{\xi_1\lambda_1(n) + \xi_2\lambda_2(n)\}(E_{12} + \mu\Delta) + \{\xi_1\lambda_1(n) - \xi_2\lambda_2(n)\}\nu\Delta.$$

Then h_2 has exactly one zero, which is simple, in F. Since $h_2 \in H_{12}^*$ this zero lies either on A or on L_i or on L_ρ. To check this we require the values of E_{12} and Δ at the points i and ρ. We have

$$E_4(i) = e_4 > 0, E_4(\rho) = 0, E_6(i) = 0, E_6(\rho) = e_6 > 0$$

so that

$$1728\Delta(i) = e_4^3,\ 1728\Delta(\rho) = -e_6^2,$$
$$691E_{12}(i) = 441e_4^3,\ 691E_{12}(\rho) = 250e_6^2.$$

See (6.1.14–16) of [5];in (6.1.16) the denominator should be 762048. It follows that

$$144e_6^{-2}h_2(\rho) = -(1723008 + 384\eta)\xi_1 - (1723008 - 348\eta)\xi_2 < 0;$$

for $384\eta < 1723008$.

Since h_2 is real on L_ρ and $h_2(\infty) = \xi_1\lambda_1 + \xi_2\lambda_2 > 0$, by (5.5), it follows that h_2 has a simple zero at a point of L_ρ. It can be shown in a similar way that the same is true for h_4. On the other hand, h_3 has a simple zero on L_i. Observe that L_ρ forms part of the set of transforms of the unit circle under $\Gamma(1)$, whereas L_i does not.

Finally, by using the fact that f_1 and f_2 are orthogonal and the asymptotic formula for $\Sigma_{n\leq x}\lambda_i^2(n)$ we can show that for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that g_n does not vanish on A. A similar result holds with A replaced by L_i and L_ρ.

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