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The zeros of certain Poincaré series


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1. Introduction

K. Wohlfahrt [6] showed in 1964 that the only zeros of the Eisenstein series $E_k$ for the modular group lie on transforms of the unit circle when $4 \leq k \leq 26$, and conjectured that this holds for all $k \geq 4$. The range of $k$ was extended to $k \leq 34$ and $k = 38$ in [4], but in [2] F.K.C. Rankin and H.P.F. Swinnerton-Dyer proved Wohlfahrt's conjecture for all $k$ by a simple argument. The purpose of the present paper is to show that similar properties hold for a wide class of meromorphic modular forms belonging to the modular group. In particular, it is shown that, if $G_k(z, m)(m \in \mathbb{Z})$ is the $m$th Poincaré series of weight $k$, then for $m \leq 1$ all its finite zeros in the standard fundamental region lie on the lower arc $A$, while for $m > 1$ at most $m - 1$ of these zeros do not lie on $A$; for $m = 0$ this reproduces the result of [2].

Throughout the paper I shall be concerned with meromorphic modular forms of even positive weight $k \geq 4$ on the upper half-plane $H = \{z: \text{Im } z > 0\}$ for the modular group

$$\Gamma(1) = \text{SL}(2, \mathbb{Z}).$$

The vector space of all such forms is denoted by $M_k$. Thus, if $f \in M_k$, $f$ has a Fourier series expansion of the form

$$f(z) = \sum_{n=-N}^{\infty} a_n e^{2\pi i n z}, \quad (1.1)$$

which is convergent when $\text{Im } z$ is sufficiently large. The subspace of
$M_k$ consisting of forms $f \in M_k$ that are holomorphic on $\mathbb{H}$ is denoted by $H_k$; for such forms the series (1.1) converges for all $z \in \mathbb{H}$. The subspace of $H_k$ consisting of cusp forms, for which we can take $N = -1$, is denoted by $C_k$.

If $f \in M_k$, then $f$ has at most a finite number of poles in any fundamental region. From the work of Petersson [1] it follows that $f(z)$ can be represented as the sum of Poincaré series

$$f(z) = \frac{1}{2} \sum_{c,d} R(e^{2\pi iTz}) \left( \frac{cz + d}{cz + d} \right)^k = : G_k(z; R). \quad (1.2)$$

Here

$$Tz = \frac{az + b}{cz + d},$$

where

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1),$$

and the summation is over all pairs of coprime integers $c, d$; for each such pair we choose a single $T \in \Gamma(1)$ with $[c, d]$ as bottom row. Here $R(t)$ is a suitably chosen rational function of $t$. The series is absolutely and uniformly convergent on every compact subset of $\mathbb{H}$ free of poles of $f$.

To illustrate this result we take two special cases. In the first place, take

$$R(t) = t^m \quad (m \in \mathbb{Z})$$

and put

$$G_k(z; R) = G_k(z, m) \quad (1.3)$$

in this case. For $m > 0$, $G_k(z, m) \in C_k$ and $C_k$ is spanned by those $G_k(z, m)$ for which

$$0 < m \leq \frac{k}{12},$$

see [5], Theorem 6.2.1. For $m = 0$, $G_k(z, 0)$ is an Eisenstein series,
being usually denoted by $E_k(z)$. For $m < 0$, $G_k(z, m) \in H_k$ and has a pole of order $m$ at $\infty$.

As a second example take

$$R(t) = (t - q)^{-n} \quad (n \in \mathbb{N}),$$

where $0 < |q| < 1$ and

$$q = e^{2\pi i w} \quad (w \in \mathbb{H}).$$

Then $G_k(z; R)$ has poles of order $n$ at those points of $H$ congruent to $w$ modulo $\Gamma(1)$.

By taking $R$ to be an appropriate linear combination of the rational functions described in the previous two paragraphs we see that any function $f \in M_k$ can be expressed in the form (1.2).

If $f \in M_k$, then $f^K \in M_k$, where

$$f^K(z) = \overline{f(-\bar{z})};$$

see §8.6 of [5]. Moreover, $f^K = f$ if and only if $f$ has real Fourier coefficients. Such a form we call a real modular form and denote by $M^\mathbb{R}$ the subset of $M_k$ consisting of such forms; similarly for $H^\mathbb{R}$ and $C^\mathbb{R}$. These are clearly vector spaces over the real field $\mathbb{R}$. Note that, if $f \in M_k$, then both

$$f + f^K \text{ and } i(f - f^K)$$

are in $M^\mathbb{R}$, so that there is, in a sense, no loss of generality in confining attention to real modular forms.

If $f \in M^\mathbb{R}$ and if $f$ has a zero or pole at a point $z \in \mathbb{H}$, then it has another of the same order at $-\bar{z}$. Further, if the rational function $R$ has real coefficients, then clearly $G_k(z; R) \in M^\mathbb{R}$; we call such a function $R$ a real rational function. When representing real modular forms as Poincaré series $G_k(z; R)$ we shall restrict our attention to real rational functions $R$. Such a function has the property that

$$R(\bar{t}) = \overline{R(t)}$$

for all $t \in \mathbb{C}$.

An arbitrary modular form may have a zero at any point of $\mathbb{H}$. However, we shall show that there is a wide class of real forms that have all their zeros on transforms of the arc
This is already known to be true for the Eisenstein series $E_k$; see [2].

2. General results

We denote by $F$ the standard fundamental region for $\Gamma(1)$. This is the subset of $\mathbb{C}$ consisting of all points $z \in \mathbb{H}$ for which either

$$|z| > 1, -\frac{1}{2} < \text{Re } z < 0$$

or

$$|z| \geq 1, 0 \leq \text{Re } z \leq \frac{1}{2},$$

and we regard $\infty$ as belonging to $F$. $F$ is bounded on its lower side by the arc $S$, but only half this arc, namely

$$A = \left\{ z = e^{i\theta}: \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \right\}$$

(2.1)

is contained in $F$.

If $f \in M_k$ and has $N$ zeros and $P$ poles in $F$, counted with appropriate multiplicities, then

$$N - P = \frac{k}{12},$$

(2.2)

see [5], Theorem 4.1.4. Here zeros or poles at $i$ are counted with weight $\frac{1}{2}$, while those at $\rho = e^{\pi i/3}$ are counted with weight $\frac{1}{3}$.

Let

$$L_i = \{ z \in \mathbb{H}: z = iy, y > 1 \},$$

(2.3)

and

$$L_\rho = \{ z \in \mathbb{H}: z = \frac{1}{2} + iy, y > \frac{1}{2}\sqrt{3} \},$$

(2.4)

so that $L_i$, $A$ and $L_\rho$ form the boundary in $\mathbb{H}$ of the right-hand half of $F$.

For $k \geq 4$ we express $k$ in the form

$$S = \left\{ z = e^{i\theta}: \frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3} \right\}$$
If $f$ is holomorphic at $i$ and $p$, then, since
we see that we must have
in the six cases, respectively. Accordingly the total weighted order of
the zeros of $f$ at $i$ and $p$ is at least $s/12$ in each case.

Let $G_k(z; R)$ be defined as in (1.2) and suppose that this function is
holomorphic on the arc $A$. We wish to count the number of its zeros
on $A$. For this purpose it is convenient to consider points on the
larger arc $S$ and put
where $0 \in [(\pi/3), (2\pi/3)] = I$, say. If we pair the terms of the Poincaré
series corresponding to $c$, $d$ and $d$, $c$, and use (1.5), we see that
$F_k(\theta; R)$ is real for $0 \in I$.

Further,
where $\theta \in [(\pi/3), (2\pi/3)] = I$, say. If we pair the terms of the Poincaré
series corresponding to $c$, $d$ and $d$, $c$, and use (1.5), we see that
$F_k(\theta; R)$ is real for $\theta \in I$.

Further,

$$F_k(\theta; R) = 2 \text{Re} \ g_k(\theta; R) + F_\dagger(\theta; R),$$

where

$$g_k(\theta; R) = e^{(1/2)k\theta} R(e^{2\pi i \theta})$$

and $F_\dagger(\theta; R)$ consists of those terms of the series defining $F_k$ for
which $c^2 + d^2 \geq 2$. Note that $g_k(\theta; R)$ arises from the terms with
$c, d = \pm 1, 0$ and $0, \pm 1$. 

\[ k = 12l + s, \quad (2.5) \]

where

\[ l = \dim C_k \geq 0 \quad (2.6) \]

and

\[ s = 4, 6, 8, 10, 0 \text{ or } 14. \quad (2.7) \]
As \( \theta \) increases from \( \pi/3 \) to \( 2\pi/3 \) the point

\[
t = e^{2\pi i \theta} = e^{-2\pi \sin \theta + 2\pi \cos \theta}
\]

describes in a clockwise direction a curve \( \gamma \) beginning at

\[
-r_0 = -e^{-\pi\sqrt{3}}
\]

which encircles the origin, passing through the point

\[
r_1 = e^{-2\pi}
\]

and returning to \(-r_0\). The curve \( \gamma \) is pear-shaped and symmetric about the real axis. It has a cusp at \(-r_0\), the two tangents there making angles of \( \pm \pi/3 \) with the positive real axis. The curve \( \gamma \) and its interior \( D_\gamma \) are entirely contained in the unit disc

\[
D = \{ t \in \mathbb{C} : |t| < 1 \}.
\]

Moreover there is a one-to-one correspondence between points \( t = e^{2\pi iz} \) in \( D_\gamma \) and points \( z \) of \( \mathbb{F} \) for which \( |z| > 1 \).

We now assume that \( R \) has no zero or pole on \( \gamma \) and that it has \( N_\gamma \) zeros and \( P_\gamma \) poles in \( D_\gamma \), counted with the appropriate multiplicities. Then the variation in the argument of \( e^{i k \theta} R(t) \) as \( t \) describes \( S \), i.e. as \( \theta \) goes from \( \pi/3 \) to \( 2\pi/3 \) is clearly

\[
2\pi \left( P_\gamma - N_\gamma + \frac{k}{12} \right),
\]

by the Argument Principle.

Because of the symmetry of \( \gamma \) about the real axis, the variation in the argument of \( e^{i k \theta} R(t) \) as \( t \) describes \( A \), i.e. as \( \theta \) goes from \( \pi/3 \) to \( \pi/2 \), is half this amount, namely

\[
\pi \left( P_\gamma - N_\gamma + \frac{k}{12} \right)
\]

Now

\[
g_k(\pi/3; R) = e^{ik\pi/6} R(-r_0)
\]

and

\[
g_k(\pi/2; R) = e^{ik\pi/4} R(r_1)
\]
Thus we may take
\[ \arg g_k(\pi/3; R) = \pi\left(n_0 + \frac{k}{6}\right), \arg g_k(\pi/2; R) = \pi\left(n_1 + \frac{k}{4}\right), \]
where \( n_0 \) and \( n_1 \) are integers and
\[ n_1 - n_0 = P_\gamma - N_\gamma \]  \hspace{1cm} (2.12)

Now suppose that \( G_k(z; R) \) has \( N_R \) zeros and \( P_R \) poles in \( F \), counted with appropriate multiplicities and weights. Then
\[ N_R - P_R = \frac{k}{12} = l + \frac{s}{12} \]  \hspace{1cm} (2.13)

We are now ready to prove our main theorems. These apply to rational functions \( R \) with certain properties. We shall say that \( R \) has property \( P_k \) if (i) \( R \) is a real rational function, (ii) all the poles of \( R \) lie in \( D_\gamma \), \( R \) has no zeros on \( \gamma \), (iii) \( l \geq N_\gamma - P_\gamma \), and (iv)
\[ |F^+ (\theta; R) | < 2|R(e^{2\pi i \theta})| \]  \hspace{1cm} (2.14)
for \( \theta \in I_0 = [\pi/3, \pi/2] \). Note that (2.14) ensures that \( G_k(z; R) \) does not vanish identically.

**Theorem 1:** Suppose that \( R \) has property \( P_k \). Then the Poincaré series \( G_k(z; R) \) has at least \( N_R - N_\gamma \) zeros at points of \( A \).

**Proof:** Note that \( N_\gamma \) is an integer, but \( N_R \) need not be. Further, by our assumptions, \( P_R = P_\gamma \). It can be checked in each of the six cases that the interval \([n_0 + k/6, n_1 + k/4]\) contains exactly
\[ n_1 - n_0 + k + 1 \] integers \( N \). Note that \( n_1 - n_0 + l \geq 0 \) by (2.12) and condition (iii).

At the corresponding points \( N \pi, 2\pi \frac{g_k(\theta; R)}{k} \) takes alternately the values \( \pm g_k(\theta; R) \), so that it follows by continuity from (2.14) that \( F^+ (\theta; R) \) vanishes at least once in each of the \( n_1 - n_0 + l \) subintervals between these points. Hence \( G_k(z; R) \) has at least
\[ n_1 - n_0 + l = P_\gamma - N_\gamma + l \] zeros at interior points of \( A \) and therefore by (2.13), at least
zeros on $A$.

As an immediate corollary we have

**Theorem 2.** Suppose that $R$ has property $P_k$ and that it does not vanish in $D_\gamma$. Then all the zeros of $G_k(z; R)$ in $F$ lie on $A$. They are all simple zeros except that, when $k \equiv 2 \pmod{6}$, there are of necessity double zeros at $\rho = e^{\pi i/3}$.

**Proof:** For $N_\gamma = 0$ and we see that in (2.8), \( \frac{1}{3} \) occurs only for $k \equiv 2 \pmod{6}$.

**Theorem 3:** Suppose that $R$ has property $P_k$ and that it has exactly one zero in $D_\gamma$, which is at the origin and is simple. Suppose also that $R$ is bounded on $D - D_\gamma$. Then $G_k(z; R)$ has a simple zero at $\infty$. All its other zeros in $F$ lie on $A$ and are simple except that, when $k \equiv 2 \pmod{6}$, there are double zeros at $\rho$.

For it is easy to see that $G_k(z; R)$ has a zero at $\infty$ whenever $R(0) = 0$.

3. Applications

Before the theorems of the previous section can be applied, it is necessary to put condition (2.14) of property $P_k$ into a more usable form. For our present purposes fairly crude estimates suffice, although we shall require more refined approximations in §4.

For $c^2 + d^2 \geq 2$ and $z = e^{i\theta} \in A$,

$$\text{Im } Tz = \frac{\sin \theta}{c^2 + d^2 + 2cd \cos \theta} = \psi_T(\theta), \quad (3.1)$$

say. Now it is easily checked, since $|cd| \geq 1$, that

$$c^2 + d^2 + 2cd \cos \theta \geq \frac{2}{\sqrt{3}} \sin \theta \quad (\pi/3 \leq \theta \leq \pi/2) \quad (3.2)$$

and hence

$$\psi_T(\theta) \leq \frac{\sqrt{3}}{2}. \quad (3.3)$$
Accordingly, 
\[ |e^{2\pi \imath z^2}| \geq e^{-\pi \sqrt{3}} = r_0 \quad (c^2 + d^2 \geq 2). \]

Define

\[ M_R = \sup \{|R(t)|: r_0 \leq |t| \leq 1\}. \]

Note that \( M \) is finite by condition (ii) of \( P_k \) since, at any pole \( t \) of \( R \), \(|t| < r_0\).

Accordingly we have

\[ |F^*_\frac{\chi}{3}(\theta; R)| \leq M_R \sum |c e^{i\theta} + d|^{-k}, \quad (3.4) \]

where, in the summation we take

\[ c > 0, \quad c^2 + d^2 \geq 2, \quad (c, d) = 1. \quad (3.5) \]

Now

\[ (c^2 + d^2 + 2cd \cos \theta)^{-k/2} + (c^2 + d^2 - 2cd \cos \theta)^{-k/2} \]

has, for \( \theta \in I_0 \), a maximum value when \( \theta = \pi/3 \) of

\[ (c^2 + cd + d^2)^{-k/2} + (c^2 - cd + d^2)^{-k/2} \]

and accordingly

\[ |F^*_\frac{\chi}{3}(\theta; R)| \leq M_R \sum (c^2 + cd + d^2)^{-k/2}, \quad (3.6) \]

subject to the same conditions (3.5). The series on the right is, apart from the omission of the terms with \( c^2 + d^2 = 1 \), a well-known Epstein zeta-function and we therefore have

\[ |F^*_\frac{\chi}{3}(\theta; R)| \leq 2M_\alpha_k, \quad (3.7) \]

where

\[ \alpha_k = \frac{3Z_3(k/2)\xi(k/2)}{2\zeta(k)} - 1. \quad (3.8) \]
Here $\zeta$ is the Riemann zeta-function and, for $s > 1$, $Z_3(s)$ is the Dirichlet $L$-series

$$Z_3(s) = 1 - 2^{-s} + 4^{-s} - 5^{-s} + 7^{-s} - 8^{-s} + \cdots.$$ 

$\alpha_k$ is a decreasing function of $k$. We have

$$\alpha_4 \leq 0.795, \alpha_6 \leq 0.568, \alpha_8 \leq 0.520, \alpha_{10} \leq 0.507, \alpha_{12} \leq 0.503,$$

while

$$\alpha_{24} \leq 0.500003$$

and for large $k$

$$\alpha_k = \frac{1}{2} + \frac{1}{2}3^{1-k/2} + O(7^{-k/2}).$$

Accordingly, condition (2.14) will be satisfied if

$$M_R \alpha_k < |R(e^{2\pi r e^{i\theta}}) (\theta \in I_0).$$

We now make a number of applications of these results.

**Case 1:** Take

$$R(t) = t^{-m}, \text{ where } m \in \mathbb{Z}, m \geq 0,$$

so that $M_R = e^{\pi m \sqrt{3}}$, while

$$|R(e^{2\pi r e^{i\theta}})| = e^{2\pi \sin \theta} \geq e^{\pi m \sqrt{3}},$$

so that (3.9) is satisfied because $\alpha_k < 1$.

Since $P_r = m$ and $N_r = 0$ it is clear that property $P_k$ holds. We deduce that the Poincaré series $G_k(z, m)$ has all its zeros in $F$ on $A$ and that they are all simple except as specified in Theorem 2. This includes the case $m = 0$ considered in [2].

**Case 2:** Let

$$R(t) = \frac{g_n(t)}{f_m(t)},$$

where $f_m$ and $g_n$ are real polynomials with leading coefficients 1 and
of degrees $m$ and $n$, respectively, where $m \geq n$. They therefore possess a total of $m + n$ non-leading coefficients all of which are real. We assume that the zeros of $f_m$ and $g_n$ lie in $D_\gamma$. Property $P_k$ then holds.

We deduce from Theorem 1 that, provided that

$$\inf\{\|R(t)\|: t \in \gamma\} > \alpha_k \sup\{\|R(t)\|: r_0 \leq |t| \leq 1\},$$

(3.10)

defines the Poincaré series $G_k(z; R)$ has at least $N_R - N_\gamma$ zeros on $A$. Now (3.10) is satisfied when $f_m(t) = t^m$, $g_n(t) = t^n$ by Case 1. Because of continuity and the compactness of the sets involved, there exists a neighbourhood $U$ of the origin in $\mathbb{R}^{m+n}$ such that, if the non-leading coefficients of $f_m$ and $g_n$ lie in $U$, then $G_k(z; R)$ has at least $N_R - N_\gamma$ zeros on $A$.

In particular, if $n = 1$, $g_n(t) = t$ and $m \geq 1$, it follows from Theorem 3 that, on some neighbourhood $V$ of the origin in $\mathbb{R}^m$ containing the non-leading coefficients of $f_m$, $G_k(z; R)$ has the properties stated in that theorem, provided that $f_n(0) \neq 0$.

CASE 3: We examine in greater detail the special case when

$$R(t) = (t - q)^{-m},$$

where $m \in \mathbb{N}$ and $q \in \mathbb{R} \cap D_\gamma$. Accordingly

$$-r_0 < q < r_1.$$

Note that

$$r_0 = 4.3334 \times 10^{-3}, r_1 = 1.8674 \times 10^{-3}.$$

Then, for $t \in \gamma$,

$$|t - q| \leq \max\{q + r_0, r_1 - q\} = r_3 + |q + r_2|,$$

where

$$r_2 = \frac{1}{2}(r_0 - r_1), r_3 = \frac{1}{2}(r_0 + r_1).$$

Also, for $|t| \geq r_0$

$$|t - q| \geq r_0 - |q|.$$
Accordingly (3.10) is satisfied whenever

\[ \frac{r_0 - |q|}{r_3 + |q + r_2|} > \beta(k, m) = \alpha^m_k, \tag{3.11} \]

and we have

\[ \frac{1}{2} < \beta = \beta(k, m) < 1. \]

Condition (3.11) is easily seen to be equivalent to

\[ -\frac{r_0 - \beta r_1}{1 + \beta} < q < \frac{1 - \beta}{1 + \beta} r_0. \]

Thus, when \( q \) lies in this interval, all the zeros of \( G_k(z; R) \) lie on \( A \), for all \( k \geq 4 \).

4. Application to cusp forms

In what follows we take

\[ R(t) = t^m \quad (m \in \mathbb{N}) \]

so that, by (1.3),

\[ G_k(z; R) = G_k(z, m). \]

We assume that

\[ k = 24 \text{ or } k \geq 28. \tag{4.1} \]

For \( G_k(z, m) \) vanishes identically for \( k = 4, 6, 8, 10, 14 \), while, for \( k = 12, 16, 18, 20, 22, 26 \), the location of its zeros is known, since

\[ G_k(z, m) = B_{k,m} \Delta(z) E_{k-12}(z), \]

where \( E_{k-12} \) is an Eisenstein series \( (E_0 = 1) \) and \( B_{k,m} \) is a constant. It is known that the functions \( G_k(z, m) \) \( (0 < m \leq l) \) span \( C_k \) and therefore do not vanish identically.

It is necessary to assume in what follows that

\[ 0 < m \leq l - 1. \tag{4.2} \]
By (2.12) and (4.2),

$$n_1 - n_0 + l = l - m \geq 1,$$

so that the interval \([n_0 + k/6, n_1 + k/4]\) contains \(l - m + 1 \geq 2\) integers and condition (iii) of property \(P_k\) holds.

To obtain the results we wish to prove we must examine the function \(F_k^*(\theta; R)\) in greater detail than previously. We consider first the terms with

\[ (c, d) = (-1, 1) \text{ and } (1, 1). \]

These give contributions

\[
\frac{(-1)^{m+1/2} e^{-nm \cot (1/2)\theta}}{(2 \sin \frac{1}{2}\theta)^k} \quad \text{and} \quad \frac{(-1)^m e^{-nm \tan (1/2)\theta}}{(2 \cos \frac{1}{2}\theta)^k}.
\]

Write

\[
g_1(\theta) = 2 \sin \theta - \cot \frac{1}{2}\theta, \quad g_2(\theta) = 2 \sin \theta - \tan \frac{1}{2}\theta
\]

and put

\[
G_1(\theta) = \frac{\exp\{n mg_1(\theta)\}}{(2 \sin \frac{1}{2}\theta)^k}, \quad G_2(\theta) = \frac{\exp\{n mg_2(\theta)\}}{(2 \cos \frac{1}{2}\theta)^k}
\]

for \(\pi/3 \leq \theta \leq \pi/2\). Then

\[
2G_1(\theta) \sin^2 \frac{1}{2}\theta = G_1(\theta)\{ \pi m \{1 + 2 \cos \theta(1 - \cos \theta)\} - \frac{1}{8} k \sin \theta \}.
\]

The expression in square brackets decreases as \(\theta\) increases taking its maximum value of \(\frac{3}{2}(2\pi m - k\sqrt{3})\) at \(\theta = \frac{1}{3}\pi\). This value is negative, so that \(G'(\theta) \leq 0\) and therefore

\[
G_1(\theta) \leq G(\frac{1}{3}\pi) = 1 \quad (\theta \in I_0). \quad (4.3)
\]

Also

\[
G_2(\theta) \cos^2 \frac{1}{2}\theta = G_2(\theta)\{ \pi m \{ \cos \theta(1 + \cos \theta) - \frac{1}{4}\} + \frac{1}{4} k \sin \theta \}
\]

which is positive since \(k \geq 12m\). Hence

\[
G_2(\theta) \leq G_2(\frac{1}{3}\pi) = \frac{e^{-nm}}{2^{k/2}}. \quad (4.4)
\]
We have $c^2 + d^2 \geq 5$ for the remaining values of $c, d$ summed over in $F_k(\theta; R)$, and $\psi_T(\theta) \equiv 0$; see (3.1). Hence, as in §3, an upper bound for the remaining terms is given by

$$
\delta_k = \frac{1}{2} \sum_{c^2 + d^2 > 5} (c^2 + d^2 + cd)^{-k/2}
$$

$$
= 3\left\{ \frac{Z_3(k)\zeta(1/2)}{\zeta(k)} - 1 - 3^{-1-k/2} \right\}.
$$

We have

$$\delta_{24} = 10^{-6} \times 3.764,$$

and, by using the approximations

$$Z_s(x) \leq 1 - 2^{-x} + 4^{-x},$$

$$\zeta(x) \leq 1 + 2^{-x} + 3^{-x} + \frac{3^{1-x}}{x-1},$$

$$\{\zeta(2x)\}^{-1} \leq 1 - 2^{-2x},$$

we find that

$$\delta_k \leq 3^{-k/2}\left( 2 + \frac{18}{k-2} \right).$$

The only condition of property $P_k$ that remains to be checked is (2.14), which now takes the form

$$e^{-2\pi m \sin \theta} \left\{ 1 + \frac{e^{\pi m}}{2^{k/2}} \right\} + \delta_k < 2e^{-2\pi m \sin \theta}.$$

For this to hold we require that

$$\frac{e^{\pi m}}{2^{k/2}} + \delta_k e^{2\pi m} < 1$$

which, by (4.6), since $12(m+1) \leq k$, reduces to

$$e^{-\pi} \left( \frac{1}{2} e^{\pi/6} \right)^{k/2} + e^{-2\pi} \left( \frac{1}{3} e^{\pi/3} \right)^{k/2} \left( 2 + \frac{18}{k-2} \right) < 1.$$
THEOREM 4: Suppose that \( l = \dim C_k \geq 1 \) and that \( 0 < m \leq l \). Then \( G_k(z, m) \) has at least \( \frac{k}{2} - m \) zeros on \( A \) and at least one at \( \infty \). In particular, all the zeros of \( G_k(1, m) \) in \( F \) are simple, except for a double zero at \( \rho = e^{\pi i / 3} \), when \( k = 2(\text{mod} \ 6) \). One of these simple zeros is at \( \infty \) and the others lie on \( A \).

In view of the preceding analysis we need only remark that the theorem is trivial when \( m = l \) since in that case there are at least \( s/12 \) zeros at \( i \) and \( \rho \).

5. Cusp forms of weight 24

Theorem 4 gives an exact estimate of the number of zeros of \( G_k(z, m) \) on \( A \) only when \( m = 1 \). For \( m > 1 \) only a lower bound is given. It would be of interest to have more precise information about the location of zeros when \( m > 1 \). In this section we examine the first such case, which arises when

\[
k = 24, \ l = 2, \ m \geq 2.
\]

The space \( C_{24} \) has a basis consisting of the newforms

\[
f_j(z) = \sum_{n=1}^{\infty} \lambda_j(n) e^{2\pi i n z} \quad (z \in \mathbb{H}; \ j = 1, 2), \tag{5.1}
\]

where the coefficients \( \lambda_j(n) \) are the eigenvalues corresponding to Hecke's operator \( T_n \). We have (see [3], (9.7))

\[
f_j = \Delta [E_{12} + \{\mu + (-1)^{j-1} \nu \} \Delta] \quad (j = 1, 2), \tag{5.2}
\]

where

\[
\mu = \frac{324204}{691}, \ \nu = 12 \sqrt{144169} = 12 \eta,
\]

and

\[
\eta = \sqrt{144169} = 379.69593.
\]

It follows from (5.1, 2) that

\[
\lambda_j(2) = 540 + (-1)^{j-1} \nu
\]
so that

\[ \lambda_1 = \lambda_1(2) = 5096.3512, \lambda_2 = \lambda_2(2) = -4016.3512. \]

For later purposes we also require the values

\[ \mu_1 = \lambda_1(3) = 169740 - 576\eta = -48964.855, \]

and

\[ \mu_2 = \lambda_2(3) = 169740 + 576\eta = 388444.85. \]

In all these and later estimates the last digit may be in doubt. Write

\[ g_k(z, m) = m^{k-1}G_k(z, m) \quad (m \in \mathbb{N}). \]

Then

\[ g_k(z, m) = g_k(z, 1)|T_m, \]

where \( T_m \) is Hecke’s operator; see [3]. If we write

\[ g_{2\ell}(z, 1) = \xi_1f_1(z) + \xi_2f_2(z), \]

then

\[ g_{2\ell}(z) = g_{2\ell}(z, n) = \xi_1\lambda_1(n)f_1(z) + \xi_2\lambda_2(n)f_2(z). \quad (5.3) \]

We are particularly interested in the location of the zeros of \( g_{2\ell} \), and therefore require to evaluate \( \xi_1 \) and \( \xi_2 \).

From [3] (p. 205) we know that \( \xi_1 \) and \( \xi_2 \) are positive and that (see [3], equation (7.4), with \( q = 20, r = 4, k = 24 \))

\[ \xi_j = \frac{\xi(20)\{\Lambda_{11}\beta(20, 4; 1) + \Lambda_{12}\beta(20, 4; 2)\}}{\alpha_1\phi_{20}^0(23)\phi_{20}^0(20)}. \quad (5.4) \]

Here

\[ \Lambda_{11} = \lambda_2/(\lambda_2 - \lambda_1), \quad \Lambda_{12} = -1/(\lambda_2 - \lambda_1), \]

\[ \Lambda_{21} = -\lambda_1/(\lambda_2 - \lambda_1), \quad \Lambda_{22} = 1/(\lambda_2 - \lambda_1), \]
and \( \alpha_n \) is the coefficient of \( e^{2\pi i z} \) in the Fourier expansion of \( E_n(z) \), and not the quantity defined in (3.8). Also

\[
\begin{align*}
\beta(20, 4, 1) &= \alpha_{20} + \alpha_4 - \alpha_{24} = 240.07005, \\
\beta(20, 4, 2) &= 9\alpha_4 + \alpha_4\alpha_{20} + 524289\alpha_{20} - 8388609\alpha_{24} \\
&= 37161.979.
\end{align*}
\]

Finally

\[
\phi(j)(s) = \sum_{n=1}^{\infty} \lambda_j(n)n^{-s} = \prod_p \{1 - \lambda_j(p)p^{-s} + p^{23-2\lambda_j^{-1}}\},
\]

where the product is carried out over all prime numbers \( p \).

By using the values of \( \lambda_j(2) \) and \( \lambda_j(3) \) and Deligne’s bounds

\[
|\lambda_j(p)| \leq 2p^{3/2}
\]

for \( p > 3 \) we find that

\[
\begin{align*}
\phi(j)(20) &= 1.00486, \quad \phi(j)(20) = 0.99629, \\
\phi(j)(23) &= 1.000607, \quad \phi(j)(23) = 0.999525,
\end{align*}
\]

which leads to the values

\[
\xi_1 = 0.45537, \quad \xi_2 = 0.54471
\]

and

\[
\xi_1\lambda_1 + \xi_2\lambda_2 = 133. \tag{5.5}
\]

From (5.3, 5) we see that \( g_2 \) has a simple zero at \( \infty \). We now show that it does not vanish on \( A \), but that its remaining zero in \( F \) lies on \( L_p \), the right-hand boundary of \( F \), and is simple. For this purpose we put

\[
g_n(z) = \Delta(z)h_n(z),
\]

where \( h_n \in H_{12} \) and

\[
h_n = \{\xi_1\lambda_1(n) + \xi_2\lambda_2(n)(E_{12} + \mu\Delta) + \{\xi_1\lambda_1(n) - \xi_2\lambda_2(n)\}n\Delta\}.
\]

Then \( h_2 \) has exactly one zero, which is simple, in \( F \). Since \( h_2 \in H_{12}^* \) this zero lies either on \( A \) or on \( L_i \) or on \( L_p \). To check this we require the
values of $E_{12}$ and $\Delta$ at the points $i$ and $\rho$. We have

$$E_4(i) = e_4 > 0, \quad E_4(\rho) = 0, \quad E_6(i) = 0, \quad E_6(\rho) = e_6 > 0$$

so that

$$1728\Delta(i) = e_4^3, \quad 1728\Delta(\rho) = -e_6^2,$$
$$691E_{12}(i) = 441e_4^3, \quad 691E_{12}(\rho) = 250e_6^2.$$

See (6.1.14–16) of [5]; in (6.1.16) the denominator should be 762048. It follows that

$$144e_6^{-2}h_2(\rho) = -((1723008 + 384\eta)\xi_1 - (1723008 - 348\eta)\xi_2 < 0;$$

for $384\eta < 1723008$.

Since $h_2$ is real on $L_p$ and $h_2(\infty) = \xi_1\lambda_1 + \xi_2\lambda_2 > 0$, by (5.5), it follows that $h_2$ has a simple zero at a point of $L_p$. It can be shown in a similar way that the same is true for $h_4$. On the other hand, $h_3$ has a simple zero on $L_i$. Observe that $L_p$ forms part of the set of transforms of the unit circle under $\Gamma(1)$, whereas $L_i$ does not.

Finally, by using the fact that $f_1$ and $f_2$ are orthogonal and the asymptotic formula for $\sum_{n \leq X} \lambda_2^2(n)$ we can show that for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $g_n$ does not vanish on $A$. A similar result holds with $A$ replaced by $L_i$ and $L_p$.

REFERENCES