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THE CLOSURE OF RADICAL CLASSES UNDER FINITE SUBDIRECT PRODUCTS

B.J. Gardner and Patrick N. Stewart

Introduction

A homomorphically closed class which is closed under subdirect products must be a variety (Kogalovskii [11]). Thus, in particular, radical classes which are closed under *arbitrary* subdirect products are rare; for associative rings they are precisely the semi-simple radical classes. Heinicke [9] has shown, however, that all hereditary radical classes are closed under *finite* subdirect products. (Associative rings were considered, but the same argument works in other situations where radical theory is viable.)

In this paper we investigate the incidence of finite subdirect product closure, showing that radical classes $\mathcal{R}$ with this property need not be hereditary, but must satisfy the condition

$$I \lhd A \in \mathcal{R},\ IA = 0 = AI \Rightarrow I \in \mathcal{R},$$

though this condition is not sufficient. In §1 we present a method of generating radical classes which are closed under finite subdirect products. §2 is devoted to the upper radical class $U(\mathcal{X})$ defined by a hereditary, homomorphically closed class $\mathcal{X}$. It is shown that when $U(\mathcal{X})$ consists of idempotent rings, $U(\mathcal{X})$ is closed under finite subdirect products if and only if it consists of rings with zero annihilators. When $\mathcal{X}$ is a class of rings satisfying a fixed polynomial identity, it is shown that $U(\mathcal{X})$ is closed under finite subdirect products if and only if it's hereditary. Other examples and counterexamples are presented in §3.

All rings considered are associative. We shall use the following

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notation: $L(\ ) =$ lower radical class, $U(\ ) =$ upper radical class; $\lhd$ indicates an ideal; ann (A) is the two-sided annihilator of a ring A; Q Z denote, respectively, the rationals and the integers (rings or additive groups), Q^0, Z^0 the zerorings on Q, Z.

1. Examples

A class $\mathcal{G}$ of ordered pairs of rings will be called a *good class* if the following conditions are satisfied.

(g1) If $(S, A) \in \mathcal{G}$, then S is a subring of A and $(f(S), \bar{A})$ is in $\mathcal{G}$ for any isomorphism $f: A \to \bar{A}$.

(g2) If $(S, A) \in \mathcal{G}$, $I \lhd A$ and $S \cap I = 0$, then $((S + I)/I, A/I) \in \mathcal{G}$.

(g3) If $(S, A) \in \mathcal{G}$ and $I \lhd A$, then $(S \cap I, I) \in \mathcal{G}$.

(g4) If $I \lhd A$, $H \lhd A$, $I \subseteq S \subseteq A$ and $(S/I, A/I) \in \mathcal{G}$, then $((S \cap H)/(I \cap H)), A/I \cap H) \in \mathcal{G}$.

1.1 Examples: Let $\mathcal{C}$ be a non-empty hereditary, isomorphism-closed class of rings. The following are examples of good classes:

(i) $\{(S, A) \mid S$ is a subring of A and $S \in \mathcal{C}\}$;

(ii) $\{(S, A) \mid S \lhd A$ and $S \in \mathcal{C}\}$;

(iii) $\{(S, A) \mid S$ is a left ideal of A and $S \in \mathcal{C}\}$;

(iv) $\{(S, A) \mid S \lhd A, S \subseteq$ ann (A) and $S \in \mathcal{C}\}$.

In each case it is easy to check that (g1), (g2) and (g3) are satisfied and that $(S \cap H)/(I \cap H)$ is the right kind of subring of $A/H \cap I$. Since $\mathcal{C}$ is hereditary and

$$(S \cap H)/(I \cap H) = (S \cap H)/(I \cap S \cap H) \cong [(S \cap H) + I]/I \lhd S/I \in \mathcal{C},$$

we also have $(S \cap H)/(I \cap H) \in \mathcal{C}$.

Note that semi-simple classes provide examples of the classes $\mathcal{C}$ discussed above.

Our first theorem shows how good classes can be a source of radical classes closed under finite subdirect products.

1.2 Theorem: *Let $\mathcal{G}$ be a good class, and let $\bar{\mathcal{G}}$ be the class of rings A such that every non-zero homomorphic image $\bar{A}$ of A satisfies the condition*

$$(S, \bar{A}) \in \mathcal{G} \Rightarrow S = 0.$$

Then $\bar{\mathcal{G}}$ is a radical class and is closed under finite subdirect products.

PROOF: Clearly $\bar{\mathscr{G}}$ is homomorphically closed. If $B \lhd A$ and if B, $A/B \in \bar{\mathscr{G}}$, let A/I be a homomorphic image of A, $(S/I, A/I) \in \mathscr{G}$. Now $(B+I)/I \lhd A/I$ and $(S/I) \cap [(B+I)/I] = [(S \cap B) + I]/I$, so by (g3), $([(S \cap B) + I]/I, (B+I)/I) \in \mathscr{G}$. Since B is in $\bar{\mathscr{G}}$, we then have $S \cap B \subseteq I$. But then $(S/I) \cap [(B+I)/I] = 0$, whence by (g2),

$$([S/I + (B+I)/I]/[(B+I)/I], (A/I)/[(B+I)/I]) \in \mathscr{G},$$

i.e. $((S+B+I)/(B+I), A/(B+I)) \in \mathscr{G}$. Now A/B is in $\bar{\mathscr{G}}$, so $S+B+I \subseteq B+I$, whence $S \subseteq S \cap (B+I) = (S \cap B) + I = I$. Thus A is in $\bar{\mathscr{G}}$ so $\bar{\mathscr{G}}$ is closed under extensions.

Now let A be a ring with an ascending chain $\{I_\lambda \mid \lambda \in \Lambda\}$ of ideals such that each I_λ is in $\bar{\mathscr{G}}$. If $(S/K, \cup I_\lambda/K) \in \mathscr{G}$, then by (g3), we have, for each $\lambda \in \Lambda$,

$$((S/K) \cap [(I_\lambda + K)/K], (I_\lambda + K)/K) \in \mathscr{G},$$

i.e. $([S \cap (I_\lambda + K)]/K, (I_\lambda + K)/K) \in \mathscr{G}$. Since each I_λ is in $\bar{\mathscr{G}}$, this means that $S \cap I_\lambda \subseteq K$ for each λ and thus that $S = \bigcup_\Lambda (S \cap I_\lambda) \subseteq K$, whence $S/K = 0$ and $\bigcup_\Lambda I_\lambda \in \bar{\mathscr{G}}$.

This shows that $\bar{\mathscr{G}}$ is a radical class. We turn now to finite subdirect products. It clearly suffices that we show that $\bar{\mathscr{G}}$ is closed under subdirect products involving only two rings.

Suppose we have a ring A, with ideals H, K such that $H \cap K = 0$, $A/H \in \bar{\mathscr{G}}$ and $A/K \in \bar{\mathscr{G}}$. We wish to show that A is in $\bar{\mathscr{G}}$. If, for some $M \lhd A$, we have $(S/M, A/M) \in \mathscr{G}$, then by (g4), $((S \cap H)/(M \cap H), A/(M \cap H)) \in \mathscr{G}$. Since $H \cap K = 0$, we have

$$\begin{aligned}[(M \cap H) + K)]/(M \cap H)] \cap [(S \cap H)/(M \cap H)] \\ &= [[(M \cap H) + K] \cap (S \cap H)]/(M \cap H) \\ &= [(S \cap H \cap K) + (M \cap H)]/(M \cap H) = 0,\end{aligned}$$

so by (g2),

$$([(S \cap H) + K]/[(M \cap H) + K], A/[(M \cap H) + K]) \in \mathscr{G}.$$

But A/K is in $\bar{\mathscr{G}}$, so $S \cap H \subseteq (M \cap H) + K$. Thus

$$\begin{aligned} S \cap H &\subseteq (S \cap H) \cap [(M \cap H) + K] \subseteq (M \cap H) + (S \cap H \cap K) \\ &= (M \cap H) \subseteq M.\end{aligned}$$

Now $(H+M)/M \lhd A/M$, and $(S/M) \cap [(H+M)/M] = [S \cap (H+M)]/M = [(S \cap H)+M]/M = 0$, so by (g2), we have $((S+M+H)/(M+H), A/(M+H)) \in \mathscr{G}$. But A/H is in $\bar{\mathscr{G}}$, so $S \subseteq M+H$, and thus

$$S = (M+H) \cap S = M + (S \cap H) \subseteq M,$$

i.e. $S/M = 0$, and we conclude that A is in $\bar{\mathscr{G}}$, as required.

We note that the proof of 1.2 actually shows that for a class $\mathscr{G}$ of ordered pairs satisfying (g1), (g2) and (g3), $\bar{\mathscr{G}}$ is a radical class. Moreover, for any radical class $\mathscr{R}$ we have $\mathscr{R} = \bar{\mathscr{G}}$ where $\mathscr{G} = \{(A, A) \mid \mathscr{R}(A) = 0\}$. Thus something in addition to (g1), (g2), (g3) is needed to make a radical class closed under finite subdirect products.

The hereditary radical classes are accounted for by 1.2: if $\mathscr{R}$ is hereditary and $\mathscr{G} = \{(S, A) \mid S \lhd A \text{ and } \mathscr{R}(S) = 0\}$, then $\mathscr{R} = \bar{\mathscr{G}}$.

Let $\mathscr{R}$ be a hereditary radical class and let

$$\mathscr{G}_{\mathscr{R}} = \{(S, A) \mid S \lhd A \text{ and } S \in \mathscr{R}\}.$$

Then $\mathscr{G}_{\mathscr{R}}$ is a good class (an instance of the class specified in 1.1(i)) so by 1.2, $\bar{\mathscr{G}}_{\mathscr{R}}$ is a radical class. Now $\bar{\mathscr{G}}_{\mathscr{R}}$ is the class of rings of which every homorphic image has no non-zero $\mathscr{R}$-ideals—the class of *strongly $\mathscr{R}$-semi-simple* rings in the terminology of [1]. Thus we have

1.3 COROLLARY: *For every hereditary radical class $\mathscr{R}$, the (radical) class $\bar{\mathscr{G}}_{\mathscr{R}}$ of strongly $\mathscr{R}$-semi-simple rings is closed under finite subdirect products.*

When a hereditary radical class $\mathscr{R}$ is supernilpotent or subidempotent, $\bar{\mathscr{G}}_{\mathscr{R}}$ is hereditary ([1], Theorems 2, 4 and 8). For hereditary radical classes in general, this need not be so.

1.4 EXAMPLE: Let $\mathscr{T}$ be the hereditary radical class of all rings with torsion additive groups. Then $\bar{\mathscr{G}}_{\mathscr{T}}$ is not hereditary. To see this, consider the ring R whose additive group is $Q \oplus Q$ and whose multiplication is given by

$$(a, b)(c, d) = (ac, ad + bc).$$

We have $I \lhd R$ with $I \cong Q^0$ and $R/I \cong Q$. The only ideals of R are 0, I and R, so the non-zero homomorphic images of R are R and Q, both of which are $\mathscr{T}$-semi-simple. Thus $R \in \bar{\mathscr{G}}_{\mathscr{T}}$. But $Q^0 \notin \bar{\mathscr{G}}_{\mathscr{T}}$, since $Q^0/Z^0 \in \mathscr{T}$.

Thus we have an example of a non-hereditary radical class which is closed under finite subdirect products.

For a radical class $\mathscr{R}$, let

$$\mathscr{G}^*_{\mathscr{R}} = \{(S, A) \mid S \lhd A \text{ and } \mathscr{R}(S) = 0\}.$$

As noted above, when $\mathscr{R}$ is hereditary, we have

$$\mathscr{R} = \bar{\mathscr{G}}^*_{\mathscr{R}}.$$

In general, $\bar{\mathscr{G}}^*_{\mathscr{R}}$ need not be hereditary.

1.5 Example: Let $\mathscr{D}$ be the (radical) class of rings with divisible additive groups. Let R be the ring in 1.4. The homomorphic images of R are 0, Q and R, while the ideals of R are 0, $I (\cong Q^0)$ and R. Thus $R \in \bar{\mathscr{G}}^*_{\mathscr{D}}$. However $I \cong Q^0 \notin \bar{\mathscr{G}}^*_{\mathscr{D}}$, as Q^0 has non-zero reduced ideals (= subrings).

The class resulting from iterations of the $\bar{\mathscr{G}}^*_{\mathscr{R}}$ construction is of some interest. For a radical class $\mathscr{R}$, let

$$\mathscr{G}^0_{\mathscr{R}} = \mathscr{G}^*_{\mathscr{R}}, \; \mathscr{G}^1_{\mathscr{R}} = \{(S, A) \mid S \lhd A \text{ and } \bar{\mathscr{G}}^0_{\mathscr{R}}(S) = 0\}$$

and in general,

$$\mathscr{G}^{n+1}_{\mathscr{R}} = \{(S, A) \mid S \lhd A \text{ and } \bar{\mathscr{G}}^n_{\mathscr{R}}(S) = 0\}.$$

1.6 Proposition: *For any radical class $\mathscr{R}$, we have*

$$\mathscr{R} \supseteq \bar{\mathscr{G}}^0_{\mathscr{R}} \supseteq \cdots \supseteq \bar{\mathscr{G}}^n_{\mathscr{R}} \supseteq \cdots,$$

and $\bigcap_n \bar{\mathscr{G}}^n_{\mathscr{R}}$ is the unique largest hereditary radical subclass of $\mathscr{R}$.

Proof: If $I \lhd A \in \bar{\mathscr{G}}^{n+1}_{\mathscr{R}}$, then $\bar{\mathscr{G}}^n_{\mathscr{R}}(I) \lhd A$ and $(I/\bar{\mathscr{G}}^n_{\mathscr{R}}(I), A/\bar{\mathscr{G}}^n_{\mathscr{R}}(I)) \in \mathscr{G}^{n+1}_{\mathscr{R}}$, so that $I/\bar{\mathscr{G}}^n_{\mathscr{R}}(I) = 0$. Thus ideals of rings in $\bar{\mathscr{G}}^{n+1}_{\mathscr{R}}$ are always in $\bar{\mathscr{G}}^n_{\mathscr{R}}$. It follows that $\bigcap_n \bar{\mathscr{G}}^n_{\mathscr{R}}$ is hereditary (and it is, of course, a radical class).

Now let $\mathscr{H}$ be a hereditary radical subclass of $\mathscr{R}$. If $A \in \mathscr{H}$ and $I \lhd A$, then every ideal of A/I is in $\mathscr{H}$ and therefore in $\mathscr{R}$, so $A \in \bar{\mathscr{G}}^0_{\mathscr{R}}$ and $\mathscr{H} \subseteq \bar{\mathscr{G}}^0_{\mathscr{R}}$. The same argument shows that $\mathscr{H} \subseteq \bar{\mathscr{G}}^{n+1}_{\mathscr{R}}$ if $\mathscr{H} \subseteq \bar{\mathscr{G}}^n_{\mathscr{R}}$. Hence $\mathscr{H} \subseteq \bigcap_n \bar{\mathscr{G}}^n_{\mathscr{R}}$.

We have seen that a radical class which is closed under finite subdirect products need not be hereditary. One can ask, then, whether some kinds of 'restricted hereditary properties' are consequences of finite subdirect product closure.

1.7 PROPOSITION: *Let $\mathcal{R}$ be a radical class which is closed under finite subdirect products. If $I \lhd A \in \mathcal{R}$ and $I \subseteq \operatorname{ann}(A)$, then $I \in \mathcal{R}$.*

PROOF: Consider the ring $A \oplus I$. We have

$$I' = \{(0, i) \mid i \in I\} \lhd A \oplus I,\ J = \{(i, -i) \mid i \in I\} \lhd A \oplus I \text{ and } I' \cap J = 0.$$

Also $A \oplus I/I' \cong A$ and $A \oplus I/J \cong A$ (*via* $(a, i) \to a + i$), so $A \oplus I$, as a subdirect product of two copies of A, is in $\mathcal{R}$. But then I is in $\mathcal{R}$ also.

This result will be useful later on. We shall say that a radical class $\mathcal{R}$ is *hereditary for annihilator ideals* if it satisfies the condition

$$I \lhd A \in \mathcal{R}, I \subseteq \operatorname{ann}(A) \Rightarrow I \in \mathcal{R}.$$

The next result, with which we end this section, is somewhat analogous to 1.6.

1.8 PROPOSITION: *Let $\mathcal{R}$ be a radical class, $\mathcal{X}$ a hereditary class such that $\mathcal{R} = U(\mathcal{X})$. Let*

$$\mathcal{G} = \{(S, A) \mid S \in \mathcal{X} \text{ and } S \subseteq \operatorname{ann}(A)\}.$$

Then $\mathcal{R} \cap \bar{\mathcal{G}}$ is the unique largest radical subclass of $\mathcal{R}$ which is hereditary for annihilator ideals.

PROOF: It is clear that $\mathcal{R} \cap \bar{\mathcal{G}}$ is in fact a radical class. Let $S \lhd A \in \mathcal{R} \cap \bar{\mathcal{G}}$ with $S \subseteq \operatorname{ann}(A)$. If $S \notin \mathcal{R}$, then for some $I \lhd S$, we have $0 \neq S/I \in \mathcal{X}$. Since $I \subseteq \operatorname{ann}(A)$, we have $I \lhd A$, and then, since $S \subseteq \operatorname{ann}(A)$, $(S/I, A/I) \in \mathcal{G}$. But A is in $\bar{\mathcal{G}}$, so $S/I = 0$ – a contradiction. Hence S is in $\mathcal{R}$. If now $J \lhd S$, then $J \lhd A$ and $J \subseteq \operatorname{ann}(A)$, so as above J is in $\mathcal{R}$. It follows that $\mathcal{R}$ contains every (annihilator) ideal of every homomorphic image of S, and thus that S is in $\bar{\mathcal{G}}$. We have shown that $\mathcal{R} \cap \bar{\mathcal{G}}$ is hereditary for annihilator ideals.

Now let $\mathcal{A}$ be any radical subclass of $\mathcal{R}$ which is hereditary for annihilator ideals. If $A \in \mathcal{A}$ and $(S/I, A/I) \in \mathcal{G}$, then since $\mathcal{A}$ is hereditary for annihilator ideals, $S/I \in \mathcal{A} \subseteq \mathcal{R}$. But as $S/I \in \mathcal{X}$, we have $S/I = 0$. Thus $A \in \bar{\mathcal{G}}$, so $\mathcal{A} \subseteq \bar{\mathcal{G}}$.

2. Upper radical classes defined by homomorphically closed classes

In this section we shall examine some upper radical classes which are closed under finite subdirect products. Our results are obtained only for upper radical classes defined by homomorphically closed (hereditary) classes, and it is not at all clear what happens when this restriction is removed. Our first theorem is closely related to a group-theoretic result of Dark and Rhemtulla ([5], Theorem 2).

2.1 THEOREM: *Let $\mathscr{X}$ be hereditary and homomorphically closed, let $\mathscr{R} = U(\mathscr{X})$ and let*

$$\mathscr{G} = \{(S, A) \mid S \in \mathscr{X} \text{ and } S \subseteq \operatorname{ann}(A)\}.$$

Then $\mathscr{R} \cap \bar{\mathscr{G}}$ is the unique largest radical subclass of $\mathscr{R}$ which is closed under finite subdirect products.

PROOF: If $\mathscr{A}$ is a radical subclass of $\mathscr{R}$ which is closed under finite subdirect products, then by 1.7, $\mathscr{A}$ is hereditary for annihilator ideals and thus by 1.8, $\mathscr{A} \subseteq \mathscr{R} \cap \bar{\mathscr{G}}$. What we need to show, therefore, is that $\mathscr{R} \cap \bar{\mathscr{G}}$ is closed under finite subdirect products.

Let A be a ring with ideals H, K such that A/H and $A/K \in \mathscr{R} \cap \bar{\mathscr{G}}$ and $H \cap K = 0$. Since $\bar{\mathscr{G}}$ is closed under finite subdirect products, A is in $\bar{\mathscr{G}}$. If $A/M \in \mathscr{X}$ for some $M \lhd A$, then $A/(M+H)$, $A/(M+K) \in \mathscr{R} \cap \mathscr{X} = \{0\}$, so $M + H = A = M + K$. Hence (since $HK = 0$) $A^2 = (M+H)(M+K) \subseteq M$, so that $(A/M, A/M) \in \mathscr{G}$. But $A \in \bar{\mathscr{G}}$, so $M = A$. It follows that $A \in U(\mathscr{X}) = \mathscr{R}$, so $A \in \mathscr{R} \cap \bar{\mathscr{G}}$ and the latter is closed under finite subdirect products.

We have not been able to determine whether or not radical classes in general have largest radical subclasses closed under finite subdirect products.

2.2 COROLLARY: *Let $\mathscr{X}$ be hereditary and homorphically closed and let*

$$\mathscr{G} = \{(S, A) \mid S \in \mathscr{X} \text{ and } S \subseteq \operatorname{ann}(A)\}.$$

Then $U(\mathscr{X})$ is closed underfinite subdirect products if and only if $U(\mathscr{X}) \subseteq \bar{\mathscr{G}}$.

Szász [16] has shown that the class $\mathscr{E}_6$ of rings whose homomorphic images have zero two-sided annihilators is a radical class.

2.3 COROLLARY: *Let $\mathcal{R}$ be a radical class such that*
(i) *$\mathcal{R} = U(\mathcal{X})$ for a hereditary homomorphically closed class $\mathcal{X}$ and*
(ii) *$\mathcal{R}$ consists of idempotent rings.*
Then $\mathcal{R}$ is closed under finite subdirect products if and only if $\mathcal{R} \subseteq \mathcal{E}_6$.

PROOF: Since $\mathcal{R}$ consists of idempotent rings, all zerorings are $\mathcal{R}$-semi-simple. If we let $\hat{\mathcal{X}}$ be the union of $\mathcal{X}$ and the class of zerorings, then $\hat{\mathcal{X}}$ is hereditary and homomorphically closed, and $\mathcal{R} = U(\hat{\mathcal{X}})$. Let

$$\mathcal{G} = \{(S, A) \mid S \in \hat{\mathcal{X}} \text{ and } S \subseteq \operatorname{ann}(A)\}.$$

Then by 2.2, $\mathcal{R}$ is closed under finite subdirect products if and only if $\mathcal{R} \subseteq \bar{\mathcal{G}}$. But in this case, $\bar{\mathcal{G}} = \mathcal{E}_6$.

Note that $\mathcal{E}_6$ itself is not hereditary, but (equal to $\bar{\mathcal{G}}$ as defined above) is closed under finite subdirect products.

We conclude this section by showing that for certain types of upper radical classes the hereditary property and finite subdirect product closure are equivalent.

2.4 THEOREM: *Let $\mathcal{X}$ be a hereditary homomorphically closed class of rings all of which satisfy a fixed polynomial identity. Then $U(\mathcal{X})$ is closed under finite subdirect products if and only if it's hereditary.*

PROOF: Of course we only need to prove the 'only if' assertion. Let $\mathcal{R} = U(\mathcal{X})$ and let $\mathcal{R}$ be closed under finite subdirect products.

Suppose firstly that $\mathcal{R}(Z^0) = 0$. Let R denote the Zassenhaus algebra over the rationals. This has a basis $\{e_\alpha \mid \alpha \in (0, 1)\}$ and multiplication given by $e_\alpha e_\beta = e_{\alpha+\beta}$ if $\alpha + \beta < 1$ and 0 if $\alpha + \beta \geq 1$. For each $\alpha \in (0, 1)$ we pick out two ideals of R,

$$I_\alpha = \left\{\sum \alpha_\beta e_\beta \mid \beta > \alpha\right\};\ \bar{I}_\alpha = \left\{\sum a_\beta e_\beta \mid \beta \geq \alpha\right\}.$$

If $R \in \mathcal{R}$, then for each α, $R/I_\alpha \in \mathcal{R}$. But $\bar{I}_\alpha/I_\alpha \subseteq \operatorname{ann}(R/I_\alpha)$, so by 1.7, $\bar{I}_\alpha/I_\alpha \in \mathcal{R}$. However, $\bar{I}_\alpha/I_\alpha \cong Q^0$, so, in effect, $Z^0 \subseteq \operatorname{ann}(R/I_\alpha)$, and another appeal to 1.7 establishes that Z^0 is in $\mathcal{R}$ – a contradiction.

Thus $R \notin \mathcal{R}$. Now consider the $n \times n$ matrix ring $M_n(R)$ over R. we have

$$M_n(\bar{I}_\alpha)/M_n(I_\alpha) \subseteq \operatorname{ann}(M_n(R)/M_n(I_\alpha)),$$

so as above, $M_n(R) \notin \mathscr{R}$ for each n. Hence for every n, $M_n(R)$ has a non-zero homomorphic image in $\mathscr{X}$.

Let I be an ideal of $M_n(R)$ satisfying the following condition.

$$\left.\begin{array}{l}\text{For every } \mu \in (0,1) \text{ there exists } \nu < \mu \text{ such that } I\\ \text{contains a matrix with an entry of the form}\\ a_\nu e_\nu + \cdots \ (a_\nu \neq 0).\end{array}\right\} \text{----}(*)$$

(Here and subsequently, linear combinations of basis elements are to be understood as commencing with the basis element of the smallest index.) Then multiplying by the scalar matrix

$$\begin{pmatrix} e_{\mu-\nu} & & 0 \\ & e_{\mu-\nu} & \\ 0 & & e_{\mu-\nu} \end{pmatrix},$$

we get, in I, a matrix with an entry

$$a_\nu e_\mu + \cdots \ (a_\nu \neq 0).$$

Thus (*) is equivalent to the following condition.

$$\left.\begin{array}{l}\text{For every } \mu \in (0,1) \text{ there is a matrix in } I\\ \text{with an entry of the form } a_\mu e_\mu + \cdots \ (a_\mu \neq 0).\end{array}\right\} \text{----}(**)$$

Take any $\alpha \in (0, 1)$, and pick β, γ, δ such that $\beta + \gamma + \delta = \alpha$. Then I contains a matrix (v_{ij}) with an entry

$$v_{kl} = a_\gamma e_\gamma + \cdots, \ a_\gamma \neq 0.$$

Let b, c be non-zero rationals and let $[x]_{rs}$ be the matrix whose (r, s) entry is x and whose others are all zero. Then I contains

$$\begin{aligned}&[be_\beta]_{1k}(v_{ij})[ce_\delta]_{l1} = [be_\beta v_{kl} ce_\delta]_{11}\\ &= [be_\beta(a_\gamma e_\gamma + \cdots)ce_\delta]_{11} = [ba_\gamma ce_\beta e_\gamma e_\delta + \cdots]_{11}\\ &= [ba_\gamma ce_\alpha + \cdots]_{11}.\end{aligned}$$

Thus for each $\alpha \in (0, 1)$, I contains a matrix whose $(1, 1)$ entry has the form

$$a_\alpha e_\alpha + \cdots, \ a_\alpha \neq 0.$$

Let J be the set of all $(1, 1)$ entries of matrices in I. Then $J \lhd R$ and for each $\alpha \in (0, 1)$ J has an element of the form

$$a_\alpha e_\alpha + \cdots, a_\alpha \neq 0.$$

From Divinsky's characterization ([6], p. 683) of the ideals of R, we can now infer that $J = R$.

Now let g be any element of $\hat{R}$. Since R is idempotent, we can express g as a sum $g = \Sigma\, uvw$. As just established, for each v, I contains a matrix (v_{ij}) with $v_{11} = v$. Thus I contains

$$\sum [u]_{i1}(v_{ij})[w]_{1j} = \sum [uv_{11}w]_{ij} = \left[\sum uvw\right]_{ij} = [g]_{ij}$$

for each i, j. But then, for any $(d_{ij}) \in M_n(R)$, we have

$$(d_{ij}) = \sum [d_{ij}]_{ij} \in I,$$

i.e. $I = M_n(R)$.

Considering (*) again, we see that if $M_n(R)/I$ is to be non-zero, then there exists an $\alpha \in (0, 1)$ such that every entry of every matrix in I has the form

$$a_\beta e_\beta + \cdots \text{ where } \beta \geq \alpha,$$

i.e. $I \subseteq M_n(\bar{I}_\alpha)$.

Thus if $0 \neq M_n(R)/I \in \mathscr{X}$, then for some α we have

$$M_n(R/\bar{I}_\alpha) \cong M_n(R)/M_n(\bar{I}_\alpha) \in \mathscr{X}.$$

Now $R/\bar{I}_\alpha$ is a ring like R, but with $(0, \alpha)$, rather than $(0, 1)$, serving as index set for a basis. We have seen that (if $\mathscr{R}(Z^\circ) = 0$) for every n there is an $\alpha_n \in (0, 1)$ such that $M_n(R/\bar{I}_{\alpha_n}) \in \mathscr{X}$. But arguing as in the proofs of Lemma 1.1 and Theorem 1.2 of [8], we see that no identity can be satisfied by all these rings, so we again arrive at a contradiction.

We therefore conclude that $\mathscr{R}(Z^0) \neq 0$. But then $Z^0 \in \mathscr{R}$, so $\mathscr{R}$ contains all prime radical rings. It follows that $\mathscr{X}$ consists of semiprime rings, and hence, being homomorphically closed, of *strongly semiprime*, or, equivalently, hereditarily idempotent rings ([1] or [4]). But Armendariz and Fisher [2] have shown that hereditarily idempotent rings with polynomial identities are regular.

Thus $\mathscr{X}$ is a class of regular rings satisfying a fixed identity. Let $\mathscr{P}$ be the class of primitive homomorphic images of rings in $\mathscr{X}$. Then since $\mathscr{X}$ is homomorphically closed, we have $\mathscr{R} = U(\mathscr{X}) = U(\mathscr{P})$. But primitive *PI* rings are simple artinian [10]; in particular they are simple unital rings. Hence $U(\mathscr{P})$ is hereditary [15].

Since the only artinian simple rings whose subrings are all strongly semiprime are the finite fields, we have:

2.5 COROLLARY (to proof): *The following conditions are equivalent for a variety* $\mathscr{V}$ of rings.

(i) $U(\mathscr{V})$ *is hereditary.*

(ii) $U(\mathscr{V})$ *is closed under finite subdirect products.*

(iii) $\mathscr{V}$ *is generated by a finite set of finite fields.*

3. More examples

In this section we gather together a miscellaneous collection of examples and counterexamples related to finite subdirect product closure. Our first result provides another source of radical classes which are closed under finite subdirect products.

3.1 PROPOSITION: *Let* $\mathscr{K}$ *be a hereditary class of prime rings. Then* $U(\mathscr{K})$ *is closed under finite subdirect products.*

PROOF: If A/H, $A/K \in U(\mathscr{K})$ and $H \cap K = 0$, suppose $A/M \in \mathscr{K} \cup \{0\}$ for some $M \lhd A$. Then $(H+M)/M \cdot (K+M)/M = 0$, so $H \subseteq M$ or $K \subseteq M$ and thus $A/M \in U(\mathscr{K})$, so $M = A$. Hence A is in $U(\mathscr{K})$, so the latter is closed under finite subdirect products.

3.1 calls for two comments. Firstly it is not clear that the non-hereditary instances of $U(\mathscr{K})$ arise from constructions like those described in 1.2. Secondly, the conclusion of 2.4 is not universally valid in the absence of a polynomial identity. This can be seen from

3.2 COROLLARY: *Let* $\mathscr{S}$ *be a class of idempotent simple rings. Then* $U(\mathscr{S})$ *is closed under finite subdirect products.*

It is known that some idempotent simple rings don't define hereditary upper radicals. It is not known precisely which ones do; the most up-to-date information can be obtained from recent work of Leavitt [12] and a survey by van Leeuwen [13].

Our next result utilises the following characterization of finite subdirect products.

3.3 THEOREM (Fuchs [7]): *Let A, B be rings $I \lhd A$ and $J \lhd B$. If there is an isomorphism $f: A/I \to B/J$ let*

$$S = \{(a, b) \in A \oplus B \mid f(a + I) = b + J\}.$$

Then S is a subdirect product of A and B. Conversely, every subdirect product of A and B arises in this way.

3.4 COROLLARY: *If $I \lhd A$, then*

$$\{(a, b) \mid a, b \in A, a - b \in I\}$$

is a subdirect product of two copies of A.

PROOF: Let $f: A/I \to A/I$ be the identity. Then $f(a + I) = b + I$ if and only if $a - b \in I$.

We have mostly been discussing upper radicals. We next present some lower radical classes which fail to be closed under finite subdirect products.

3.5 THEOREM: *Let A be a non-simple ring with identity $\neq 0$ such that*

(i) *$I \cap J \neq 0$ for all non-zero ideals I and J and*
(ii) *$A/K \cong A \Rightarrow K = 0$.*

Then $L(\{A\})$ is not closed under finite subdirect products.

PROOF: Let M be an ideal of A, $0 \neq M \subsetneq A$. Let $S = \{(a, b) \in A \oplus A \mid a - b \in M\}$. Then by 3.4, S is a subdirect product of two copies of A. Suppose some homomorphism $f: A \to S$ has accessible (and therefore, since A is idempotent, ideal) image. Let e denote the identity of A. Then $e - e \in M$, so (e, e) is the identity of S. Let (u, v) be a central idempotent of S. Then u and v are idempotents. If $a \in A$, then for any $m \in M$, we have $(a, a + m) \in S$ and thus $(ua, v(a + m)) = (u, v)(a, a + m) = (a, a + m)(u, v) = (au, (a + m)v)$, so $ua = au$, i.e. u is a central idempotent of A. Similarly we see that v is a central idempotent. Now by (i) A is indecomposable, and hence the only central idempotents of A are e and 0. But then $(0,0)$, $(e,0)$, $(0,e)$ and (e,e) are the only possible central idempotents of S. Since $M \neq A$, $(e,0)$ and $(0,e)$ are not elements of S. Thus $f(e) = (0,0)$ or (e,e).

Suppose $f(e) = (e,e)$. Then f is surjective. Let $\pi_1, \pi_2 : S \to A$ be the coordinate projections. Then $\pi_1 f$, π_2 are surjective maps from A to A, so by (ii), $\pi_1 f$ and $\pi_2 f$ are injective. Thus $A \cong S$. But $M \oplus 0$, $0 \oplus M$ are non-zero ideals of S and $(M \oplus 0) \cap (0 \oplus M) = 0$. This violates (i), so we conclude that $f(e) = (0,0)$. But then $f = 0$, so that $S \notin L(\{A\})$.

An obvious example of a ring satisfying the conditions of 3.5 is Z. However, $L(\{Z\})$ is accounted for by 1.7 ($\operatorname{ann}(Z/4Z) \neq 0$, for example). What we really need are *hereditarily idempotent* rings satisfying (i) and (ii). Examples of such rings are prime unital regular rings with finite ideal lattices. These include the full rings of linear transformations of vector spaces of dimension $\aleph_n$ for finite n. (See, e.g. [3], pp. 197–199). Other examples are given by Raphael ([14], pp. 558 *et seqq.*). Note that there must be at least three ideals: simple rings determine lower radical classes which are hereditary and therefore closed under finite subdirect products.

It is also worth noting that when A is a hereditarily idempotent ring satisfying (i) and (ii) of 3.5, $L(\{A\}) \subseteq \mathscr{C}_6$ (cf. 2.3). In particular, such a class $L(\{A\})$ is hereditary for annihilator ideals, so the converse to 1.7 is false.

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