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JAN VAN MILL

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HOMOGENEOUS SUBSETS OF THE REAL LINE

Jan van Mill

Abstract

If $A \subset \mathbb{R}$ then A is homogeneous provided that $A + Q = A$. As an application we give an elementary proof of Menu's theorem that the real line can be decomposed in two homogeneous homeomorphic subsets. We also show that such a decomposition is not topologically unique. There are homogeneous $A, B \subset \mathbb{R}$ with $A \approx \mathbb{R}\backslash A$, $B \approx \mathbb{R}\backslash B$ but $A \not\approx B$.

1. Introduction

In [2], J. Menu showed that the real line $\mathbb{R}$ can be decomposed in two homeomorphic subsets which are topologically homogeneous[1]. This gave an answer to a problem which was well-known among Dutch topologists. Menu's construction is extremely complicated which motivated the author to try to find an easier proof of this interesting result. The aim of this paper is to do that.

We will show that whenever $A \subset \mathbb{R}$ and $\{A + q : q \in Q\} = A$, where Q denotes the rational numbers, then A is homogeneous. We then construct such a set which is homeomorphic to its complement. This gives a proof of Menu's theorem.

The fact that a decomposition of $\mathbb{R}$ in two homogeneous homeomorphic parts exists, suggests the question of whether such a decomposition is topologically unique. We answer this question in the negative by constructing two sets $A, B \subset \mathbb{R}$ which are homogeneous and homeomorphic to their complements such that A contains a Cantor set but B contains no Cantor set.

[1] A space X is called *homogeneous* provided that for all $x, y \in X$ there is a homeomorphism h from X onto X such that $h(x) = y$.

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2. Homogeneous subsets of the real line

In this section we will give a simple algebraic criterion for a subset $A \subset \mathbb{R}$ which implies that A is topologically homogeneous. All results in this paper depend on this observation.

If $A, B \subset \mathbb{R}$ then define, as usual,

$$A + B = \{a + b : a \in A, b \in B\}.$$

Similarly, define $A - B$, xA, $x + A$, etc.

A clopen set refers to a set which is both closed and open.

2.1. Homogeneity lemma: *Let X be a zero-dimensional separable metric space and let $x, y \in X$. If for each $\epsilon > 0$ there are clopen neighborhoods U and V of, respectively, x and y such that*

(1) diam $U < \epsilon$, diam $V < \epsilon$,

(2) *U is homeomorphic to V,*

then there is a homeomorphism $h : X \to X$ with $h(x) = y$.

Proof: Without loss of generality, diam $X < 1$. For each $n \geq 0$ we will construct a clopen neighborhood U_n of y and a clopen neighborhood V_n of x and for each $n \geq 1$ a homeomorphism $h_n : X \to X$ such that

(1) diam $U_n < 2^{-n}$, diam $V_n < 2^{-n}$,

(2) $U_n \cup h_{n-1} \circ \cdots \circ h_1(V_n) \subset U_{n-1}$ and $h_1^{-1} \circ \cdots \circ h_{n-1}^{-1}(U_n) \cup V_n \subset V_{n-1}$,

(3) $h_n \circ \cdots \circ h_1(V_n) = U_n$.

(4) $h_n \mid (X \setminus U_{n-1}) = \mathrm{id}$.

Put $U_0 = V_0 = X$. Suppose that we have defined V_i and U_i for all $0 \leq i \leq n-1$ and h_i for all $1 \leq i \leq n-1$. If $h_{n-1} \circ \cdots \circ h_1(x) = y$ then let V be any clopen neighborhood of x of diameter less than 2^{-n} for which $h_{n-1} \circ \cdots \circ h_1(V) \subset V_{n-1}$ has diameter less than 2^{-n}. Put $V_n = V$, $U_n = h_{n-1} \circ \cdots \circ h_1(V)$ and $h_n = \mathrm{id}$. Suppose therefore that $h_{n-1} \circ \cdots \circ h_1(x) \neq y$. Let $V \subset V_{n-1}$ be a clopen neighborhood of x and $U \subset U_{n-1}$ be a clopen neighborhood of y such that

(5) diam V $< 2^{-n}$, diam $U < 2^{-n}$,

(6) V is homeomorphic to U,

(7) $h_{n-1} \circ \cdots \circ h_1(V) \subset U_{n-1}$,

(8) $h_{n-1} \circ \cdots \circ h_1(V) \cap U = \emptyset$.

Let $f : h_{n-1} \circ \cdots \circ h_1(V) \to U$ be a homeomorphism. Define $U_n = U$,

$V_n = V$ and $h_n : X \to X$ by

$$\begin{cases} h_n(x) = x \text{ if } x \notin U_n \cup V_n, \\ h_n(x) = f(x) \text{ if } x \in V_n, \\ h_n(x) = f^{-1}(x) \text{ if } x \in U_n. \end{cases}$$

It is clear that U_n, V_n and h_n satisfy our inductive requirements.

If $p \in X$ and $d(p, x) > 2^{-n}$ then $p \notin V_n$ and consequently, $h_n \circ \cdots \circ h_1(p) \notin h_n \circ \cdots \circ h_1(V_n) = U_n$. Therefore, by (4),

$$h_k \circ \cdots \circ h_1(p) = h_n \circ \cdots \circ h_1(p) \tag{5}$$

for all $k \geq n$. Consequently, if we define $h : X \to X$ by

$$h = \lim_{n \to \infty} h_n \circ \cdots \circ h_1$$

then h is well defined. Observe that by (2) and (3), $h(x) = y$. We claim that h is a homeomorphism. If $d(p, x) > 2^{-n}$ then, as remarked above, $h_{n-1} \circ \cdots \circ h_1(p) \notin U_n$ which implies, because of (5), that $h(p) \notin U_n$. Since $h(x) = y$, we may therefore conclude that h is one to one. In addition, if $d(p, y) > 2^{-n}$, then $p \notin U_n$ and therefore, by (4), if $q = h_1^{-1} \circ \cdots \circ h_n^{-1}(p)$ then $h(q) = p$. Consequently, h is onto. Also, by (1) and (4), h is continuous. It is clear that h is open in all points but x. We therefore only check oppenness of h at x. We can then conclude that h is a homeomorphism. To this end, let V be any neighborhood of x. Find $n \geq 0$ with $V_n \subset V$. Then clearly $y = h(x) \in U_n = h(V_n) \subset h(V)$. We therefore conclude that $h(V)$ is a neighborhood of y, which implies that $h(V)$ is open. □

2.2. Remark: It is trivial to formulate and, using the above technique, to prove a version of this Lemma for the class of first countable spaces.

The Homogeneity Lemma is used in the proof of the following:

2.3. Theorem: *Let $A \subset \mathbb{R}$ be such that $A + Q = A$. Then A is homogeneous.*

Proof: We clearly may assume that $\mathbb{R} \setminus A \neq \emptyset$ for if not, then there is nothing to prove. Take $x_0 \in \mathbb{R} \setminus A$ arbitrarily. We claim that $x_0 + Q \subset$

$\mathbb{R}\backslash A$. For if $x_0+q\in A$ for certain $q\in Q$ then $x_0=(x_0+q)-q\in A$ which is not the case. Observe that x_0+Q is dense in $\mathbb{R}$ and that this implies that A is zero-dimensional. We are therefore in a position to apply the Homogeneity Lemma. To this end, take $a, b\in A$ arbitrarily and let $\epsilon>0$. We can obviously find points $q_1, q_2, q_3, q_4\in Q$ such that

(a) $d(q_1, q_2)=d(q_3, q_4)<\frac{1}{2}\epsilon$,

(b) $a\in(x_0+q_1, x_0+q_2)$ and $b\in(x_0+q_2, x_0+q_3)$.

Let $q=q_3-q_2$ and define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x+q$. Clearly $f(A)=A$ and consequently

$$f([x_0+q_1, x_0+q_2]\cap A)=[x_0+q_3, x_0+q_4]\cap A.$$

We have therefore found two homeomorphic clopen neighborhoods of a and b of diameter less than ϵ. Applying the Homogeneity Lemma yields the desired result. $\square$

3. The first example

We will construct a homogeneous $\tilde{A}\subset\mathbb{R}$ such that $\mathbb{R}\backslash\tilde{A}\approx\tilde{A}$. By Theorem 2.3 we only need to find a subset $\tilde{A}\subset\mathbb{R}$ such that $\tilde{A}+Q=\tilde{A}$ and $\tilde{A}\approx\mathbb{R}\backslash\tilde{A}$.

As usual, let $\mathbb{Z}=\{0,\pm1,\pm2,\pm3,\dots\}$. Define

$$\mathcal{A}=\{A\subset\mathbb{R}: Q\subset A, A+Q=A \text{ and } A\cap\bigcup_{n\in\mathbb{Z}\backslash\{0\}}A+n\pi=\emptyset\}.$$

Observe that $\mathcal{A}\neq\emptyset$ since $Q\in\mathcal{A}$. If $\mathcal{K}\subset\mathcal{A}$ is a chain, then clearly $\bigcup\mathcal{K}\in\mathcal{A}$. Consequently, by the Kuratowski–Zorn Lemma, we can find an $A_0\in\mathcal{A}$ such that whenever $B\in\mathcal{A}$ and $A_0\subset B$ then $A_0=B$.

3.1. Lemma: *If $x\in\mathbb{R}\backslash A_0$ then there exists on $n\in\mathbb{Z}\backslash\{0\}$ and an $a\in A_0$ such that $x=a+n\pi$.*

Proof: Let $B=A_0\cup\{x+q:q\in Q\}$. Clearly $B\neq A_0$ since $x\notin A_0$. By maximality of A_0 and by the fact that $B+Q=B\supset Q$ we can find a $b\in B$ and an $n\in\mathbb{Z}\backslash\{0\}$ such that $b+n\pi\in B$.

Case 1: $b\in A_0$.

Since $b+n\pi\notin A_0$ for all $b\in A_0$ it is true that $b+n\pi=x+q$ for certain $q\in Q$. Consequently, $x=(b-q)+n\pi$. Since $b-q\in A_0$, we are done.

CASE 2: $b \notin A_0$.

Take $q \in Q$ such that $b = x + q$. If $b + n\pi \notin A_0$ then we can find $s \in Q$ with

$$x + q + n\pi = x + s.$$

Since $n \neq 0$ this shows that $\pi \in Q$ which is a contradiction. Therefore, $b + n\pi = a \in A_0$. We conclude that $x + q + n\pi = a$, and since $a - q \in A_0$, the desired result follows. □

For each $n \in \mathbb{Z}\backslash\{0\}$ let $A_n = \{x \in \mathbb{R}\backslash A_0 : \exists a \in A_0 \text{ such that } x = a + n\pi\}$.

3.2. LEMMA: *If $n \neq m$ then $A_n \cap A_m = \emptyset$.*

PROOF: Suppose not. Then there are $a, b \in A_0$ with $a + n\pi = b + m\pi$. Consequently, $a = b + (m - n)\pi$. Since $m - n \neq 0$ this is a contradiction. □

3.3. COROLLARY: *$\{A_n : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$.*

PROOF: By Lemma 3.1. $\bigcup_{n \in \mathbb{Z}} A_n = \mathbb{R}$ and by Lemma 3.2, the collection $\{A_n : n \in \mathbb{Z}\}$ is pairwise disjoint. □

Observe that $A_n \neq \emptyset$ for all $n \in \mathbb{Z}$ since $n\pi \in A_n$.

3.4. LEMMA: *If $n \in \mathbb{Z}$ then $A_n + Q = A_n$.*

PROOF: Take $x \in A_n$ and $q \in Q$. Choose $a \in A_0$ such that $x = a + n\pi$. Consequently, $x + q = (a + q) + n\pi$. Since $a + q \in A_0$ this shows that $x + q \in A_n$. □

Now define $\tilde{A} = \bigcup\{A_n : n \in \mathbb{Z} \text{ and } n \text{ is even}\}$ and $B = \bigcup\{A_n : n \in \mathbb{Z} \text{ and } n \text{ is odd}\}$. By Lemma 3.4, $\tilde{A} + Q = \tilde{A}$. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(r) = r + \pi$. Then $\phi(\tilde{A}) = B = \mathbb{R}\backslash\tilde{A}$ (Corollary 3.3). Consequently, $\tilde{A} \approx \mathbb{R}\backslash\tilde{A}$.

3.5. REMARK: Using the same ideas as in this section it is easy to find for each $n \in \mathbb{N}$ a partition $\{A_1, \ldots, A_n\}$ of $\mathbb{R}$ such that each A_i is homogeneous while moreover $A_1 \approx A_2 \approx \cdots \approx A_n$.

4. A Cantor set

We will show that there is a Cantor set $K \subset \mathbb{R}$ such that the set

$$\{x - y : x, y \in K\}$$

is also a Cantor set. We will use this Cantor set to construct a homogeneous $A \subset \mathbb{R}$ with $\mathbb{R} \setminus A \approx A$ such that A contains a Cantor set.

It is not entirely clear that K exists. For if C is the Cantor middle third set in $[0, 1]$, i.e.

$$C = \left\{x = \sum_{i=1}^{\infty} \frac{2x_i}{3^i} : x_i \in \{0, 1\} \text{ for } i \in \mathbb{N}\right\},$$

then it is easy to see that $C - C = \{x - y : x, y \in C\} = [-1, 1]$.

We will construct K by a standard argument. Define

$$F_0 = [0, 1]$$

$$F_1 = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right],$$

$$F_2 = \left[0, \frac{1}{16}\right] \cup \left[\frac{3}{16}, \frac{4}{16}\right] \cup \left[\frac{12}{16}, \frac{13}{16}\right] \cup \left[\frac{15}{16}, \frac{16}{16}\right],$$

$$F_3 = \cdots$$

$$\vdots$$

i.e. at each stage we remove from each interval previously constructed an interval of the same length as the remaining two intervals together (in a regular way of course). Put $K = \bigcap_{n=0}^{\infty} F_n$. Clearly K is a Cantor set. It is easily seen that

$$F_0 - F_0 = [-1, 1]$$

$$F_1 - F_1 = \left[-1, -\frac{1}{2}\right] \cup \left[-\frac{1}{4}, \frac{1}{4}\right] \cup \left[\frac{1}{2}, 1\right],$$

$$F_2 - F_2 = \left[-1, -\frac{14}{16}\right] \cup \left[-\frac{13}{16}, -\frac{11}{16}\right] \cup \left[-\frac{10}{16}, -\frac{1}{2}\right] \cup \left[-\frac{1}{4}, -\frac{2}{16}\right]$$
$$\cup \left[-\frac{1}{16}, \frac{1}{16}\right] \cup \left[\frac{2}{16}, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{10}{16}\right] \cup \left[\frac{11}{16}, \frac{13}{16}\right] \cup \left[\frac{14}{16}, 1\right],$$

$$F_3 - F_3 = \cdots.$$

$$\vdots$$

This easily implies that

$$K - K = \cap_{n=0}^{\infty} F_n - F_n$$

is a Cantor set.

4.1. Proposition: *There is a Cantor set* $K \subset \mathbb{R}$ *and an irrational number* $\tilde{\pi} \in \mathbb{R}$ *such that if* $x_1, x_2 \in K$, $q \in Q$ *and* $n \in \mathbb{Z}\backslash\{0\}$ *then* $n\tilde{\pi} \neq (x_1 - x_2) + q$.

Proof: For each $q \in Q$ and $n \in \mathbb{Z}\backslash\{0\}$ define

$$X_n^q = \frac{1}{n}((K - K) + q),$$

where K is defined as above. Clearly, $X_n^q \approx K - K$ which implies that X_n^q is nowhere dense in $\mathbb{R}$. By the Baire Category Theorem we can therefore find a point

$$\tilde{\pi} \in \mathbb{R}\backslash(\bigcup_{q \in Q} \bigcup_{n \in Z\backslash\{0\}} X_n^q \cup Q).$$

It is clear that $\tilde{\pi}$ is as required. □

5. The second example

We will now construct an example of a homogeneous $A \subset \mathbb{R}$ with $\mathbb{R}\backslash A \approx A$ such that A contains a Cantor set.

Let K be the Cantor set and let $\tilde{\pi}$ be the irrational number of Proposition 4.1. Put $\tilde{K} = K + Q$.

5.1. Lemma: $K \cap \bigcup_{n \in Z\backslash\{0\}} (\tilde{K} + n\tilde{\pi}) = \emptyset$.

Proof: Suppose not. Then we can find $x, y \in K$, $q, s \in Q$ and $n \in \mathbb{Z}\backslash\{0\}$ such that

$$x + q = (y + s) + n\tilde{\pi}.$$

Consequently, $n\tilde{\pi} = (x - y) + (q - s)$, which contradicts Proposition 4.1. □

Define

$$\mathscr{A} = \{A \subset \mathbb{R} : \tilde{K} \subset A,\ A + Q = A, \text{ and } A \cap \bigcup_{n \in Z\backslash\{0\}} (A + n\tilde{\pi}) = \emptyset\}.$$

By Lemma 5.1, $\tilde{K} \in \mathscr{A}$. Now find a maximal $A_0 \in \mathscr{A}$ and proceed in precisely the same way as in section 3 (replace π by $\tilde{\pi}$). This gives the required example.

5.1. THEOREM: *Let $B \subset \mathbb{R}$ be a Cantor set. Then there is a homogeneous $A \subset \mathbb{R}$ containing B such that $A \approx \mathbb{R} \backslash A$.*

PROOF: By Kuratowski [1] there is a homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(K) = B$. The rest is clear. $\square$

6. The third example

We will now construct a homogeneous $\tilde{A} \subset \mathbb{R}$ such that $\mathbb{R} \backslash \tilde{A} \approx \tilde{A}$ while in addition $\tilde{A}$ contains no Cantor set.

Let $\{C_\alpha : \alpha < \mathfrak{c}\}$ enumerate all Cantor sets in $\mathbb{R}$. Without loss of generality, assume that $\pi \in C_0$. By transfinite induction we will construct for every $\alpha < \mathfrak{c}$ a point $x_\alpha \in C_\alpha$ such that for all $\beta < \alpha < \mathfrak{c}$,

$$(*) \qquad (x_\beta + Q + \pi\mathbb{Z}) \cap (x_\alpha + Q + \pi\mathbb{Z}) = \emptyset.$$

Define $x_0 = \pi$ and assume that for each $\beta < \alpha$ we have found x_β.

6.1. LEMMA: $\left|\bigcup_{\beta<\alpha} (x_\beta + Q + \pi\mathbb{Z})\right| < \mathfrak{c}$.

PROOF: This is clear since $|\alpha| < \mathfrak{c}$ and each set $x_\beta + Q + \pi\mathbb{Z}$ is countable. $\square$

Put $Z = \bigcup_{\beta<\alpha} (x_\beta + Q + \pi\mathbb{Z})$.

6.2. LEMMA: *There is an $x \in C_\alpha$ such that $Z \cap (x + Q + \pi\mathbb{Z}) = \emptyset$.*

PROOF: Suppose not. Then for any $x \in C$ we can find $q_x \in Q$ and $n_x \in \mathbb{Z}$ such that

$$x + q_x + \pi n_x \in Z.$$

Since $|C_\alpha| = \mathfrak{c}$ and since $\mathfrak{c}$ has uncountable cofinality (i.e. $\mathfrak{c}$ is not the sum of countably many smaller cardinals), we can find a subset $A \subset C_\alpha$ of cardinality $\mathfrak{c}$ such that for all $x, y \in A$ it is true that

$$q_x = q_y \quad \text{and} \quad n_x = n_y.$$

This implies that for distinct $x, y \in A$,

$$x + q_x + \pi n_x \neq y + q_y + \pi n_y,$$

which is a contradiction since $|A| = \mathfrak{c}$ and $|Z| < \mathfrak{c}$ (Lemma 6.1). □

Now define $x_\alpha = x$, where x is as in Lemma 6.2. Then x_α is clearly as required, which completes the construction.

For each $\alpha < \mathfrak{c}$ put $a_\alpha = x_\alpha - \pi$ and put $A' = \{a_\alpha : \alpha < \mathfrak{c}\}$ and $B = A' + Q$.

6.3. LEMMA: *$Q \subset B$, $B + Q = B$ and $B \cap \bigcup_{n \in Z \setminus \{0\}} (B + n\pi) = \emptyset$.*

PROOF: Since $x_0 = \pi$, $0 \in A'$ which implies that $Q \subset A' + Q = B$. Clearly $B + Q = B$. Now assume there are $\alpha, \beta < \mathfrak{c}$, $q, s \in Q$ and $n \in \mathbb{Z} \setminus \{0\}$ such that

$$x_\alpha - \pi + q = x_\beta - \pi + s + n\pi.$$

Consequently, $x_\alpha = x_\beta + (s - q) + n\pi$. If $\alpha = \beta$ then, since $n \neq 0$, $\pi \in Q$, which is not the case. Therefore, without loss of generality, $\beta < \alpha$. Then

$$x_\alpha = x_\beta + (s - q) + n\pi$$

contradicts (*). □

As in section 3, define

$$\mathscr{A} = \{A \subset \mathbb{R} : Q \subset A, A + Q = A \text{ and } A \cap \bigcup_{n \in Z \setminus \{0\}} (A + n\pi) = \emptyset\}.$$

By Lemma 6.3, $B \in \mathscr{A}$. Let $A_0 \in \mathscr{A}$ be a maximal element containing B (obviously, by the Kuratowski–Zorn Lemma, each $A \in \mathscr{A}$ is contained in a maximal $\tilde{A} \in \mathscr{A}$). For each $n \in \mathbb{Z}$ put $A_n = \{x \in \mathbb{R} : \exists a \in A_0$ such that $x = a + n\pi\}$. By Corollary 3.3, $\{A_n : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$. As in Section 3 put $\tilde{A} = \cup\{A_n : n \in \mathbb{Z}$ and n is even$\}$. We claim that $\tilde{A}$ is as required.

6.4. LEMMA: *$\{x_\alpha : \alpha < \mathfrak{c}\} \cap \tilde{A} = \emptyset$.*

PROOF: Since $a_\alpha = x_\alpha - \pi \in A' \subset \tilde{A}$,

$$x_\alpha = a_\alpha + \pi \in A_1.$$

Therefore, by Lemma 3.2, $x_\alpha \notin \tilde{A}$. $\square$

Since $\{x_\alpha : \alpha < \mathfrak{c}\}$ intersects all Cantor sets in $\mathbb{R}$, $\tilde{A}$ cannot contain a Cantor set. By the same technique as in section 3, $\tilde{A}$ is a homogeneous and $\tilde{A} \approx \mathbb{R}\backslash\tilde{A}$.

6.5. REMARK: It is possible by using more sophisticated techniques, to construct a family $\mathscr{A}$ of $2^{\mathfrak{c}}$ pairwise nonhomeomorphic homogeneous subsets of $\mathbb{R}$ which are all homeomorphic to their complements. The examples in section 5 and this section have the pleasant property that they are not homeomorphic by an obvious reason.

7. Remarks

Perhaps the reader feels that with a more careful construction we can construct the examples A_0 and $\tilde{A}$ of section 3 in such a way that one of them is a subgroup of $\mathbb{R}$, thus getting topological homogeneity for free. This is not the case however. If A_0 is a subgroup then $\tilde{A}$ is also a subgroup. So we only need to show that $\tilde{A}$ is not a subgroup. If $\frac{1}{2}\pi \in \tilde{A}$ then, since

$$\tfrac{1}{2}\pi + \tfrac{1}{2}\pi = \pi \in A_1 \subset \mathbb{R}\backslash\tilde{A},$$

$\tilde{A}$ is clearly not a subgroup. Therefore, $\frac{1}{2}\pi \in \mathbb{R}\backslash\tilde{A}$ which implies that

$$-\tfrac{1}{2}\pi = \tfrac{1}{2}\pi - \pi \in \tilde{A}$$

which also shows that $\tilde{A}$ is not a subgroup since $-\frac{1}{2}\pi - \frac{1}{2}\pi = -\pi \in A_{-1} \subset \mathbb{R}\backslash\tilde{A}$ (I am indebted to E. Wattel for bringing this to my attention).

A zero-dimensional subset $A \subset \mathbb{R}$ is called *strongly homogeneous* provided that all nonempty clopen subspaces of A are homeomorphic. Strongly homogeneous spaces have the pleasant property that any homeomorphism between closed and nowhere dense subsets can be extended to an autohomeomorphism of the whole space, [3], and since this property is important for various reasons [3], [4], [5] one in naturally led to the following.

7.1. QUESTION: *Is there a strongly homogeneous $A \subset \mathbb{R}$ such that $\mathbb{R}\backslash A \approx A$?*

I don't know how to answer this question. The following question is due to M.A. Maurice.

7.2. QUESTION: *Is there a homogeneous subset* $A \subset \mathbb{R}$ *with* $A \approx \mathbb{R} \setminus A$ *such that for* $a, b \in A$ *with* $a < b$ *there is an order preserving homeomorphism* $h : (a, b) \cap A \to A$?

Let L be a compact connected ordered space. The space L is said to be *order homogeneous* provided that for all $a, b \in L$ with $a < b$ there is an order preserving homeomorphism $\phi : [a, b] \to L$.

7.3. QUESTION: *Is there a compact connected order homogeneous space* L *such that* $L \setminus A \not\approx A$ *for all homogeneous* $A \subset L$?

After this paper was written, Jeroen Bruyning suggested to me the following possibility to decompose the real line in two homogeneous and homeomorphic parts. Consider $\mathbb{R}$ to be a vector space over Q. Let $\{\lambda_i : i \in I\}$ be a Hamel basis for $\mathbb{R}$ such that for certain $i, j \in I$ we have that $\lambda_i = 1$ and $\lambda_j = \pi$. Put $A = \{x \in \mathbb{R} : \text{the } j^{\text{th}} \text{ coordinate of } x \text{ belongs to } \bigcup_{n \in Z} [2n, 2n+1) \cap Q\}$ and $B = \mathbb{R} \setminus A$. Clearly $A = A + Q$, so A is homogeneous by Theorem 2.3, and the map $h(x) = x + \pi$ sends A onto B. It is not clear how to use this decomposition to get examples such as in sections 5 and 6.

In response to two of my questions, Evert Wattel has shown that if $(A, \mathbb{R} \setminus A)$ is the decomposition of section 5 then there is a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h(A)$ is measurable. In addition, he has used our decomposition in section 3 and the Homogeneity Lemma 2.1 of this paper, to show that $\mathbb{R}^n$ can be decomposed in $n+1$ homeomorphic and homogeneous zero-dimensional parts.

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(Oblatum 21–I–1981)

Subfaculteit Wiskunde
Vrije Universiteit
De Boelelaan 1081
Amsterdam
The Netherlands