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THE VOLUMES OF SMALL GEODESIC BALLS FOR A METRIC CONNECTION

V. Miquel

§1. Introduction

Let M be a real-analytic Riemannian manifold of dimension n. Let $V_m^\nabla(r)$ denote the volume of the geodesic ball with center $m \in M$ and radius r, where ∇ denotes the Levi–Civita connection. Then $V_m^\nabla(r)$ can be expanded in a power series in r. In 1848 Bertrand–Diguet–Puiseux [3] computed the first two terms for surfaces in $\mathbb{R}^3$. Vermeil [14] in 1917 and Hotelling [11] in 1939 generalized it to arbitrary Riemannian manifolds. Recently, the third and fourth term have been computed by A. Gray [5] and by A. Gray and L. Vanhecke [6], respectively.

To obtain that expansion, it is necessary to discuss general power expansions of tensor fields in normal coordinates as used for example for harmonic spaces (see [13]).

The volumes of tubes about submanifolds of $\mathbb{R}^n$, $\mathbb{C}^n$, S^n, $\mathbb{C}P^n$ have been computed by H. Weyl [15], R.A. Wolf [17], F. J. Flaherty [4], P. A. Griffiths [9]. The expansions of volumes of tubes about submanifolds of arbitrary Riemannian manifolds are given in [11], [7], [8].

In this note we consider a metric connection D on M. Let $V_m^D(r)$ denote the volume of the D-geodesic ball $\bar{B}_r^D(m)$ of center m and radius r. Then $\bar{B}_r^D(m) \subseteq \bar{B}_r^\nabla(m)$ (see §2). We compute the first non trivial term C_1^D of the expansion of $V_m^D(r)$. This is our main theorem 5.4. If M is C^∞, we can compute the Taylor expansion of $V_m^D(r)$, since it is the same as in the analytic case, although it may not be convergent.

We shall show that the difference $C_1^D - C_1^\nabla$ with the case $D = \nabla$ has constant sign and it vanishes only if ∇ and D have the same geodesics

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(Corollary 5.5). On the total volume function, this result ($V_m^D(r) \leq V_m^\nabla(r)$) is a consequence of the inclusion of the D-balls in the ∇-balls. This fact also implies that the volumes coincide only if ∇ and D have the same geodesics. The Corollary 5.5 shows that it is also true with the weaker hypothesis $C_1^D = C_1^\nabla$.

These results were announced in [12].

I express my gratitude to A. Gray and A.M. Naveira for the many talks we had. Without them this work could never have been completed. I wish also to thank L. Vanhecke for his advice and the referee for several helpful suggestions, in particular the use of elementary representation theory of the orthogonal group.

§2. Geodesic balls for a metric connection

Let $\langle\ ,\ \rangle$ be the metric tensor of M, $\chi(M)$ the algebra of vector fields over M and M_m the tangent space to M at the point $m \in M$.

A metric connection D over M is a linear connection which satisfies

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle \text{ for every } X, Y, Z \in \chi(M). \tag{2.1}$$

By a normal coordinate system $(U; x^1, \ldots, x^n)$ at m with respect to D we take a normal coordinate system in the sense of [10] such that the local vector fields $X_i = \partial/\partial x^i$ are orthonormal at m. Then, if $\exp_m : B_r(0) \to U$ is the exponential map associated to D, the normal coordinates are given by $x^i(\exp_m(\Sigma_{j=1}^r a^j e_j)) = a^i$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of M_m.

In this paper we always work in the domain U of a normal coordinate system.

The injectivity radius r_D of (M, D) at m is the supremum of the positive real numbers r such that $\exp_m$ is a diffeomorphism of $B_r(0)$ onto its image.

Let $\mathcal{U}$ be the open set $\mathcal{U} = \exp_m B_{r_D}(0)$. For any p in $\mathcal{U}$, there exists a unique D-geodesic arc joining m and p. Then, we define $\delta^D(m, p)$ as the length of this geodesic arc. Then, since the velocity vector of a geodesic for a metric connection has constant length,

$$\delta^D(m, p) = \|\exp_m^{-1}(p)\|. \tag{2.2}$$

Let r be a positive real number such that $r < r_D$. We call a

D-geodesic ball of center m and radius r, the set $\bar{B}_r^D(m) = \{p \in \mathcal{U} / \delta^D(m,p) \leq r\}$. By (2.2) we have $\bar{B}_r^D(m) = \exp_m(\bar{B}_r(0))$.

Now, we examine the inclusion relation between $\bar{B}_r^D(m)$ and $\bar{B}_r^\nabla(m)$. It is well known that, if d is the standard distance function for the Riemannian manifold M, and $d(m,p) = r < r_\nabla$, there exists a unique arc of ∇-geodesic σ from m to p of length r. Moreover, if α is another arc of curve from m to p, then the length of α is greater or equal than r.

Let r be a real number such that $r < \min(r_D, r_\nabla)$. If $p \in \bar{B}_r^D(m)$, there exists an arc of D-geodesic α joining m and p, and another arc of ∇-geodesic σ from m to p. As we have indicated above, we have $d(m,p) = \text{length of } \sigma \leq \text{length of } \alpha = \delta^D(m,p) \leq r$. Then $p \in \bar{B}_r^\nabla(m)$ and, consequently, $\bar{B}_r^D(m) \subseteq \bar{B}_r^\nabla(m)$. It implies $V_m^D(r) \leq V_m^\nabla(r)$.

We are going to obtain an integral formula for $V_m^D(r)$, the volume of $\bar{B}_r^D(m)$. In [6], it was done for the Levi–Civita connection ∇, by using the Gauss lemma. This approach fails for a general metric connection, and we require the use of polar coordinates as defined in [1] and [2] for a new proof of this formula.

2.1. PROPOSITION: *Let M be orientable, ω, the standard volume form on M and $\omega_{1\ldots n} = \omega(X_1, \ldots, X_n)$. For any $r_0 < r_D$ we have*

$$V_m^D(r_0) = \int_0^{r_0} r^{n-1} \left(\int_{S^{n-1}} \omega_{1\ldots n}(\exp_m(ru))\sigma \right) dr, \tag{2.3}$$

where σ is the standard volume form on S^{n-1}.

PROOF: The definition of $V_m^D(r_0)$ gives

$$V_m^D(r) = \int_{\bar{B}_{r_0}^D(m)} \omega = \int_{\bar{B}_{r_0}(0)} \exp_m^* \omega = \int_{B_{r_0}(0)} (\omega_{1\ldots n} \circ \exp_m)\theta,$$

where θ is the standard volume form on M_m.

Let be $f: S^{n-1} \times]0, r_0[\to B_{r_0}(0) - \{0\}$ the map defining the polar coordinates (u, r). It is well known [2] that $f^*\theta = r^{n-1}\, dr \Lambda \sigma$, so

$$V_m^D(r) = \int_{S^{n-1} \times]0, r_0[} (\omega_{1\ldots n} \circ \exp_m(ru)) r^{n-1}\, dr\, \Lambda\, \sigma.$$

From this, (2.3) follows immediately. □

§3. Power expansions in normal coordinates of a *r*-covariant tensor

Let S be the curvature operator of D given by

$$S_{XY} = D_{[X,Y]} - [D_X, D_Y], \quad S_{XYZW} = \langle S_{XY}Z, W\rangle.$$

We denote by T the torsion of D, and

$$D^p_{X_1 \dots X_p} Y = D_{X_1}(D_{X_2} \dots (D_{X_p} Y) \dots).$$

We say that $X \in \chi(M)$ is a coordinate vector field at m if there exists constants $a^1, \dots, a^n$ such that, in $\mathcal{U}$, $X = \Sigma_{i=1}^n a^i X_i$. From now on $X, Y, Z, \dots$ will denote coordinate vector fields and $a, b, c, \dots$ their corresponding integral curves with initial conditions $a(0) = b(0) = c(0) = \dots = m$. Thus, $a, b, c, \dots$ are geodesics starting at m, and $a'(t) = X_{a(t)}$ wherever $a(t)$ is defined. Moreover we have $S_{XY}Z = -D^2_{XY}Z + D^2_{YX}Z$, $T_XY = D_XY - D_YX$, and $T_{XYZ} = \langle T_XY, Z\rangle$.

Then, we have the following results, whose proofs follow closely the ones given in [5] for the corresponding ones.

3.1. Lemma:

$$(3.1.1) \qquad (D^p_{X \dots X} X)_{a(t)} = 0 \quad p = 1, 2, \dots$$

$$(3.1.2) \qquad (D_X Y)_m = \frac{1}{2}(T_X Y)_m.$$

3.2. Lemma:

$$(3.2.1) \qquad (D^p_{X \dots X} Y)_m + \sum_{k=1}^p (D^p_{X \dots \overset{k}{\smile}{Y} \dots X} X)_m = 0.$$

$$(3.2.2) \qquad \sum_{k=1}^p (D^p_{X \dots \overset{k}{Y} \dots X} Y)_m + \sum_{k \neq 1 = 1}^p (D^p_{X \dots X\overset{k}{Y}X \dots X\overset{1}{Y}X \dots X} X)_m = 0.$$

. . . .

$$(3.2.\text{p-1}) \qquad \sum_{k \neq 1 = 1}^p (D^p_{Y \dots Y\overset{k}{X}Y \dots Y\overset{1}{X}Y \dots Y} Y)_m + \sum_{k=1}^p (D^p_{Y \dots \overset{k}{X} \dots Y} X)_m = 0.$$

$$(3.2.\text{p}) \qquad \sum_{k=1}^p (D^p_{Y \dots \overset{k}{X} \dots Y} Y)_m + (D^p_{Y \dots Y} X)_m = 0.$$

3.3. LEMMA: *At m, we have*

$$(3.3)\quad (p+1)D^p_{X\dots X}Y - pD^{p-1}_{X\dots X}(T_XY) + (p-1)D^{p-2}_{X\dots X}(S_{XY}X) = 0.$$

From (3.1.1), (3.1.2) and (3.3) we have

$$(3.4)\qquad (D^2_{XX}Y)_m = \left\{-\frac{1}{3}S_{XY}X + \frac{2}{3}D_X(T)_XY + \frac{1}{3}T_XT_XY\right\}_m.$$

The same method works for $p \geq 3$ to get $D^p_{X\dots X}Y$.

From now on, we assume that the manifold M and any mathematical object defined on M are real-analytic. (The expansions are the same for the C^∞ case).

Let W be a r-covariant tensor field on a neighbourhood of m. We denote $W(X_{a_1},\dots,X_{a_r})$ by $W_{a_1\dots a_r}$ and D_{X_i} by D_i. The power series expansion of $W_{a_1\dots a_r}$ is then

$$(W_{a_1\dots a_r})_x = \sum_{k=0}^{\infty}\sum_{i_1,\dots,i_k=1}^{n}\frac{1}{k!}(X_{i_1}\dots X_{i_k}W_{a_1\dots a_r})_m x^{i_1}\dots x^{i_k},$$

where $x^1,\dots,x^n$ are the coordinates of the point $x\in M$.

Notice that

$$(3.5)\qquad (X^pW_{a_1\dots a_r})_m = \sum_{\nu_1+\dots+\nu_{r+1}=p}\frac{p!}{\nu_1!\dots\nu_{r+1}!}$$
$$\times D^{\nu_{r+1}}_{X\dots X}(W)\,(D^{\nu_1}_{X\dots X}X_{a_1},\dots,D^{\nu_r}_{X\dots X}X_{a_r})_m.$$

Then, it is possible to determine (3.5) as a function of S, T and their covariant derivatives. We can also determine the coefficients of the power series expansion of $W_{a_1\dots a_r}$ by linearizing the left hand side of (3.5).

3.4. THEOREM: *For any point x in U we have the following expansion*:

$$W_{a_1\dots a_r}(x) = W_{a_1\dots a_r}(m) + \sum_{i=1}^{n}\Big\{D_i(W)_{a_1\dots a_r}$$
$$+\frac{1}{2}\sum_{s=1}^{r}\sum_{q=1}^{n}T_{ia_sq}W_{a_1\dots a_{s-1}qa_{s+1}\dots a_r}\Big\}(m)$$
$$\times x^i + \frac{1}{2}\sum_{i,j=1}^{n}\Big\{D^2_{ij}(W)_{a_1\dots a_r}$$

$$+\sum_{s=1}^{r}\sum_{q=1}^{n} T_{ia_sq}D_j(W)_{a_1\ldots a_{s-1}qa_{s+1}\ldots a_r}$$
$$+\frac{1}{3}\sum_{s=1}^{r}\sum_{q=1}^{n}\left(-S_{ia_sjq}+2D_i(T)_{ja_sq}+\sum_{\beta=1}^{n}T_{i\beta q}T_{ja_s\beta}\right)$$
$$\times W_{a_1\ldots a_{s-1}qa_{s+1}\ldots a_r}+\frac{1}{4}\sum_{s\neq t=1}^{r}\sum_{q,h=1}^{n}T_{ia_sq}T_{ja_th}$$
$$\left.\times W_{a_1\ldots a_{s-1}qa_{s+1}\ldots a_{t-1}ha_{t+1}\ldots a_r}\right\}(m)x^ix^j.$$

PROOF: From (3.1.2), (3.4) and (3.5) we get

$$X_i(W_{a_1\ldots a_r})(m)=\left\{D_i(W)_{a_1\ldots a_r}+\sum_{s=1}^{r}W(X_{a_1},\ldots,D_iX_{a_s},\ldots,X_{a_r})\right\}_m$$
$$=\left\{D_i(W)_{a_1\ldots a_r}+\sum_{s=1}^{r}\sum_{q=1}^{n}\frac{1}{2}T_{ia_sq}W_{a_1\ldots a_{s-1}qa_{s+1}\ldots a_r}\right\}_m,$$

which is the coefficient of x^i, and

$$X_i^2(W_{a_1\ldots a_r})(m)=\left\{D_{ii}^2(W)_{a_1\ldots a_r}+\sum_{s=1}^{r}\sum_{q=1}^{n}T_{ia_sq}D_i(W)_{a_1\ldots a_{s-1}qa_{s+1}\ldots a_r}\right.$$
$$+\frac{1}{3}\sum_{s=1}^{r}\sum_{q=1}^{n}\left(-S_{ia_siq}+2D_i(T)_{ia_sq}\right.$$
$$\left.+\sum_{\beta=1}^{n}T_{i\beta q}T_{ia_s\beta}\right)W_{a_1\ldots a_{s-1}qa_{s+1}\ldots a_r}$$
$$\left.+\frac{1}{4}\sum_{s\neq t=1}^{r}\sum_{q,h=1}^{n}T_{ia_sq}T_{ia_sh}W_{a_1\ldots a_{s-1}qa_{s+1}\ldots a_{t-1}ha_{t+1}\ldots a_r}\right\}_m.$$

Linearizing the last expression, we get the coefficient of x^ix^j. □

We apply this expansion to the metric tensor. Let $g_{ij}=\langle X_i, X_j\rangle$, then $g_{ij}(m)=\delta_{ij}$ and, since $D\langle\ ,\ \rangle=0$, we get

3.5. PROPOSITION: *For any x in U and $A, B=1,\ldots,n$, we have*

$$g_{AB}(x)=\delta_{AB}+\frac{1}{2}\sum_{i=1}^{n}(T_{iAB}+T_{iBA})_m x^i$$
$$+\frac{1}{6}\sum_{i,j=1}^{n}\left\{-(S_{iAjB}+S_{iBjA})+2(D_i(T)_{jAB}+D_i(T)_{jBA}\right.$$
$$+\sum_{\beta=1}^{n}(T_{i\beta A}T_{jB\beta}+T_{i\beta B}T_{jA\beta})$$
$$\left.+\frac{3}{4}\sum_{\beta=1}^{n}(T_{iA\beta}T_{jB\beta}+T_{iB\beta}T_{jA\beta})\right\}_m x^ix^j+\cdots.$$

In the remainder of this paper we assume that M is orientable. This is not a real restriction, since we are always working locally.

We choose the normal coordinates in such a way that $\{X_1, \ldots, X_n\}$ is a positively-oriented local frame. As $X_1, \ldots, X_n$ are orthonormal at m, we have $\omega_{1\ldots n}(m) = 1$. Clearly then $D\omega = 0$.

Let ρ be the Ricci tensor of the connection D. Then, for any local orthonormal frame $\{E_1, \ldots, E_n\}$, $\rho(X, Y) = \sum_{i=1}^{n} S_{XE_iYE_i}$.

3.6. Proposition: *Applying 3.4 to ω, we get, for any x in U,*

$$\omega_{1\ldots n}(x) = 1 + \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{\beta=1}^{n} T_{i\beta\beta}\right)_m x^i + \frac{1}{6}\sum_{i,j=1}^{n}\left(-\rho_{ij} + 2\sum_{\beta=1}^{n} D_i(T)_{j\beta\beta} + \frac{1}{4}\sum_{\beta,\delta=1}^{n} T_{i\delta\beta}T_{j\beta\delta} + \frac{3}{4}\sum_{\beta,\delta=1}^{n} T_{i\beta\beta}T_{j\delta\delta}\right)_m x^i x^j + \cdots.$$

§4. Relationship between T and B

Let B be the difference tensor of the connections D and ∇, $B_XY = D_XY - \nabla_XY$. We define $B_{XYZ} = \langle B_XY, Z\rangle$. Then

$$T_{XYZ} = B_{XYZ} - B_{YXZ}. \tag{4.1}$$

It is well known (see [10]) that the connections D and ∇ have the same geodesics if and only if $B_XY = -B_YX$. Moreover [18], a connection D is a metric connection if and only if $B_{XYZ} = -B_{XZY}$. Then the connections D and ∇ have the same geodesics if and only if $T_{XYZ} = -T_{XZY}$. In fact, D and ∇ have the same geodesics if and only if $B_XY = -B_YX$, i.e., $\langle T_XY, Y\rangle = \langle B_XY - B_YX, Y\rangle = 2\langle B_XY, Y\rangle = 0$. In this case the torsion T and the tensor B belong to the irreducible subspace Λ^3V^* of $\Lambda^2V^* \otimes V^*$ and $V^* \otimes \Lambda^2V^*$, respectively, where $V = M_m$.

It is also useful to have an expression of B in terms of T. Yano [18] proved

$$B_{XYZ} = \frac{1}{2}(T_{XYZ} + T_{ZXY} - T_{YZX}). \tag{4.2}$$

Here, we give another proof of this formula by using elementary representation theory. Later we shall use the same method to obtain a good formula for the expansion of $V_m^D(r)$.

The tensor T_m belongs to $\Lambda^2V^* \otimes V^*$ and B_m to $V^* \otimes \Lambda^2V^*$. Since the map $\alpha : V^* \otimes \Lambda^2V^* \to \Lambda^2V^* \otimes V^*$ given by $\alpha(B_m) = T_m$ is $\mathrm{Gl}(n)$-invariant, it is a multiple of the intertwining operator between the $\mathrm{Gl}(n)$-irreducible subspaces of $V^* \otimes \Lambda^2V^*$ and those of $\Lambda^2V^* \otimes V^*$. The space $W = \Lambda^2V^* \otimes V^* \cong V^* \otimes \Lambda^2V^*$ decomposes in the form

$$W = \Lambda^3V^* \otimes Y_1^2,$$

where Y_1^2 is the irreducible representation of $\mathrm{Gl}(n)$ with Young diagram . The projection β on Λ^3V^* is given by

$$\beta(A)_{XYZ} = \frac{1}{3}(A_{XYZ} + A_{ZXY} + A_{YZX})$$

and $\ker \beta$ identifies itself with Y_1^2.

To obtain the inverse of α, we observe that the restriction to Λ^3V^* is given by

$$\alpha(A)_{XYZ} = A_{XYZ} - A_{YXZ} = 2A_{XYZ},$$

and, the restriction to $Y_1^2 = \ker \beta$ is

$$\alpha(A)_{XYZ} = A_{XYZ} - A_{YXZ} = A_{XYZ} + A_{ZYX} + A_{XZY} = -A_{ZXY}.$$

Then

$$B_{XYZ} = \alpha^{-1}(T)_{XYZ} = \frac{1}{2}\beta(T)_{XYZ} - (T - \beta(T))_{YZX}$$

$$= \frac{1}{2}(T_{XYZ} + T_{ZXY} - T_{YZX}).$$

§5. Power series expansion of the volume function of a *D*-geodesic ball

5.1. Proposition: *For any r such that $0 < r < r_D$, we have:*

$$V_m^D(r) = \frac{(\pi r^2)^{n/2}}{(n/2)!}\left\{1 + \frac{1}{n+2}\left(-\frac{1}{6}\tau_D - \frac{1}{3}\partial\bar{T}\right.\right.$$

$$\left.\left.+\frac{1}{24}\check{T} + \frac{1}{8}\|\bar{T}\|^2\right)r^2 + O(r^4)\right\}$$

where

$$\tau_D = \sum\nolimits_{i,j=1}^{n} S_{ijij}(m) \quad (\textit{scalar curvature of } D \textit{ at } m),$$

$$-\partial\bar{T} = \sum\nolimits_{i,j=1}^{n} D_i(T)_{iij}(m), \quad \|T\|^2 = \sum\nolimits_{i,j=1}^{n} T_{ijk}^2(m)$$

$$\check{T} = \sum\nolimits_{i,j,k=1}^{n} T_{ijk}T_{ikj}(m), \quad \|\bar{T}\|^2 = \sum\nolimits_{i=1}^{n} \bar{T}_i^2(m),$$

$\bar{T}$ being the one-form defined by $\bar{T}_X = \Sigma_{j=1}^{n} T_{XE_jE_j}$ for any local orthonormal frame $\{E_1, \ldots, E_n\}$.

The proof – which makes use of 2.1 and 3.6 – follows closely the one given in [6] for the Levi–Civita connection. □

Some geometric formulas will prove useful to eliminate $\delta\bar{T}$ in 5.1. For this, we need the following lemmas:

5.2. LEMMA: *If R is the curvature tensor of ∇, at m, we have*:

$$\begin{aligned} S_{XYXY} = R_{XYXY} &- \frac{1}{2}(\|B_XY\|^2 + \|B_YX\|^2) + \langle B_XX, B_YY\rangle + \|T_XY\|^2 \\ &+ \frac{1}{2}(\langle T_XT_XY, Y\rangle + \langle T_YT_YX, X\rangle) + \langle D_X(T)_XY, Y\rangle \\ &+ \langle D_Y(T)_YX, X\rangle. \end{aligned}$$

PROOF:

$$\begin{aligned} S_{XYXY} &= -\langle D^2_{XY}X, Y\rangle + \langle D^2_{YX}X, Y\rangle \\ &= R_{XYXY} - \langle B_X\nabla_YX, Y\rangle - \langle \nabla_X(B_YX), Y\rangle \\ &\quad - \langle B_XB_YX, Y\rangle + \langle B_Y\nabla_XX, Y\rangle + \langle \nabla_Y(B_XX), Y\rangle + \langle B_YB_XX, Y\rangle. \end{aligned}$$

But, at m, by (3.1.2)

$$\nabla_YX = D_YX - B_YX = \frac{1}{2}T_YX - B_YX = -\frac{1}{2}(B_XY + B_YX).$$

Then, using (4.1) and (4.2) we have

$$\begin{aligned} S_{XYXY} = R_{XYXY} &- \frac{1}{2}\langle B_XY, B_YX + B_XY\rangle - X\langle T_YX, Y\rangle \\ &- \frac{1}{2}\langle B_YX, B_XY + B_YX\rangle \\ &+ \langle B_XY, B_YX\rangle + \langle B_YY, B_XX\rangle + Y\langle T_YX, X\rangle; \end{aligned}$$

and the lemma follows by a direct computation, using (3.1.2). □

5.3. LEMMA: *If τ_∇ is the scalar curvature of ∇, at m, we have*

$$\tau_D = \tau_\nabla + \frac{1}{4}\|T\|^2 + \frac{1}{2}\check{T} + \|\bar{T}\|^2 - 2\,\partial\bar{T}.$$

PROOF: From 5.2 we see that

$$\tau_D = \tau_\nabla - \|B\|^2 - \|\bar{B}\|^2 + \|T\|^2 + \check{T} - 2\,\partial\bar{T}, \tag{5.1}$$

where $\|B\|^2 = \Sigma_{i,j,k=1}^n B_{ijk}^2(m)$ and $\|\bar{B}\|^2 = \Sigma_{i=1}^n (\Sigma_{j=1}^n B_{ijj})^2(m)$. The result is then immediate, since from (4.2) we have

$$\|B\|^2 = \frac{3}{4}\|T\|^2 + \frac{1}{2}\check{T}, \tag{5.2}$$

$$\|\bar{B}\|^2 = \|\bar{T}\|^2. \quad \square \tag{5.3}$$

Now, we consider the decomposition of $\Lambda^2 V^* \otimes V^*$ into $O(n)$-irreducible subspaces. $\Lambda^3 V^*$ is already $O(n)$-irreducible, but Y_1^2 decomposes into two subspaces, namely $\tilde{Y}_1^2 = \{A \in Y_1^2 / \bar{A}_X = 0 \text{ for any } X \in V\}$ and

$$\bar{Y}_1^2 = \left\{A \in Y_1^2 / A_{XYZ} = \frac{1}{n-1}(-\langle X, Z\rangle \bar{A}_Y + \langle Y, Z\rangle \bar{A}_X)\right\}$$

(where $\bar{A}_X = \Sigma_{i=1}^n A_{Xe_ie_i}$, $\{e_i\}$ being an orthonormal basis of V). (cfr [16]).

If we split $T = T^1 + T^2 + T^3$, with T^1 belonging to $\Lambda^3 V^*$, T^2 to $\tilde{Y}_1^2$ and T^3 to $\bar{Y}_1^2$, obviously $\|T\|^2 = \|T^1\|^2 + \|T^2\|^2 + \|T^3\|^2$.

If $\tilde{\alpha} : \Lambda^2 V^* \otimes V^* \to \Lambda^2 V^* \otimes V^*$ is the map given by $\tilde{\alpha}(A)_{XYZ} = A_{ZYX} - A_{ZXY}$, then $\tilde{\alpha}|_{\Lambda^3 V^*} = -2I$ and $\tilde{\alpha}|_{Y_1^2} = I$ (here I is the identity map). Moreover, $\tilde{\alpha}$ is Gl(n)-invariant, and $\check{T} = (1/2)\langle T, \tilde{\alpha}(T)\rangle$. Then

$$\check{T} = -\|T^1\|^2 + \frac{1}{2}\|T^2\|^2 + \frac{1}{2}\|T^3\|^2. \tag{5.4}$$

We also get $\bar{T} = \overline{T^1} + \overline{T^2} + \overline{T^3} = \overline{T^3}$, hence

$$\|\bar{T}\|^2 = \|\overline{T^3}\|^2 = \frac{n-1}{2}\|T^3\|^2. \tag{5.5}$$

Now 5.1 can be reformulated as follows:

5.4. THEOREM: *For any r such that $0 < r < r_D$ we have*

$$V_m^D(r) = \frac{(\pi r^2)^{(n/2)}}{(n/2)!}\left\{1 - \frac{1}{6(n+2)}\left(\tau_\nabla + \frac{3}{8}\|T^2\|^2 + \frac{n+2}{8}\|T^3\|^2\right) r^2 + O(r^4)\right\}.$$

If $V_0 = \dfrac{\pi^{(n/2)}}{(n/2)!}$ (the volume of the unit ball in $\mathbb{R}^n$), 5.4 can be rewritten

$$V_m^D(r) = V_0 r^n\{1 + C_1^D r^2 + C_2^D r^4 + \cdots + C_n^D r^{2n} + \cdots\}$$

and we can state the following corollaries:

5.5. COROLLARY: *D and ∇ have the same geodesics if and only if $C_1^D = C_1^\nabla$ for any m in M.*

PROOF: $C_1^D = C_1^\nabla$ implies $T^2 = T^3 = 0$, so $T = T^1$, i.e., T lies on $\Lambda^3 V^*$ and, as we have indicated in §4, D and ∇ have the same geodesics. □

5.6. COROLLARY: *If M has non-negative Ricci curvature ρ_∇ and $C_1^D = C_2^D = 0$ for any m in M, then M is locally flat* $(R = 0)$.

PROOF: $\rho_\nabla(X, X) \geq 0$ gives $\tau_\nabla \geq 0$. Since $C_1^D = 0$, from 5.4 we have $\tau_\nabla = 0$ and $T = T^1$. Then D and ∇ have the same geodesics, $V_m^D(r) = V_m^\nabla(r)$ and $C_1^D = C_2^D = C_1^\nabla = C_2^\nabla = 0$. But in [6] it is proved that if $\rho_\nabla(X, X) \geq 0$ and $C_1^\nabla = C_2^\nabla = 0$, then $R = 0$. □

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