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MINIMAL MODELS FOR ELLIPTIC CURVES WITH COMPLEX MULTIPLICATION

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Let R be the ring of integers in an algebraic number field F. An abelian variety A of dimension g over F determines an element c_A in the ideal class group R in the following manner. Let N denote the Néron model of A over R [4]; the space $\underline{\omega}_{N/R}$ of invariant differentials on N is a projective R-module of rank g. We may define c_A to be the class of $\overset{g}{\Lambda}\underline{\omega}_{N/R}$ in $\mathrm{Pic}(R)$.

When $\dim A = 1$ Tate has given an alternate description of the class c_A in terms of minimal Weierstrass models [5]. We use this formulation, and some classical results of Deuring [1] and Hasse, to calculate c_A for some elliptic curves with complex multiplication.

§1. Minimal models of elliptic curves

Let A be an elliptic curve over F, a number field with ring of integers R. The space $\underline{\omega}_{A/F} = H^0(A, \Omega^1/F)$ of invariant differentials is an F-vector space of dimension 1. Associated to any non-zero differential ω we have its discriminant $\Delta_\omega \in F^*$ [5]. If $\omega' = u^{-1}\omega$ then $\Delta_{\omega'} = u^{12}\Delta_\omega$; hence A determines a coset $\Delta_A \in F^*/F^{*12}$.

For any discrete valuation v of F, let ω_v and $\Delta_v = \Delta_{\omega_v}$ be the differential and discriminant of a minimal Weierstrass equation for A at v [5]. We define the discriminant ideal $\mathscr{D}_A$ by the formula:

(1.1) $$\mathscr{D}_A = \prod_v \mathscr{P}_v^{v(\Delta_v)},$$

where $\mathscr{P}_v$ is a prime ideal at the place v. For any non-zero differential

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ω on A over F we define the ideal δ_ω by the formula:

$$\delta_\omega = \prod_v \mathcal{P}_v^{v(\omega/\omega_v)}. \tag{1.2}$$

One then has the equality of ideals in R:

$$(\Delta_\omega)\delta_\omega^{12} = \mathcal{D}_A. \tag{1.3}$$

The class of the ideal δ_ω in $\mathrm{Pic}(R)$ is independent of the choice of ω. We denote this class by δ_A; then A has a global differential ω with $(\Delta_\omega) = \mathcal{D}_A$ if and only if $\delta_A \sim 1$ in $\mathrm{Pic}(R)$. In this case one can find a global minimal model for A: i.e., an equation for A over R which is simultaneously minimal at all places v.

By (1.3) one has:

$$\delta_A^{12} \sim \mathcal{D}_A \quad \text{in } \mathrm{Pic}(R). \tag{1.4}$$

Hence a necessary condition for the existence of a global minimal model is that the ideal $\mathcal{D}_A$ be principal. By (1.4) this is also sufficient when the group $\mathrm{Pic}(R)$ has no 12-torsion.

It is not difficult to compare δ_A with the class c_A of Néron differentials defined in the introduction. Let X be the minimal regular model for A over R_v; X is a regular projective scheme over R_v which can be obtained by resolving the possible singularity on a minimal Weierstrass equation for A over R_v [4, pp. 94–101]. The Néron minimal model N is a smooth group scheme over R_v; it is obtained by removing all fibres of multiplicity greater than one on X and all singular points in the remaining fibres. The pull-back of a minimal Weierstrass differential ω_v on A/R_v is everywhere non-zero on N. Hence we find:

$$\underline{\omega}_{N/R_v} = \omega_v R_v \subset \underline{\omega}_{A/F_v}, \tag{1.5}$$

so globally we have the identity:

$$\underline{\omega}_{N/R} = \omega\delta_\omega^{-1} \subset \underline{\omega}_{A/F}. \tag{1.6}$$

To sum up, we have the following

PROPOSITION 1.7:
(1) $c_A \sim \delta_A^{-1}$ *in* Pic(R).
(2) *The following statements are equivalent*
 (a) $c_A \sim \delta_A \sim 1$ *in* Pic(R).
 (b) *A has a global minimal Weierstrass model over R.*
 (c) *A has a non-zero differential* ω *with* $(\Delta_\omega) = \mathcal{D}_A$.
 (d) $\underline{\omega}_{N/R}$ *is a free R-module of rank* 1.

§2. Elliptic curves with complex multiplication

We now assume that A is an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}$ of an imaginary quadratic field K. We assume further that the field F of definition for A is H, the Hilbert class field of K. Then all endomorphisms of A are defined over H, and the curve A is determined up to isomorphism by its modular invariant j_A and the associated Hecke character χ_A on the idèles I_H of H [2; 9.1.3].

PROPOSITION 2.1: *Both the ideal* $\mathcal{D}_A$ *and the class* δ_A *depend only on the character* χ_A, *and not on the modular invariant* j_A.

PROOF: Let B be another elliptic curve over F with $\chi_B = \chi_A$; then $j_B = j_A^\sigma$ with $\sigma \in \mathrm{Aut}(H)$. The group $\mathrm{Hom}_H(B, A)$ is described in [2, 9.4.2]: for any integral ideal $\mathfrak{a}$ of K such that $\sigma = \sigma_\mathfrak{a}^{-1}$ in Aut(H) we have an isogeny $\phi_\mathfrak{a} : B \to A$ with kernel isomorphic to $\mathcal{O}/\mathfrak{a}$. More precisely, we may choose an embedding of H into $\mathbb{C}$ so that the following diagram commutes:

$$\text{(2.2)} \qquad \begin{array}{ccc} B(\mathbb{C}) & \xrightarrow{\phi_\mathfrak{a}} & A(\mathbb{C}) \\ {\scriptstyle \int \phi_\mathfrak{a}^*\omega}\downarrow & & \downarrow{\scriptstyle \int \omega} \\ \mathbb{C}/\Omega\mathfrak{a} & \xrightarrow{p} & \mathbb{C}/\Omega\mathcal{O} \end{array}$$

where ω is a non-zero differential on A, $\Omega \in \mathbb{C}^*$ is a fixed integral period of ω, and p is the natural projection.

Now let v be a fixed place of H and choose $\mathfrak{a}$ with $\sigma_\mathfrak{a}^{-1} = \sigma$ and $\mathbb{N}\mathfrak{a}$ prime to v (this is always possible). Then the induced map $\phi_\mathfrak{a}^* : \underline{\omega}_{B/R_v} \to \underline{\omega}_{A/R_v}$ on the spaces of local Néron differentials is an isomorphism. Hence to show that $\mathcal{D}_A = \mathcal{D}_B$ it suffices to show that $v(\Delta_{\omega_v}) = v(\Delta_{\phi_\mathfrak{a}^*\omega_v})$. But by (2.2), if we compute over $\mathbb{C}$,

$$\Delta_\omega = \frac{\Delta(\mathcal{O})}{\Delta(\mathfrak{a})} \Delta_{\phi_\mathfrak{a}^*\omega}. \tag{2.3}$$

It is well-known that $\Delta(\mathcal{O})/\Delta(\mathfrak{a})$ is an algebraic integer in H which generates the ideal $\mathfrak{a}^{12}$ [1, p. 33], [3, p. 165]. Since this is prime to v, the minimal discriminants have the same valuation.

Now let ω be any non-zero differential on A over H and put $\nu = \phi_{\mathfrak{a}}^{*}(\omega)$. Then by (1.3) and the above paragraph:

$$(\Delta_\omega)\delta_\omega^{12} = \mathcal{D}_A = \mathcal{D}_B = (\Delta_\nu)\delta_\nu^{12}.$$

Since $\Delta_\omega/\Delta_\nu = \Delta(\mathcal{O})/\Delta(\mathfrak{a})$ by (2.3), we have

$$(\delta_\nu/\delta_\omega)^{12} = (\Delta(\mathcal{O})/\Delta(\mathfrak{a})) = \mathfrak{a}^{12}.$$

Hence $\delta_\nu = \delta_\omega \cdot \mathfrak{a}$ as ideals of H. But the ideal $\mathfrak{a}$ of K capitulates in H; hence $\delta_A \sim \delta_B$ in $\mathrm{Pic}(R)$.

Note: If we assume that the Hecke character $\chi_A : I_H \to K^*$ is $\mathrm{Gal}(H/K)$-equivariant, then by Proposition 2.1 the ideal $\mathcal{D}_A$ is fixed by $\mathrm{Gal}(H/K)$. Since H is unramified over K, any fixed ideal is represented by an ideal of K. But all ideals of K capitulate in H, so $\mathcal{D}_A \sim 1$ in $\mathrm{Pic}(R)$. Is $\delta_A \sim 1$ in $\mathrm{Pic}(R)$? We will show this is the case when K has prime discriminant.

§3. A global minimal model for $A(p)$

We now specialize to the case where the multiplication field $K = \mathbb{Q}(\sqrt{-p})$ has *prime* discriminant.

LEMMA 3.1: *For any fractional ideal $\mathfrak{a}$ of K, the ratio $\Delta(\mathcal{O})/\Delta(\mathfrak{a})$ is a 12^{th} power in H^*.*

PROOF: By Deuring [1, p. 14, 41] the ratio $\Delta(\mathcal{O})/\Delta(\mathfrak{b}^2)$ is a 24^{th} power in H^* when $(6, \mathfrak{b}) = 1$. When K has prime discriminant, its class group has *odd* order. Hence we may find an ideal $\mathfrak{b}$ prime to 6 such that $(\alpha)\mathfrak{a} = \mathfrak{b}^2$. Then

$$\Delta(\mathcal{O})/\Delta(\mathfrak{a}) = \alpha^{12} \cdot \Delta(\mathcal{O})/\Delta(\mathfrak{b}^2) \equiv 1 \pmod{H^{*12}}.$$

We can now answer affirmatively a question posed by D. Zagier. Assume that $p > 3$ and let $A(p)$ denote $\mathbb{Q}$-curve over the field $F = \mathbb{Q}(j_{A(p)})$ studied in chapter 5 of [2]. Recall that $A(p)$ has good reduction outside p and has minimal discriminant ideal $\mathcal{D}_{A(p)} = (-p^3)$. The fact that this ideal is principal raises the possibility of a global minimal model.

PROPOSITION 3.2: *The curve $A(p)$ has a global minimal model over the field $F = \mathbb{Q}(j_{A(p)})$ with discriminant $\Delta = -p^3$. The associated differential $\omega(p)$ is determined up to sign.*

PROOF: In §23 of [2] we constructed a pair (A, ω) over F with $j_A = j_{A(p)}$, $\Delta_\omega = -p^3$, and sign $c_6 = \left(\frac{2}{p}\right)$. Recall that A is given by the equation

$$y^2 = x^3 + \frac{mp}{2^4 \cdot 3}x - \frac{np^2}{2^5 \cdot 3^3} \tag{3.3}$$

where

$$m^3 = j_{A(p)}$$

$$n^2 = (j_{A(p)} - 1728)/-p, \quad \text{sign}\, n = \left(\frac{2}{p}\right), \tag{3.4}$$

The differential $\omega = dx/2y$ on A has $\Delta_\omega = -p^3$. To prove Proposition 3.2 we will show that A is *isomorphic* to $A(p)$ over F. We will then have a global minimal model by Proposition 1.7, as $(\Delta_\omega) = \mathcal{D}_{A(p)}$. The differential $\omega = \omega(p)$ with $\Delta_\omega = -p^3$ is determined up to sign, as $\mu(F^*) = \langle \pm 1 \rangle$.

In summary, we are reduced to proving:

PROPOSITION 3.5: *The elliptic curve A defined by equations* (3.3–3.4) *is a $\mathbb{Q}$-curve which is isomorphic over F to the curve $A(p)$.*

PROOF: Consider the map

$$f_A : \text{Gal}(H/\mathbb{Q}) \to \text{Hom}(I_H, K^*)$$
$$\sigma \mapsto \chi_A^{\sigma - 1}$$

where all Homs refer to continuous homomorphisms of topological groups. Then f_A is a 1-cocycle, which takes values in the group $\text{Hom}(I_H/H^*, K^*)$. Since K^* is totally disconnected, this group may be identified with the group $\text{Hom}(\text{Gal}(\bar{H}/H), K^*)$ via the Artin homomorphism of global class field theory. Since $\text{Gal}(\bar{H}/H)$ is compact and K^* is discrete, any continuous homomorphism takes values in the finite group $\mu(K^*) = \langle \pm 1 \rangle$. Finally, we may identify

$$\mathrm{Hom}(\mathrm{Gal}(\bar{H}/H), \pm 1) \simeq H^*/H^{*2},$$

by Kummer theory, and view f_A as a map

$$f_A : \mathrm{Gal}(H/\mathbb{Q}) \to H^*/H^{*2}. \tag{3.5}$$

To show A is a $\mathbb{Q}$-curve is equivalent to showing that $f_A(\sigma) \equiv 1$ for all $\sigma \in \mathrm{Gal}(H/\mathbb{Q})$. Since A is defined over F we have $f_A(\tau) \equiv 1$. Hence, it suffices to show $f_A(\sigma) = 1$ for all $\sigma \in \mathrm{Gal}(H/K)$.

For this, we need a concrete description of $f_A(\sigma)$ in H^*/H^{*2}. Embed F in $\mathbb{C}$ via its real place, and let $\mathfrak{a}$ be an integral ideal of K with $\sigma = \sigma_{\mathfrak{a}}^{-1}$. There is an isogeny $\phi_{\mathfrak{a}}$ defined over $\bar{\mathbb{Q}}$ which makes the following diagram commutative:

$$\begin{array}{ccc} A^{\sigma} & \xrightarrow{\ \phi_{\mathfrak{a}}\ } & A \\ {\scriptstyle \int \phi_{\mathfrak{a}}^{*}\omega} \downarrow & & \downarrow {\scriptstyle \int \omega} \\ \mathbb{C}/\Omega\mathfrak{a} & \xrightarrow[\ p\ ]{} & \mathbb{C}/\Omega\mathcal{O}. \end{array}$$

If we write $\phi_{\mathfrak{a}}^{*}(\omega) = h_{\mathfrak{a}} \cdot \omega^{\sigma}$ with $h_{\mathfrak{a}} \in \bar{\mathbb{Q}}^*$, then the isogeny $\phi_{\mathfrak{a}}$ is defined over the extension $H(h_{\mathfrak{a}})$. The identities:

$$c_4(\mathcal{O})/c_4(\mathfrak{a}) = h_{\mathfrak{a}}^{4} \cdot c_4^{1-\sigma}$$
$$c_6(\mathcal{O})/c_6(\mathfrak{a}) = h_{\mathfrak{a}}^{6} \cdot c_6^{1-\sigma}$$

show that $h_{\mathfrak{a}}^{2} \in H^*$ [3, p. 158]. In fact, we have the formula

$$f_A(\sigma) \equiv h_{\mathfrak{a}}^{2} \pmod{H^{*2}}. \tag{3.6}$$

On the other hand, we have the identity:

$$\Delta(\mathcal{O})/\Delta(\mathfrak{a}) = h_{\mathfrak{a}}^{12} \cdot \Delta^{1-\sigma} = h_{\mathfrak{a}}^{12}$$

as $\Delta = -p^3$ is *fixed* by $\mathrm{Gal}(H/\mathbb{Q})$. By Lemma 3.1, $h_{\mathfrak{a}}^{12}$ is a 12th power in H^*. Since $h_{\mathfrak{a}}^{2} \in H^*$, we must have $h_{\mathfrak{a}} \in H^*\mu_4$ and $f_A(\sigma) \equiv \pm 1 \pmod{H^{*2}}$. But f_A is a *cocycle* and the order of $\mathrm{Gal}(H/K)$ is *odd*. Hence $f_A(\sigma) \equiv 1$ and A is a $\mathbb{Q}$-curve.

Since $v_{\mathfrak{P}}(\Delta_{\omega}) = 3$ we see $A \simeq A(p)^d$ with $(p, d) = 1$ [2, 12.3.2]. But $\mathfrak{D}_A = \mathfrak{b}^{12}(-p^3)$ and $\mathfrak{D}_{A(p)^d} = \mathfrak{c}^{12}(-p^3 d^6)$ where $\mathfrak{b}$ and $\mathfrak{c}$ are ideals of H.

Hence $(d) = (\mathfrak{b}/\mathfrak{c})^2$ is the square of an ideal of H. Since H is unramified over K and d is a quadratic discriminant, there are only two possibilities: $d = 1$ and $d = -4$. But the curve $A(p)^{-4}$ has the wrong sign of c_6, so $A \simeq A(p)$.

§4. Global minimal models for K-curves

Let $\omega(p)$ be one of the differentials on $A(p)$ given by Proposition 3.2. For any integral ideal $\mathfrak{a}$ of K we may define $h_\mathfrak{a}$ in $H^*/\pm 1$ by the formula:

$$\phi_\mathfrak{a}^*(\omega(p)) = h_\mathfrak{a} \cdot \omega(p)^{\sigma_\mathfrak{a}^{-1}}. \tag{4.1}$$

The ambiguity in sign is caused by the ambiguity in the choice of isogeny $\phi_\mathfrak{a}$; we will discuss a choice of the sign in §5. In $H^*/\pm 1$ we have the cocycle relations

$$\begin{aligned} h_{\mathfrak{a}\mathfrak{b}} &= h_\mathfrak{a}^{\sigma_\mathfrak{b}^{-1}} \cdot h_\mathfrak{b} \\ h_{\mathfrak{a}^\tau} &= h_\mathfrak{a}^\tau \end{aligned} \tag{4.2}$$

We have seen in §3 that when F is embedded into $\mathbb{C}$ via its real place we have the complex identity:

$$h_\mathfrak{a}^{12} = \Delta(\mathcal{O})/\Delta(\mathfrak{a}).$$

Hence $h_\mathfrak{a}$ is *integral* in H and generates the ideal $\mathfrak{a}$. The same is true for $h_\mathfrak{a}^\sigma$ for any $\sigma \in \mathrm{Gal}(H/K)$.

LEMMA 4.1: *For all* $\sigma \in \mathrm{Gal}(H/K)$, $h_\mathfrak{a}^{\sigma-1} \equiv 1 \pmod{H^{*2}}$.

PROOF: First note that this identity makes sense, independent of the choice of sign for $h_\mathfrak{a}$. We have seen, in the proof of Lemma 3.1, that $\Delta(\mathcal{O})/\Delta(\mathfrak{b}^2) = h_{\mathfrak{b}^2}^{12}$ is a 24th power in H^*. Hence $h_{\mathfrak{b}^2} = \pm 1 \pmod{H^{*2}}$. Since we may find $\mathfrak{b}$ such that $\mathfrak{a} = (\alpha)\mathfrak{b}^2$, we find from (4.2) that $h_\mathfrak{a} \equiv \pm\alpha \pmod{H^{*2}}$. Hence $h_\mathfrak{a}^{\sigma-1} \equiv \pmod{H^{*2}}$ for any $\sigma \in \mathrm{Gal}(H/K)$.

LEMMA 4.2: *Let K' be a quadratic extension of K with conductor $\mathfrak{a}$. Then we may choose the sign of $h_\mathfrak{a}$ so that $HK' = H(\sqrt{h_\mathfrak{a}})$.*

PROOF: Write $K' = K(\sqrt{\alpha})$. Since $\mathfrak{a}$ is the discriminant ideal of K'/K and α is the discriminant of the specific K-basis $\langle 1, \sqrt{\alpha}/2\rangle$ we find $(\alpha)\mathfrak{b}^2 = \mathfrak{a}$ with $\mathfrak{b}$ an ideal of K. Raising this identity to the h^{th} power and writing $(\beta) = \mathfrak{b}^h$ we find $(\alpha^h\beta^2) = \mathfrak{a}^h = (\mathrm{N}_{H/K}h_{\mathfrak{a}})$. Since h is odd and $\mathcal{O}_K^* = \langle \pm 1 \rangle$, we may choose the sign of $h_{\mathfrak{a}}$ so that $\alpha \equiv \mathrm{N}_{H/K}h_{\mathfrak{a}}$ (mod K^{*2}). Then $K' = K(\sqrt{\mathrm{N}_{H/K}h_{\mathfrak{a}}})$ and $HK' = H(\sqrt{\mathrm{N}_{H/K}h_{\mathfrak{a}}})$.

By Lemma 4.1, $h_{\mathfrak{a}} \equiv h_{\mathfrak{a}}^{\sigma}$ (mod H^{*2}) so multiplying over the entire Galois group we find $h_{\mathfrak{a}}^h \equiv \mathrm{N}_{H/K}h_{\mathfrak{a}}$ (mod H^{*2}). Since h is odd, $h_{\mathfrak{a}} \equiv h_{\mathfrak{a}}^h \equiv \mathrm{N}_{H/K}h_{\mathfrak{a}}$ (mod H^{*2}) and $HK' = H(\sqrt{h_{\mathfrak{a}}})$ as claimed.

Now let A be an elliptic curve over H such that χ_A is Gal(H/K) equivariant. By [2, 12.3.1] we may write $A = A(p)^{\psi}$ with

$$\psi \in \mathrm{Hom}(\mathrm{Gal}(\bar{H}/H), \pm 1)^{\mathrm{Gal}(H/K)} \simeq \mathrm{Hom}(\mathrm{Gal}(\bar{K}/K), \pm 1).$$

Let $\mathfrak{a}$ be the conductor of ψ and write the associated quadratic extension $H' = H(\sqrt{h_{\mathfrak{a}}})$ as permitted by Lemma 4.2. For simplicity, assume that $\mathfrak{a}$ is prime to p. Let ρ be a generator of Gal(H'/H); we then have the identification

$$\underline{\omega}_{A/H} = \{\omega \in \underline{\omega}_{A(p)/H'} : \omega^{\rho} = -\omega\}.$$

Hence the differential $\omega_A = (1/\sqrt{h_{\mathfrak{a}}}) \cdot \omega(p)$ descends to A over H.

PROPOSITION 4.3: *Either ω_A or $2\omega_A$ is a global minimal differential on A/H.*

PROOF: We clearly have $\Delta_{\omega_A} = -p^3h_{\mathfrak{a}}^6$ so $(\Delta_{\omega_A}) = (-p^3)\mathfrak{a}^6$. This is equal to $\mathfrak{D}_A$ except in the case when $\left(\frac{2}{p}\right) = -1$ and $8 \mid \mathfrak{a}$ [2, 14.1.1]. In that case it is equal to $(2^{12})\mathfrak{D}_A$.

COROLLARY 4.4: *If K has prime discriminant and the Hecke character χ_A of A is* Gal(H/K) *equivariant, then $\delta_A \sim c_A \sim 1$ in* Pic(R).

Indeed, the minimal differential given in Proposition 4.3 is determined up to sign.

§5. The sign of $h_{\mathfrak{a}}$

When the ideal $\mathfrak{a}$ of K is prime to (p), we may normalize the sign of $h_{\mathfrak{a}}$ by insisting that $\mathrm{N}_{H/K}h_{\mathfrak{a}}$ is a *square* (mod $\sqrt{-p}$). Then the following identities hold in H^*:

$$h_{\mathfrak{a}\mathfrak{b}} = h_{\mathfrak{a}}^{\sigma_{\mathfrak{b}}^{-1}} h_{\mathfrak{b}}$$

$$(5.1) \qquad h_{\mathfrak{a}^{\tau}} = h_{\mathfrak{a}}^{\tau}$$

$$h_{(\alpha)} = \alpha \quad \text{if } \alpha \equiv 1 \ (\mathrm{mod}\ \sqrt{-p}).$$

Hence there is a unique continuous 1-cocycle

$$\phi : I_K \to H^*$$

which is the identity on principal idèles and satisfies $\phi(a) = \Pi_{v \nmid p,\infty} h_{\mathfrak{a}_v}^{v(a)}$ for all idèles which are trivial at ∞ and congruent to 1 (mod $\sqrt{-p}$). (The group I_K acts on H^* via its quotient $I_K/K^* \cdot (\mathbb{C}^* \times \Pi_v \mathcal{O}_v^*) = \mathrm{Gal}(H/K)$, and the cocycle ϕ is τ-equivariant.)

Recall the elements $t_\mathfrak{a}$ in $T^*/\pm 1$ defined in [2, 15.2.5]. Again, when $\mathfrak{a}$ is prime to (p) we may normalize the sign of $t_\mathfrak{a}$ by insisting that $t_\mathfrak{a}^h$ is a square (mod $\sqrt{-p}$). We then have the identities in T^*:

$$\begin{aligned} t_{\mathfrak{a}\mathfrak{b}} &= t_\mathfrak{a} t_\mathfrak{b} \\ t_{\mathfrak{a}^\tau} &= t_\mathfrak{a}^\tau \qquad\qquad (5.2)\\ t_{(\alpha)} &= \alpha \quad \text{if } \alpha \equiv 1\,(\sqrt{-p}). \end{aligned}$$

Since $(t_\mathfrak{a}) = \mathfrak{a}$ we find:

PROPOSITION 5.3: *The elements* $u_\mathfrak{a} = t_\mathfrak{a}/h_\mathfrak{a}^{\sigma_\mathfrak{a}}$ *are units in the field HT which satisfy the identities*

$$\begin{aligned} u_{\mathfrak{a}\mathfrak{b}} &= u_\mathfrak{a} \cdot u_\mathfrak{b}^{\sigma_\mathfrak{a}} \\ u_{\mathfrak{a}^\tau} &= u_\mathfrak{a}^\tau \\ u_{(\alpha)} &= 1. \end{aligned}$$

Since $u_\mathfrak{a}$ depends only on the class of $\mathfrak{a}$ in $\mathrm{Pic}(\mathcal{O})$ it is convenient to write $u_{\sigma_\mathfrak{a}}$ for the unit $u_\mathfrak{a}$. By Proposition 5.3 the assignment

$$\begin{aligned} \sigma &\to u_\sigma \\ \tau &\to 1 \end{aligned}$$

gives a 1-cocycle f on $\mathrm{Gal}(HT/T^+) \simeq \mathrm{Gal}(H/\mathbb{Q})$ with values in the units U of $(HT)^*$.

QUESTION 5.4: *Is* $f \sim 1$ *in* $H^1(\mathrm{Gal}(HT/T^+), U)$?

As a stronger question, one can ask if $\epsilon = \Sigma_\sigma u_\sigma$ is a *unit* of HT.

REFERENCES

[1] M. DEURING: Die Klassenkörper der Komplexen Multiplication. *Ency. der Math. Wiss. Band I, 2. Teil, Heft 10, Teil II* (1958).

[2] B. GROSS: Arithmetic on elliptic curves with complex multiplication. Springer Lecture Notes 776 (1980).

[3] S. LANG: *Elliptic functions.* Reading: Addison-Wesley (1973).

[4] A. NÉRON: Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. *IHES Publ. Math. No. 21 (1964)* 361–483.

[5] J.T. TATE: Algorithm for determining the type of singular fiber in an elliptic pencil. *Springer Lecture Notes 476* (1975) 33–52.

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