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COMPLEX GEODESICS

Edoardo Vesentini

Let D be a domain in a complex, locally convex, Hausdorff vector space \mathcal{E} . The Carathéodory and Kobayashi invariant pseudo-distances have been introduced in D , together with the corresponding infinitesimal pseudo-metrics [14]. A holomorphic map of the unit disc Δ of \mathbb{C} into D which is an isometry for the Poincaré distance of Δ and the Carathéodory or the Kobayashi pseudo-distances of D is called a complex geodesic. It is easily seen that complex geodesics do not always exist. However, their existence turns out to be a useful tool in the investigation of the group of all holomorphic automorphisms of D .

The primary purpose of this paper is that of establishing basic properties of complex geodesics, clarifying in particular the relationship between the Carathéodory pseudo-distance and the Carathéodory pseudo-metric along a complex geodesic. This latter result will lead to conditions whereby holomorphic maps of D which preserve complex geodesics are necessarily affine maps. As a consequence, some results obtained in [10] and in [13] will be improved.

All these facts will be established in nn.3 and 4. Some basic material will be collected in n.1, and in n.2 the relationship between the "size" of D and the behaviour of the Kobayashi pseudo-metric will be investigated; as a consequence, classical results of the theory of conformal mappings will be extended to the domain D in \mathcal{E} .

1. Let \mathcal{E} and \mathcal{E}_1 be two complex locally convex Hausdorff vector spaces, and let D and D_1 be domains in \mathcal{E} and in \mathcal{E}_1 respectively. By definition (cf. e.g. [8]), a holomorphic map $F: D \rightarrow D_1$ is a continuous, Gateaux-analytic map F of D into \mathcal{E}_1 such that $F(D) \subset D_1$. The

symbol $\text{Hol}(D, D_1)$ will denote the set of all holomorphic maps of D into D_1 .

If D is an open convex neighborhood of 0 in \mathcal{E} , the support function m of D is a convex, continuous function in \mathcal{E} . Let Δ be the open unit disc in \mathbb{C} , and let $f \in \text{Hol}(\Delta, D)$. For any $\zeta \in \Delta$ and any sufficiently small $r > 0$,

$$f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta + re^{i\theta}) d\theta. \quad \bullet$$

Hence

$$m(f(\zeta)) = m\left(\frac{1}{2\pi} \int_0^{2\pi} f(\zeta + re^{i\theta}) d\theta\right) \leq \frac{1}{2\pi} \int_0^{2\pi} m(f(\zeta + re^{i\theta})) d\theta.$$

That proves

LEMMA 1.1: *For any $f \in \text{Hol}(\Delta, D)$, the function $\zeta \mapsto m(f(\zeta))$ is subharmonic on Δ .*

REMARK: If the domain D is furthermore a balanced neighborhood of 0 , then m is a semi-norm and the function $\zeta \mapsto \log m(f(\zeta))$ is subharmonic [12].

By lemma 1.1 the function $m \circ f$ satisfies a maximum principle, whereby, if $f(\Delta) \cap \partial D \neq \emptyset$, then $f(\Delta) \subset \partial D$. In the following a stronger form of the maximum principle will be needed, which was established by E. Thorp and R. Whitley in [11] for complex Banach spaces. As it was shown in [11] (cf. also [12]), there exists a constant c , $0 < c < 1$, such that, for every positive integer n , complex numbers z_1, \dots, z_n can be found satisfying the following conditions:

$$|z_\alpha| < \frac{1}{c} \quad (\alpha = 1, \dots, n); \quad \sum_{\alpha=1}^n z_\alpha = n; \quad \sum_{\alpha=1}^n z_\alpha^p = 0 \quad \text{for } p = 2, \dots, n.$$

The proof of the following lemma—given originally in [11] in the case of a Banach space—can be adapted to locally compact spaces.

LEMMA 1.2: *For every non-constant holomorphic function $f: \Delta \rightarrow \mathcal{E}$, there is an element $a \in \mathcal{E} \setminus \{0\}$ such that the image of the open disc $\{\zeta \in \mathbb{C}: |\zeta| < c\}$ by the affine map $\zeta \mapsto f(0) + \zeta a$ belongs to the closure of the convex hull of $f(\Delta)$.*

PROOF: Assuming $f(0) = 0$, let

$$f(\zeta) = \zeta a_1 + \zeta^2 a_2 + \dots$$

be the power series expansion of f in Δ . If p is any continuous semi-norm on \mathcal{E} , the series $\sum_{\nu=1}^{\infty} p(a_\nu) t^\nu$ converges for $0 \leq t < 1$. Then, choosing n complex numbers z_1, \dots, z_n as before,

$$\sum_{\alpha=1}^n f(z_\alpha \zeta) = n \zeta a_1 + h_n(\zeta),$$

where

$$h_n(\zeta) = \left(\sum_{\alpha=1}^n z_\alpha^{n+1} \right) \zeta^{n+1} a_{n+1} + \left(\sum_{\alpha=1}^n z_\alpha^{n+2} \right) \zeta^{n+2} a_{n+2} + \dots$$

Since

$$p\left(\frac{1}{n} h_n(\zeta)\right) \leq \sum_{\nu=1}^{\infty} p(a_\nu) \left(\frac{|\zeta|}{c}\right)^\nu,$$

then

$$\lim_{n \rightarrow \infty} p\left(\frac{1}{n} h_n(\zeta)\right) = 0,$$

for $|\zeta| < c$ and for any continuous semi-norm p . Therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} h_n(\zeta) = 0,$$

i.e.

$$\lim_{n \rightarrow \infty} \frac{1}{n} f(z_\alpha \zeta) = \zeta a_1 = \zeta f'(0) \quad \text{for } |\zeta| < c.$$

That proves the theorem when $f'(0) \neq 0$. If $f'(0) = 0$, let

$$f(\zeta) = \zeta^n a_n + \zeta^{n+1} a_{n+1} + \dots \quad (a_n \neq 0)$$

be the power series expansion of f . Denoting by $\epsilon_1, \dots, \epsilon_n$ the n -roots of 1, the holomorphic function $g: \Delta \rightarrow \mathcal{E}$ defined by

$$g(\zeta) = \frac{1}{n} (f(\epsilon_1 \zeta^{1/n}) + \dots + f(\epsilon_n \zeta^{1/n}))$$

belongs to the convex envelope of $f(\Delta)$ and is such that

$$g'(0) = a_n. \quad \text{QED}$$

Recall that a point $x_0 \in \partial D$ is a complex extreme point of \bar{D} if $y = 0$ is the only vector such that $x_0 + \Delta y \subset \bar{D}$. Lemmas 1.1 and 1.2 yield

PROPOSITION 1.3: *Let D be an open, convex neighborhood of 0 in \mathcal{E} . If $f \in \text{Hol}(\Delta, D)$ is such that $f(\Delta) \cap \partial D \neq \emptyset$, then $f(\Delta) \subset \partial D$. If $f(\Delta) \cap \partial D$ contains a complex extreme point of \bar{D} , then f is constant.*

REMARK: Proposition 1.3 extends a similar statement proved in [14] in the case in which D is balanced. The proof given in [14] is a straightforward consequence of a simplified version of Thorp and Whitley's original argument, which was established by L.A. Harris in [5] (cf. also [3]).

2. The Poincaré metric of Δ

$$ds^2 = (1 - |\zeta|^2)^{-2} |d\zeta|^2 \quad (\zeta \in \Delta)$$

has Gaussian curvature -4 . For $\zeta \in \Delta$, $\tau \in \mathbb{C}$, set

$$\langle \tau \rangle_\zeta = (1 - |\zeta|^2)^{-1} |\tau|.$$

For ζ_1, ζ_2 in Δ , the Poincaré distance is expressed by

$$\omega(\zeta_1, \zeta_2) = \frac{1}{2} \log \frac{1 + \left| \frac{\zeta_1 - \zeta_2}{1 - \bar{\zeta}_1 \zeta_2} \right|}{1 - \left| \frac{\zeta_1 - \zeta_2}{1 - \bar{\zeta}_1 \zeta_2} \right|}.$$

Given any two points x and y in the domain $D \subset \mathcal{E}$, an analytic chain joining x and y in D consists of $\nu + 1$ points $\zeta_0, \dots, \zeta_\nu$ in Δ , and of ν functions $f_j \in \text{Hol}(\Delta, D)$ ($j = 1, \dots, \nu$) such that:

$$f_1(\zeta_0) = x, f_j(\zeta_j) = f_{j+1}(\zeta_j) (j = 1, \dots, \nu - 1), f_\nu(\zeta_\nu) = y.$$

The open set D being connected, analytic chains joining x and y in D do exist. The Kobayashi pseudo-distance $k_D(x, y)$ is defined by

$$k_D(x, y) = \inf \sum_{j=1}^{\nu} \omega(\zeta_{j-1}, \zeta_j),$$

where the infimum is taken over all analytic chains joining x and y in D .

The Carathéodory pseudo-distance $c_D(x, y)$ is, by definition,

$$c_D(x, y) = \sup\{\omega(h(x), h(y)): h \in \text{Hol}(D, \Delta)\}$$

It turns out [14] that

$$c_D(x, y) \leq k_D(x, y),$$

and

$$c_\Delta = k_\Delta = \omega.$$

The infinitesimal versions of k_D and c_D are defined as follows: the Kobayashi pseudo-metric is given, for $x \in D$, $v \in \mathcal{E}$, by

$$(1) \quad \kappa_D(x; v) = \inf\{\langle \tau \rangle_\zeta: \zeta \in \Delta, \tau \in \mathbb{C}, f \in \text{Hol}(\Delta, D), f(\zeta) = x, \tau f'(\zeta) = v\},$$

and the Carathéodory pseudo-metric by

$$\gamma_D(x; v) = \sup\{\langle dh(x)v \rangle_{h(x)}: h \in \text{Hol}(D, \Delta)\}.$$

The domain D being open, the set appearing on the right-hand side of (1) is not empty. Furthermore [3]

$$\gamma_D(x; v) \leq \kappa_D(x; v)$$

and

$$\gamma_\Delta(\zeta; \tau) = \kappa_\Delta(\zeta; \tau) = \langle \tau \rangle_\zeta \quad (\zeta \in \Delta, \tau \in \mathbb{C}).$$

These pseudo-distances and pseudo-metrics are all contracted by holomorphic maps: for $F \in \text{Hol}(D, D_1)$

$$c_{D_1}(F(x), F(y)) \leq c_D(x, y), \quad k_{D_1}(F(x), F(y)) \leq k_D(x, y), \\ \gamma_{D_1}(F(x); dF(x)v) \leq \gamma_D(x; v), \quad \kappa_{D_1}(F(x); dF(x)v) \leq \kappa_D(x; v)$$

for all $x, y \in D$, $v \in \mathcal{E}$.

For further details on the above defined pseudo-distances and pseudo-metrics see e.g. [3] in the case of domains in complex Banach spaces, and [14] for the general case.

For any continuous semi-norm p on \mathcal{E} , any $x \in \mathcal{E}$ and any $r \geq 0$, $B_p(x, r)$ denotes the open ball with center x and radius r for the pseudo-distance defined by p . Then ([3], [13], [14])

$$(2) \quad \begin{aligned} \gamma_{B_p(x, r)}(x; v) &= \kappa_{B_p(x, r)}(x; v) = \frac{p(v)}{r}, \\ c_{B_p(x, r)}(x, y) &= k_{B_p(x, r)}(x, y) = \omega\left(0, \frac{p(y-x)}{r}\right) \end{aligned}$$

for all $y \in B_p(x, r)$, $v \in \mathcal{E}$.

The following theorem links the behaviour of the Kobayashi pseudo-metric κ_D of a domain $D \subset \mathcal{E}$ to the “size” of D .

THEOREM 1: *Let $x_0 \in D$, $v_0 \in \mathcal{E}$ be such that $\kappa_D(x_0; v_0) > 0$. Then*

$$(3) \quad (\mathcal{E} \setminus D) \cap \left\{ x_0 + \frac{\zeta}{\kappa_D(x_0; v_0)} v_0 : \zeta \in \bar{\Delta} \right\} \neq \emptyset.$$

Furthermore, there is no continuous semi-norm p on \mathcal{E} such that D is completely interior to $B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0; v_0)}\right)$; i.e. there is no continuous semi-norm p such that

$$(4) \quad D \subset B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0; v_0)}\right)$$

and

$$(5) \quad \inf\left\{ p(x-y) : x \in D, y \notin B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0; v_0)}\right) \right\} > 0$$

PROOF: If (3) does not hold, there is some ϵ such that $0 < \epsilon < \kappa_D(x_0; v_0)$ and

$$\left\{ x_0 + \frac{\zeta}{\kappa_D(x_0; v_0) - \epsilon} v_0 : \zeta \in \Delta \right\} \subset D.$$

Since the holomorphic function $f: \zeta \mapsto x_0 + \frac{\zeta}{\kappa_D(x_0; v_0) - \epsilon} v_0$ maps Δ into D , and moreover

$$\varphi(0) = x_0, (\kappa_D(x_0; v_0) - \epsilon)f'(0) = v_0,$$

then, by (1),

$$\kappa_D(x_0; v_0) \leq \kappa_D(x_0; v_0) - \epsilon,$$

which is a contradiction.

Let p be a continuous semi-norm on \mathcal{E} for which (4) and (5) hold. Then, there is some $\epsilon > 0$ such that

$$D \subset B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0; v_0) + \epsilon}\right),$$

and therefore, setting $A = B_p\left(x_0, \frac{p(v_0)}{\kappa_D(x_0; v_0) + \epsilon}\right)$,

$$\kappa_D(x_0; v_0) \geq \kappa_A(x_0; v_0) = \frac{p(v_0)}{\frac{p(v_0)}{\kappa_D(x_0; v_0) + \epsilon}} = \kappa_D(x_0; v_0) + \epsilon.$$

Contradiction.

QED

Let D be a simply connected domain $D \subsetneq \mathbb{C}$. For any $x \in D$, let h be the conformal map of D onto Δ with $h(x) = 0, h'(x) > 0$. Then

$$\kappa_D(x; 1) = \langle h'(x) \rangle_{h(x)} = h'(x),$$

and therefore $\kappa_D(x; 1)^{-1}$ is the conformal radius of D at x . Theorem 1 becomes in this case a classical result of complex function theory (cf. [1], [4]).

The relative topology of D in \mathcal{E} is finer than the topologies defined by the pseudo-distances c_D and k_D . If the relative topology coincides with the topology defined by k_D , D is called a *hyperbolic domain*. A bounded domain is hyperbolic [14]. Since \mathcal{E} is a Hausdorff space, on a hyperbolic domain k_D is a distance. Moreover the pseudo-metric κ_D cannot degenerate, as is shown by the following

THEOREM 2: *Let D be a hyperbolic domain. For $x_0 \in D$ let $r > 0$ and let p be a continuous semi-norm on \mathcal{E} such that*

$$B_p(x_0, r) \subset D.$$

There exists a positive constant c such that

$$\kappa_D(x_0; v) \geq cp(v)$$

for all $v \in \mathcal{E}$.

PROOF: For $s > 0$ let $B_k(x_0, s)$ be the open ball with center x_0 and radius s for the distance k_D . The domain D being hyperbolic, there is $s > 0$ such that

$$B_k(x_0, s) \subset B_p(x_0, r).$$

If the conclusion of the theorem is false, there is a sequence $\{v_\nu\}$ in \mathcal{E} such that $p(v_\nu) = 1$ and $\lim_{\nu \rightarrow \infty} \kappa_D(x_0; v_\nu) = 0$. Assume $\kappa_D(x_0; v_\nu) < 1$ for all ν , and let ϵ_ν be such that

$$\kappa_D(x_0; v_\nu) < \epsilon_\nu < 1,$$

and $\lim_{\nu \rightarrow \infty} \epsilon_\nu = 0$. Let $f_\nu \in \text{Hol}(\Delta, D)$ and $\tau_\nu \in \mathbb{C}$ be such that

$$f_\nu(0) = x_0, \quad \tau_\nu f'_\nu(0) = v_\nu, \quad |\tau_\nu| < \epsilon_\nu.$$

The latter condition implies that

$$(6) \quad p(f'_\nu(0)) > \frac{1}{\epsilon_\nu}.$$

Let $\delta_\nu = \epsilon_\nu^{1/2}$. Being, for $\nu \geq 0$,

$$f'_\nu(0) = \frac{1}{2\pi\delta_\nu} \int_0^{2\pi} e^{-i\theta} f'_\nu(\delta_\nu e^{i\theta}) d\theta,$$

then

$$p(f'_\nu(0)) \leq \frac{1}{\delta_\nu} \sup\{p(f_\nu(\zeta)): |\zeta| < \delta_\nu\},$$

and (6) yields

$$\sup\{p(f_\nu(\zeta)): |\zeta| < \delta_\nu\} > \frac{\delta_\nu}{\epsilon_\nu} = \epsilon_\nu^{-1/2}.$$

Since $\lim_{\nu \rightarrow \infty} \epsilon_\nu^{-1/2} = +\infty$, there is an index ν_0 and—for every $\nu > \nu_0$ —

some $|\zeta_\nu| \leq \delta_\nu$, such that $p(f_\nu(\zeta_\nu)) > r$. Therefore $k_D(x_0, f_\nu(\zeta_\nu)) > s$. But this is a contradiction, for

$$k_D(x_0, f_\nu(\zeta_\nu)) \leq \omega(0, \zeta_\nu) = \frac{1}{2} \log \frac{1 + |\zeta_\nu|}{1 - |\zeta_\nu|} \leq \frac{1}{2} \log \frac{1 + \delta_\nu}{1 - \delta_\nu} \rightarrow 0$$

as $\nu \rightarrow \infty$.

QED

If \mathcal{E} is locally bounded, then for any $v \in \mathcal{E} \setminus \{0\}$ the continuous semi-norm p can be chosen in such a way that $p(v) > 0$.

COROLLARY: *If D is a hyperbolic domain in a complex Banach space \mathcal{E} , for any $x_0 \in D$ there is a positive constant c such that*

$$\kappa_D(x_0; v) \geq c\|v\|$$

for all $v \in \mathcal{E}$.

The corollary improves Proposition V.1.9 of [3].

3. Let D be a domain in \mathcal{E} and let $f \in \text{Hol}(\Delta, D)$. For all $\zeta_0, \zeta \in \Delta$,

$$(7) \quad c_D f(\zeta_0), f(\zeta) \leq k_D(f(\zeta_0), f(\zeta)) \leq \omega(\zeta_0, \zeta).$$

If there is $\zeta_0 \in \Delta$ such that $c_D(f(\zeta_0), f(\zeta)) = \omega(\zeta_0, \zeta)$ ($k_D(f(\zeta_0), f(\zeta)) = \omega(\zeta_0, \zeta)$) for all $\zeta \in \Delta$, f is called a *complex geodesic* for c_D (respectively for k_D) at $f(\zeta_0)$. Inequality (7) yields the first part of the following lemma (the second part is trivial).

LEMMA 3.1: *If f is a complex geodesic for c_D , then f is a complex geodesic for k_D . If $k_D(c_D)$ is a distance and if f is a complex geodesic for $k_D(c_D)$ at $f(\zeta_0)$, then $f(\Delta)$ is closed in D for the k_D -topology (c_D -topology).*

REMARK: Let D be a non simply connected bounded domain in \mathbb{C} . Since any nonconstant $f \in \text{Hol}(\Delta, D)$ is an open map, the above lemma shows that there are no complex geodesics for either c_D or k_D .

PROPOSITION 3.2: *If $f \in \text{Hol}(\Delta, D)$ and $\zeta_0 \in \Delta$ are such that*

$$\gamma_D(f(\zeta_0); f'(\zeta_0)) = \langle 1 \rangle_{\zeta_0},$$

then f is a complex geodesic for c_D at $f(\zeta_0)$ in D .

PROOF: By composing f on the right with the Moebius transformation

$$\zeta \mapsto \frac{\zeta + \zeta_0}{1 + \overline{\zeta_0} \zeta},$$

there is no restriction in assuming $\zeta_0 = 0$, and therefore

$$\gamma_D(f(0); f'(0)) = 1.$$

By definition, there is a sequence $\{h_\nu\}$ ($h_\nu \in \text{Hol}(D, \Delta)$) such that $h_\nu(f(0)) = 0$, and

$$\lim_{\nu \rightarrow \infty} |dh_\nu(f(0))f'(0)| = 1.$$

In view of Montel's theorem, the sequence $\{h_\nu \circ f\}$ of holomorphic maps $h_\nu \circ f \in \text{Hol}(\Delta, \Delta)$ contains a subsequence $\{h_{\nu_j} \circ f\}$ normally convergent on compact sets of Δ to a map $g \in \text{Hol}(\Delta, \Delta)$. Since $g(0) = \lim_{j \rightarrow \infty} h_{\nu_j}(f(0)) = 0$, $|g'(0)| = \lim_{j \rightarrow \infty} |dh_{\nu_j}(f(0))f'(0)| = 1$, then, by the Schwarz lemma, g is a holomorphic automorphism of Δ . The sequence of inequalities

$$\omega(0, \zeta) \geq c_D(f(0), f(\zeta)) \geq \omega(h_{\nu_j}(f(0)), h_{\nu_j}(f(\zeta)))$$

yields, as j tends to infinity,

$$\omega(0, \zeta) \geq c_D(f(0), f(\zeta)) \geq \omega(g(0), g(\zeta)) = \omega(0, \zeta),$$

for all $\zeta \in \Delta$.

QED

A similar argument leads to the following

PROPOSITION 3.3: *If there are two distinct points ζ_0, ζ_1 in Δ , such that*

$$c_D(f(\zeta_0), f(\zeta_1)) = \omega(\zeta_0, \zeta_1),$$

then $f: \Delta \rightarrow D$ is a complex geodesic for c_D at $f(\zeta_0)$.

PROOF: By definition there is a sequence $h_\nu \in \text{Hol}(D, \Delta)$ such that

$$\lim_{\nu \rightarrow \infty} \omega(h_\nu(f(\zeta_0)), h_\nu(f(\zeta_1))) = \omega(\zeta_0, \zeta_1).$$

By Montel’s theorem, the sequence $\{h_n \circ f\}$ contains a subsequence $\{h_{n_j} \circ f\}$ normally convergent on compact sets of Δ to a map $g \in \text{Hol}(\Delta, \Delta)$ for which

$$\omega(g(\zeta_0), g(\zeta_1)) = \omega(\zeta_0, \zeta_1).$$

Hence, by the Schwarz–Pick lemma g is a holomorphic automorphism of Δ , and therefore

$$\begin{aligned} \omega(\zeta_0, \zeta) &\geq c_D(f(\zeta_0), f(\zeta)) \geq \lim_{j \rightarrow \infty} \omega(h_{n_j}(f(\zeta_0)), h_{n_j}(f(\zeta))) = \\ &= \omega(g(\zeta_0), g(\zeta)) = \omega(\zeta_0, \zeta) \end{aligned}$$

for all $\zeta \in \Delta$.

QED

COROLLARY: *If $f: \Delta \rightarrow D$ is a complex geodesic for c_D at some point $f(\zeta_0)$ ($\zeta_0 \in \Delta$), then f is a complex geodesic for c_D at any point $\zeta \in \Delta$.*

If $f \in \text{Hol}(\Delta, D)$ is a complex geodesic for k_D or c_D , and if φ is any Moebius transformation of Δ , then $f \circ \varphi$ is a complex geodesic for k_D or c_D respectively. To discuss the converse to this statement, let f and g be two injective holomorphic maps of Δ into D , having the same range: $f(\Delta) = g(\Delta)$. This fact, together with injectivity, sets up a bi-jjective map φ of Δ onto Δ , defined by $g = f \circ \varphi$. To prove that φ is holomorphic, let λ be a continuous linear form on \mathcal{E} such that $\lambda \circ f$ is not constant. The maps $\zeta \mapsto \lambda(f(\zeta))$ and $\zeta \mapsto \lambda(g(\zeta))$ ($\zeta \in \Delta$) are holomorphic. If $(\lambda \cdot f)'(\tau_0) = \lambda(f'(\tau_0)) \neq 0$ at some point $\tau_0 \in \Delta$, then, by the inverse function theorem, φ is holomorphic in a neighborhood of $\varphi^{-1}(\tau_0)$. The set of points $\{\tau \in \Delta: (\lambda \circ f)'(\tau) = 0\}$ is discrete in Δ . Since $|\varphi|$ is bounded in Δ , the Riemann extension theorem implies that φ is holomorphic on Δ , and—being bijective—is therefore a holomorphic automorphism of Δ . The above argument yields

PROPOSITION 3.4: *Let f and g be two complex geodesics for k_D (or for c_D). Then f and g have the same range if, and only if, there is a Moebius transformation φ of Δ such that $g = f \circ \varphi$.*

Let p be a continuous semi-norm on \mathcal{E} , and let $B_p = B_p(0, 1)$ be the open unit ball for p . A direct application of (2), of the Hahn–Banach theorem and of Proposition 1.3 yields the following lemma [13], [14].

LEMMA 3.5: *For any $x \in B_p$ for which $p(x) > 0$ the map $f \in \text{Hol}(\Delta, B)$ defined by $f(\zeta) = \frac{\zeta}{p(x)} x$ is a complex geodesic for c_{B_p} . If*

$f(\bar{\Delta}) \cap \partial B_p$ contains a complex extreme point of \bar{B}_p , then f is (up to a change of parameter expressed by a Moebius transformation in Δ) the unique complex geodesic for c_{B_p} (unique also for k_{B_p}) whose range contains 0 and x .

EXAMPLE: Let B be the open unit ball of a complex Hilbert space \mathcal{H} . Every boundary point of B is a real (hence complex) extreme point of \bar{B} . Lemma 3.5 implies that for any $x \in B \setminus \{0\}$ the map $\zeta \mapsto \frac{\zeta}{\|x\|} x$ of Δ into B is the unique complex geodesic for c_B whose range contains 0 and x . The group $\text{Aut}(B)$ of all holomorphic automorphisms of B acts transitively on B (cf. e.g. [3], Proposition VI.1.5, pp. 148–149). Hence, for any $y \in B$, there is $F \in \text{Aut}(B)$ such that $F(y) = 0$. Thus, given $x \in B \setminus \{y\}$, the map

$$(8) \quad \zeta \mapsto F^{-1} \left(\frac{\zeta}{\|F(x)\|} F(x) \right)$$

of Δ into B is the unique complex geodesic for c_B whose range contains x and y . By Theorem VI.1.7 (p. 150) of [3], the range of the map (8) is the intersection of B with a complex affine line of \mathcal{H} . To describe F , let T be the continuous linear operator of \mathcal{H} defined by

$$T(z) = \frac{(z, y)}{1 + \alpha(y)} y + \alpha(y)z \quad (z \in \mathcal{H}),$$

where (\cdot, \cdot) is the scalar product of \mathcal{H} and $\alpha(y) = \sqrt{1 - \|y\|^2}$.

The map F is defined by

$$F(z) = \frac{1}{1 - (z, y)} (T(z) - y) \quad (z \in B)$$

and

$$F^{-1}(z) = \frac{1}{1 + (z, y)} (T(z) + y) \quad (z \in B).$$

Hence the complex geodesic (8) is given by

$$\zeta \mapsto \frac{1}{\|F(x)\| + \zeta(F(x), y)} (\zeta T(F(x)) + \|F(x)\|y).$$

Since

$$(T(x), y) = (x, y),$$

then

$$(F(x), y) = \frac{(x, y) - \|y\|^2}{1 - (x, y)},$$

and

$$T(F(x)) = \frac{(F(x), y)}{1 + \alpha(y)} y + \alpha(y)F(x) = \frac{1 - \|y\|^2}{1 - (x, y)} x - y.$$

Finally, the norm $\|F(x)\|$ is given by ([3], p. 156):

$$\|F(x)\|^2 = \frac{1}{|1 - (x, y)|^2} (|1 - (x, y)|^2 - (1 - \|x\|^2)(1 - \|y\|^2)).$$

PROPOSITION 3.5: *Let $x_0 \in D$, $v_0 \in \mathcal{E} \setminus \{0\}$ and $h \in \text{Hol}(D, \Delta)$ be such that*

$$(9) \quad \langle dh(x_0)v_0 \rangle_{h(x_0)} = \kappa_D(x_0; v_0).$$

If the domain D is bounded, convex and if \bar{D} is complete and compact for the weak topology, then there is a complex geodesic $f \in \text{Hol}(\Delta, D)$ for c_D in D such that $f(0) = x_0$ and $f'(0)$ is collinear to v_0 .

PROOF: Let p be a continuous semi-norm on \mathcal{E} such that $B_p(x_0, r) \subset D$ for some $r > 0$, and such that $p(v_0) > 0$. The domain D , being bounded, is hyperbolic. Thus, by Theorem 2, $\kappa_D(x_0; v_0) > 0$. By definition,

$$\kappa_D(x_0; v_0) = p(v_0) / \sup\{p(g'(0)) : g \in \text{Hol}(\Delta, D), g(0) = x_0, g'(0)$$

collinear to $v_0\}$.

Let $\{g_\nu\}$ be a sequence of functions $g_\nu \in \text{Hol}(\Delta, D)$ such that: $g_\nu(0) = x_0$, $g'_\nu(0) \neq 0$ is collinear to v_0 , and

$$(10) \quad \lim_{\nu \rightarrow \infty} p(g'_\nu(0)) = \frac{p(v_0)}{\kappa_D(x_0; v_0)} \neq 0.$$

Since D is bounded and the ranges of the functions g_ν lie in a complete weakly compact subset of \mathcal{E} , then there is a subsequence $\{g_{\nu_j}\}$ weakly convergent on compact subsets of Δ to a function $f \in \text{Hol}(\Delta, D)^{(1)}$. The range of f belongs to the weak closure of D , which coincides with the closure \bar{D} . Thus $f(\Delta) \subset \bar{D}$. Suppose now that $f(\Delta) \cap \partial D \neq \emptyset$. Denoting by m the support function of D , lemma 1.1 implies that $m(f(\zeta)) = 1$ for all $\zeta \in \Delta$. On the other hand, for every continuous linear form λ on \mathcal{E} ,

$$\lambda(f(0)) = \lim_{j \rightarrow \infty} \lambda(g_{\nu_j}(0)) = \lambda(x_0).$$

Therefore $f(0) = x_0$ and by consequence $m(f(0)) < 1$. This contradiction proves that $f(\Delta) \subset D$.

If $f'(0)$ is not collinear to v_0 , there is a continuous linear form λ on \mathcal{E} such that

$$\lambda(f'(0)) = 0 \quad \lambda(v_0) \neq 0.$$

Since $g'_\nu(0) = c_\nu v_0$ with $c_\nu \in \mathbb{C} \setminus \{0\}$, then

$$(11) \quad \lambda(f'(0)) = \lim_{j \rightarrow \infty} \lambda(g'_{\nu_j}(0)) = \lambda(v_0) \lim_{j \rightarrow \infty} c_{\nu_j}.$$

Thus $\lim_{j \rightarrow \infty} c_{\nu_j} = 0$, contradicting (10). Then there exists $c \in \mathbb{C} \setminus \{0\}$ such that $f'(0) = cv_0$.

The same computation (11), for every continuous linear form λ on \mathcal{E} , yields

$$c = \lim_{j \rightarrow \infty} c_{\nu_j},$$

and therefore, by (10),

$$p(f'(0)) = |c|p(v_0) = \lim_{j \rightarrow \infty} p(c_{\nu_j}v_0) = \lim_{j \rightarrow \infty} p(g'_{\nu_j}(0)) = \frac{p(v_0)}{\kappa_D(x_0; v_0)},$$

¹ That is theorem 3.14.2 (pp. 105–106) of [7] in the case where \mathcal{E} is a complex Banach space. The proof of the theorem as given in [7] carries over with no change to the case of a bounded, convex domain D in a locally convex complex space \mathcal{E} , such that \bar{D} is complete. This latter condition (which is automatically satisfied in the case of Banach spaces) ensures the applicability—as in the original proof given in [7]—of the Eberlein–Smulian theorem [9, pp. 187–188].

i.e.

$$|c| \kappa_D(x_0; v_0) = 1.$$

Since $h \circ f \in \text{Hol}(\Delta, \Delta)$, and

$$h \circ f(0) = h(x_0),$$

$$\langle (h \circ f)'(0) \rangle_{h(x_0)} = \langle dh(x_0)f'(0) \rangle_{h(x_0)} = |c| \langle dh(x_0)v_0 \rangle_{h(x_0)} = |c| \kappa_D(x_0; v_0) = 1,$$

by the Schwarz–Pick lemma $h \circ f$ is a holomorphic automorphism of Δ . Thus

$$1 = \kappa_D(x_0; f'(0)) \geq \gamma_D(x_0; f'(0)) \geq \langle dh(x_0)f'(0) \rangle_{h(x_0)} = 1,$$

and therefore $\gamma_D(x_0; f'(0)) = 1$. Proposition 3.2 yields the conclusion.

QED

COROLLARY: *Let \mathcal{E} be a reflexive Banach space, and let D be a convex bounded domain in \mathcal{E} . If $x_0 \in D$, $v_0 \in \mathcal{E} \setminus \{0\}$, and $h \in \text{Hol}(D, \Delta)$ satisfy (9), then there exists a complex geodesic f for c_D such that $f(0) = x_0$ and $f'(0)$ is collinear to v_0 .*

4. LEMMA 4.1: *Let \mathcal{E}_1 and \mathcal{E}_2 be normed spaces over \mathbb{C} . Let B_1 and B_2 be the open unit balls for \mathcal{E}_1 and \mathcal{E}_2 , and assume that every boundary point of B_2 is a complex extreme point of \bar{B}_2 . If $F \in \text{Hol}(B_1, B_2)$ is such that*

$$(12) \quad \|F(x)\| = \|x\| \quad \text{for all } x \in B_1,$$

then F is a linear $\| \cdot \|$ -isometry.

PROOF: Condition (12) is equivalent to

$$c_{B_2}(0, F(x)) = c_{B_1}(0, x) \quad \text{for all } x \in B_1.$$

For $u \in \mathcal{E}_1$, $\|u\| = 1$, the map $\zeta \mapsto \zeta u$ is a complex geodesic for c_{B_1} . Thus by (2) and (12) $\zeta \mapsto F(\zeta u)$ is a complex geodesic for c_{B_2} at 0. By lemma 3.4. $\zeta \mapsto F(\zeta u)$ is linear, i.e.

$$F(\zeta u) = \zeta dF(0)u.$$

Let

$$F(x) = dF(0)x + P_2(x) + P_3(x) + \dots$$

be the power series expansion of F around 0, where P_2, P_3, \dots are continuous homogeneous polynomials of degrees 2, 3, \dots from \mathcal{E}_1 to \mathcal{E}_2 . Then $P_\nu(u) = 0$ for all $u \in \mathcal{E}_1$ and all $\nu = 2, 3, \dots$. Therefore

$$F(x) = dF(0)x \quad \text{for all } x.$$

and the conclusion follows.

QED

The hypothesis concerning the complex extreme points cannot be dropped as is shown by the map $\zeta \mapsto (\zeta, \zeta^2)$ of Δ into the bi-disc $\Delta \times \Delta$. However, according to a result of L.A. Harris [5][3], that hypothesis can be avoided if $dF(0)$ is assumed to be a linear isometry of \mathcal{E}_1 onto \mathcal{E}_2 .

The above lemma could be compared with the theorem of Mazur–Ulam [2, pp. 166–168]. Does a statement similar to Lemma 4.1 hold for real analytic mappings?

The following result is a direct consequence of Lemma 4.1.

THEOREM 3: *Let \mathcal{E}_1 and \mathcal{E}_2 be two locally convex, locally bounded, complex vector spaces. Let D_1 and D_2 be two bounded, convex, balanced open neighborhoods of 0 in \mathcal{E}_1 and \mathcal{E}_2 , and let $F \in \text{Hol}(D_1, D_2)$ be such that $F(0) = 0$ and that either*

$$\gamma_{D_2}(0; dF(0)u) = \gamma_{D_1}(0; u) \quad \text{for every } u \in \mathcal{E}_1,$$

or any one of the following conditions

$$k_{D_2}(0, F(x)) = k_{D_1}(0, x),$$

$$c_{D_2}(0, F(x)) = c_{D_1}(0, x)$$

holds for all $x \in D_1$. If every point of ∂D_2 is a complex extreme point of \bar{D}_2 , then F is (the restriction to D_1 of) a linear map of \mathcal{E}_1 into \mathcal{E}_2 :

$$F(x) = dF(0)x \quad \text{for all } x \in D_1.$$

COROLLARY: *Under the same hypotheses of Theorem 3 for D_1 and D_2 , let $F \in \text{Hol}(D_1, D_2)$ be such that any one of the following conditions holds:*

$$\gamma_{D_2}(F(0); dF(0)u) = \gamma_{D_1}(0; u) \quad \text{for all } u \in \mathcal{E}_1,$$

or

(13)
$$c_{D_2}(F(0), F(x)) = c_{D_1}(0, x) \quad \text{for all } x \in D_1.$$

If the semi-group S_c of all holomorphic c_{D_2} -isometries is transitive on D_2 , then for any $g \in S_c$ such that $g(F(0)) = 0$, $g \circ F$ is linear and $g(F(D_1))$ is the intersection of D_2 with a linear subspace of \mathcal{E}_2 .

A similar conclusion holds if condition (13) is replaced by

$$k_{D_2}(F(0), F(x)) = k_{D_1}(0, x) \quad \text{for all } x \in D_1,$$

and if the semi-group S_k of all holomorphic k_{D_2} -isometries is transitive on D_2 .

The above conditions concerning S_c and S_k are fulfilled if the group $\text{Aut}(D_2)$ of all bi-holomorphic automorphisms of D_2 acts transitively on D_2 . That is the case if, e.g., D_2 is the open unit ball B of a complex Hilbert space \mathcal{H} . Since every boundary point of B is a real (hence complex) extreme point of \bar{B} , all the hypotheses of the above corollary are satisfied. Furthermore, for any linear subspace \mathcal{P} of \mathcal{H} and any $g \in \text{Aut}(B)$ there is an affine subvariety \mathcal{L} of \mathcal{H} such that $g(\mathcal{P} \cap B) = \mathcal{L} \cap B$. Moreover, for all $y_1, y_2 \in \mathcal{L} \cap B$, $c_{B \cap \mathcal{L}}(y_1, y_2) = k_{B \cap \mathcal{L}}(y_1, y_2) = c_B(y_1, y_2) = k_B(y_1, y_2)$ [3].

COROLLARY: Let D_1 be a bounded, convex, balanced open neighborhood of 0 in \mathcal{E}_1 , and let $F \in \text{Hol}(D_1, B)$ be such that any one of the following conditions

$$(14) \quad \gamma_B(F(0); dF(0)x) = \gamma_{D_1}(0; x),$$

$$(15) \quad c_B(F(0), F(x)) = c_{D_1}(0, x),$$

$$(16) \quad k_B(F(0), F(x)) = k_{D_1}(0, x),$$

holds for all $x \in D_1$. Then there is an affine sub-variety \mathcal{L} of \mathcal{H} such that $F(D_1) = \mathcal{L} \cap B$ and,

$$c_{B \cap \mathcal{L}}(F(0), F(x)) = c_{D_1}(0, x) \quad (x \in D_1)$$

if either (14) or (15) holds, or

$$k_{B \cap \mathcal{L}}(F(0), F(x)) = k_{D_1}(0, x) \quad (x \in D_1)$$

if (16) holds. Moreover for any $g \in \text{Aut}(B)$ such that $g \circ F(0) = 0$, $g \circ F$ is a continuous linear map.

5. The following result, concerning the non-homogeneous case, im-

proves previous statements of [13]. Let Ξ be a σ -algebra on a set M and let μ be a positive measure on Ξ . Let B be the open unit ball in the complex Banach space $L^1(M, \Xi, \mu)$ and let D be a convex hyperbolic domain in $L^1(M, \Xi, \mu)$.

PROPOSITION 5.1: *Let $F \in \text{Hol}(B, D)$ be such that $dF(0)$ has a continuous inverse and moreover*

$$(17) \quad \gamma_D(F(0); dF(0)u) = \gamma_B(0; u) \text{ for all } u \in L^1(M, \Xi, \mu).$$

If $\dim_c L^1(M, \Xi, \mu) > 1$, and if, for every $s > 0$, there is $0 < r < s$ such that the open set $D_r = \{y \in D: c_D(F(0), y) < r\}$ is bounded and convex, then F is an affine map

$$F(x) = F(0) + dF(0)x \quad (x \in B)$$

of B onto D .

PROOF: There is no restriction in assuming $F(0) = 0$. By the inverse mapping theorem (cf. e.g. [3]) there is an open neighborhood U of 0 in B such that $F(U)$ is an open neighborhood of 0 in D and the restriction $F|_U$ is a holomorphic diffeomorphism of U onto $F(U)$. The domain D being hyperbolic, in view of the hypothesis there is some $r > 0$ such that the set $D_r = \{y \in D: c_D(0, y) < r\}$ is bounded, convex and contained in $F(U)$. For any $y \in D_r \setminus \{0\}$, let $x \in U$ be the unique point of U such that $F(x) = y$. The map $\zeta \mapsto \frac{\zeta}{\|x\|} x$ ($\zeta \in \Delta$) is a complex geodesic at 0 for c_B whose range contains x ²). By Proposition 3.2 $\zeta \mapsto F\left(\frac{\zeta}{\|x\|} x\right)$ is a complex geodesic at 0 for c_D , whose range contains $y = F(x)$. Hence

$$c_D(0, y) = c_D(0, F(x)) = \omega(0, \|x\|) = c_B(0, x),$$

and therefore $x \in B' := \{x \in L^1(M, \Xi, \mu): c_B(0, x) < r\} =$

$$= \left\{ x \in L^1(M, \Xi, \mu): \|x\| < \frac{e^{2r} - 1}{e^{2r} + 1} \right\}.$$

²Since every boundary point of B is a complex extreme point of \bar{B} [11], that map is actually the *unique* complex geodesic at 0 for c_B whose range contains 0 (Lemma 4.3 of [13]).

Since F contracts the Carathéodory distances, then

$$F(B') = D_r,$$

and $F|_{B'}$ is a holomorphic diffeomorphism of the ball B' onto the convex domain D_r . By a theorem of T.J. Suffridge [10, Theorem 7] $F|_{B'}$ is linear. Hence F itself is linear.

For any $u \in L^1(M, \Xi, \mu)$ with $\|u\| = 1$, the map $\zeta \mapsto F(\zeta u) = \zeta dF(0)u$ is a complex geodesic for c_D at 0 in the convex open set D . By Lemma 3.1, its range is closed in D . Since $dF(0)$ has a continuous inverse, that implies that $F(B) = D$. QED

COROLLARY: *Let $F \in \text{Hol}(B, D)$ be such that $dF(0)$ has a continuous inverse and that (17) holds. If $\dim_c L^1(M, \Xi, \mu) > 1$, and if D is an open, convex, balanced, hyperbolic neighborhood of $F(0)$, then F is an affine map of B onto D .*

The following result, which is a consequence of the above corollary, improves Theorem II of [13].

THEOREM 4: *Let $F \in \text{Hol}(B, B)$ be such that $dF(0)$ has a continuous inverse and that*

$$\gamma_B(F(0); dF(0)x) = \gamma_B(0, x) \quad \text{for all } x \in B.$$

If $\dim_c L^1(M, \Xi, \mu) > 1$, F is the restriction to B of a linear isometry of $L^1(M, \Xi, \mu)$ onto itself.

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