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REAL ANALYTIC APPROXIMATION OF LOCALLY EMBEDDABLE CR MANIFOLDS

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dedicated to: Aldo Andreotti

0. Introduction

One of the most useful tools in analysis is the approximation of smooth functions by real analytic ones. In this work we prove that C^s, $s \geq 2$, non-generic CR manifolds of type (m, ℓ) embedded in $\mathbf{C}^N$, can be locally approximated by real analytic non-generic CR manifolds of the same type, see Theorem 1, Section 2. The authors have an application for this theorem, namely to extend the results of finding families of analytic discs with boundaries on CR manifolds and building a manifold of one higher dimension, see [4] and [5], to non-generic manifolds and thereby to extend CR functions in a very natural way (see [8] and [6]).

Theorem 2, see Section 2, states that any C^s, CR function on a C^s CR manifold can be approximated by holomorphic polynomials restricted to the manifold. The proof of Theorem 2 is a corollary of the proof of Theorem 1. In [7] and [9], statement of special cases of our results have been given, considering CR-manifolds restricted to have a certain Levi convexity. However, in [9], the proof uses an induction argument that makes use of certain forms being $\bar{\partial}$ closed (see [9], p. 352) in the last step of the induction. When the CR codimension is ≥ 3, this is false, and the proofs given in [9] break down. This error recurs in the work of [7], and it does not seem that a correction has been written down. Our argument is completely different.

The proofs of both the theorems are found in Section 3. They rely

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heavily on a theorem of Baouendi and Treves [2], which we modify for our Proposition 2. We are indebted to both Prof. Baouendi and Prof. Treves for their discussion of their work.

1. Notation and preliminary concepts

The formal definition of an *abstract* CR structure on a manifold will be deferred until it is needed in Section 2. Here we shall be concerned with what we now prefer to call a *concrete* CR structure on a manifold M – it means an abstract CR structure on M which has the additional property of being *locally embeddable* at each point of M; see Section 2. Actually in what follows, we shall be considering such a concrete CR structure on some portion M of a manifold, such that M has an embedding in some complex number space.

To be precise: let M be a real d dimensional differentiable manifold which is embedded as a locally closed real differentiable submanifold of $\mathbb{C}^N \cong \mathbb{R}^{2N}$. Let $T_p(M)$ denote the *real tangent space* to M at a point $p \in M$; the *complex part of tangent space* to M at p is defined by

$$\mathcal{H}T_p(M) = T_p(M) \cap JT_p(M). \tag{1.1}$$

Here J is the operator of multiplication by $\sqrt{-1}$ which defines the complex structure of $\mathbb{C}^N$. Thus, $\mathcal{H}T_p(M)$ is the largest complex linear subspace of $T_p(\mathbb{C}^N)$ that is contained in $T_p(M)$. We shall make the requirement

$$\dim_{\mathbb{C}} \mathcal{H}T_p(M) = m(p) \equiv m, \quad \forall p \in M, \tag{1.2}$$

i.e., that this space has *constant dimension*, and set

$$\ell = d - 2m. \tag{1.3}$$

We will refer to the situation just described by saying that M is a *locally embedded* CR manifold of *type* (m, ℓ). Often m will be called the CR *dimension*, and ℓ will be called the CR *codimension* of M.

In what follows, we set

$$n \equiv m + \ell = d - m$$
$$k \equiv N - n = N - (d - m)$$
$$q \equiv 2N - d = \ell + 2k.$$

Then, the real codimension of M in $\mathbb{C}^N$ is equal to q; thus $k \geq 0$ automatically, and $k = 0$ if and only if the dimension of the embedding space is the minimum possible. When $k = 0$, the embedding is called *generic*.

M can be locally defined by a system of (real) equations

$$\rho_i(z) = 0 \quad (1 \leq i \leq q)$$

where $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and $d\rho_1 \wedge d\rho_2 \wedge \cdots \wedge d\rho_q \neq 0$ on M. The space of *holomorphic tangent vectors* at $p \in M$ is characterized by

$$\text{(1.4)} \quad HT_p(M) = \left\{ X \equiv \sum_{j=1}^{N} A_j \frac{\partial}{\partial z_j} \,\middle|\, \sum_{j=1}^{N} A_j \frac{\partial \rho_i}{\partial z_j}(p) = 0 \quad \text{for } 1 \leq i \leq q \right\}$$

where the A_j are complex numbers. The map $I - iJ$ takes $\mathcal{H}T_p(M)$ isomorphically onto $HT_p(M)$. It intertwines the operator J on $\mathcal{H}T_p(M)$ with $\sqrt{-1}HT_p(M)$. The *antiholomorphic* tangent space, $\overline{HT}_p(M)$, is defined by complex conjugation: let

$$X_\alpha(p) \equiv \sum_{j=1}^{N} A_\alpha^j(p) \frac{\partial}{\partial z_j} \quad (1 \leq \alpha \leq m)$$

be a basis for $HT_p(M)$, then

$$\bar{X}_\alpha(p) \equiv \sum_{j=1}^{N} \bar{A}_\alpha^j(p) \frac{\partial}{\partial \bar{z}_j} \quad (1 \leq \alpha \leq m)$$

is a basis for $\overline{HT}_p(M)$. The type of M at p is determined by

$$\text{(1.5)} \qquad m(p) = N - \operatorname{rank}\left[\frac{\partial \rho}{\partial z}(p)\right],$$

and (1.3), where $[\partial\rho/\rho z]$ denotes the matrix $[\partial\rho_i/\partial z_j]$, with $i = 1, \ldots, q$ and $j = 1, \ldots, N$. It follows that genericity is an open condition, and that, in general, $m(p)$ is an upper-semicontinuous function of p. Assumption (1.2) amounts to a constant rank assumption; i.e., that the number of linearly independent $(0, 1)$ forms among $\bar{\partial}\rho_1, \ldots, \bar{\partial}\rho_q$ is constant on M.

Associated to M is the system of homogeneous *tangential Cauchy Riemann* equations, which we write as

$$\text{(1.6)} \qquad \bar{\partial}_M f = 0,$$

where f is a complex valued function defined on M. Since the dimension of the space of holomorphic tangent vectors is constant on M, one can select mC^∞ vector fields X_α in a neighborhood U of any point $p \in M$ giving a basis for $HT_{p'}(M)$, $p' \in U$. The function f is called a CR function at $p \in M$ if

$$\overline{X_\alpha} f(p') = 0$$

for all p' in some neighborhood of p. Solutions f of (1.6) are equivalent to functions which are CR at every point $p \in M$. Such functions are called CR functions on M and are denoted by $f \in CR(M)$.

We call a transformation $F: X \to Y$, $X \subset C^r$, $Y \subset C^s$, between two CR manifolds, a CR transformation or CR map at $p \in X$ if

$$dF_p : T_p(X) \to T_{f(p)}(Y) \tag{1.7}$$

is such that

$$dF_{p'} : HT_{p'}(X) \to HT_{f(p')}(Y) \tag{1.8}$$

for p' in a neighborhood of p in X. Note that dF_p, in equation (1.8) is actually the complexified differential. Equation (1.8) is equivalent to

$$dF_{p'} : \mathscr{H}T_{p'}(X) \to \mathscr{H}T_{f(p')}(Y) \tag{1.9}$$

with the extra condition that $dF_{p'}$ be complex linear, i.e., $dF_{p'}$ commutes with J. We say F is a CR map on X if it is a CR map at each point $p \in X$. If F has components $F_1, \ldots, F_s$ then F is a CR map if and only if each component F_j, is a CR function on X. In fact if $Z \in HT_p(X)$ with

$$dF_p(Z) = \sum_{j=1}^{s} (ZF_j) \frac{\partial}{\partial F_j} + \sum_{j=1}^{s} Z\bar{F}_j \frac{\partial}{\partial \bar{F}_j}$$

one has that $dF_p(Z) \in HT_{F(p)}(Y)$ if and only if $Z\bar{F}_j = 0 \, \forall_j, \forall Z$, that is, if and only if $\bar{Z}F_j = 0 \, \forall_j, \forall Z$, i.e., if and only if $F_j \in CR(X)$.

2. Set up and statement of the main theorem

Let M' be a C^s, $s \geq 1$, CR manifold of type (m, ℓ) embedded in $\mathbb{C}^N$. Suppose M' is not generic at $p \in M'$ (and hence, since M' is locally

CR, M' is not generic at any point p' in a neighborhood of p). Without loss of generality, we can assume that $p = 0$,

$$T_0(M') = \{y_1 = \cdots = y_\ell = 0, z_{n+1} = \cdots = z_N = 0\}$$

and

$$\mathscr{H}T_0(M') = \{z_1 = \cdots = z_\ell = 0, z_{n+1} = \cdots = z_N = 0\}$$

where $z_j = x_j + iy_j$ $(j = 1, 2, \ldots, N)$ are holomorphic coordinates in a neighborhood of 0 in $\mathbb{C}^N$. We can also represent M' locally as the graph over its tangent space $T_0(M')$. Thus, since all our work is local, we can restrict M' sufficiently so that, with U' a neighborhood of the origin in $\mathbb{C}^N$, we have

$$(2.1)\quad M' = \begin{array}{r} (z_1, \ldots, z_N) \in U' : y_\eta = h_\eta(x_1, \ldots, x_\ell, z_{\ell+1}, \ldots, z_n), \\ \eta = 1, \ldots, \ell \\ x_{n+\nu} = h'_\nu(x_1, \ldots, x_\ell, z_{\ell+1}, \ldots, z_n), \\ y_{n+\nu} = h''_\nu(x_1, \ldots, x_\ell, z_{\ell+1}, \ldots, z_n), \\ \nu = 1, \ldots, k \end{array}$$

where $\eta_\eta, h'_\nu, h''_\nu$ are real valued functions defined in a neighborhood Ω' of 0 in $\mathbb{R}^\ell \times \mathbb{C}^m \cong T_0(M')$, which are as smooth as M', and which vanish at least to second order at 0. To simplify writing we shall use the symbol H to denote the triplet of functions (h, h', η'') which defines M'.

What we shall prove in this work is the following two theorems.

THEOREM 1: *Given an embedded CR manifold $M' \subset \mathbb{C}^N$ of type (m, ℓ) and class C^s, $2 \le s \le \infty$, M' can be approximated by a sequence of real analytic (in fact polynomial in $(x, z, \bar{z})$) CR manifolds $M'_j \subset \mathbb{C}^N$, all of type (m, ℓ), in the sense that if $H : \Omega' \to \mathbb{R}^\ell \times \mathbb{C}^n$ defines M' as in* (2.1) *there exist $\Omega^0 \subset \Omega'$ and $H_j : \Omega^0 \to R^\ell \times C^k$, each defining M'_j, as in* (2.1) *such that $H_j \to H$ in $C^{s-2}(\Omega^0)$.*

As a corollary to the proof we have, setting $M' = V \cap M'$ with V open in $\mathbb{C}^N$.

THEOREM 2: *Let M' be as above, an embedded CR manifold of type (m, ℓ) and class C^s. Then there exists an open neighborhood U of the origin in $\mathbb{C}^N$, with $U \subset V$, such that: If $f \in CR(M')$ is of class C^t, $2 \le t \le s \le \infty$, then there exists a sequence of polynomials p^j such that $p^j \to f$ in $C^{t-1}(U \cap M')$.*

REMARK: If M' is real analytic, then in particular we have that there exists a sequence of real analytic CR functions defined on M which converge to f in the C^{t-1} norm.

Before beginning the proof of the theorems, we first want to discuss the connection between a non-generic manifold M' and its associated generic manifold M. In general, let $H = (h, h', h'')$ be any triplet of functions defined and at least of class C^1 on Ω', a neighborhood of the origin in $\mathbb{R}^\ell \times \mathbb{C}^m$. Assume, moreover, that

$$h : \Omega' \to \mathbb{R}^\ell$$
$$h' : \Omega' \to \mathbb{R}^k$$
$$h'' : \Omega' \to \mathbb{R}^k$$

all vanish to 2nd order at the origin. We can choose $U' \subset \mathbb{C}^N = \mathbb{C}^\ell \times \mathbb{C}^m \times \mathbb{C}^k$ a neighborhood of the origin such that its projection onto $R^\ell \times \mathbb{C}^m$ is contained in Ω'. Defining $M' \subset \mathbb{C}^N$ by (2.1) we have that $\dim T_p(M') = \ell + 2m$ but we know nothing about the type of M'. Let $\pi : \mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$ be the projection onto the first n components and let $M = \pi(M')$. Then M is a manifold of the same smoothness as M'. Restricting Ω' so that $\pi|_{M'}$ and $d(\pi|_{M'})$ are both injective, we easily see that M is generic at 0 and of type (m, ℓ). Since genericity is an open condition we can shrink U' sufficiently so that for $U = \pi(U')$ and

$$M = \{(z_1, \ldots, z_n) \in U : y_\eta = h_\eta(x_1, \ldots, x_\ell, z_{\ell+1}, \ldots, z_n),\ \eta = 1, \ldots, \ell\}, \tag{2.2}$$

with the h_η's as above, M is an embedded generic manifold of type (m, ℓ) in $\mathbb{C}^n$. Defining $\phi \equiv (\pi|_{M'})^{-1} : M \to M' \subset \mathbb{C}^n$ we have

$$\phi_\eta(z_1, \ldots, z_n) = \begin{cases} z_\eta & \eta \leq n \\ (h'_{\eta-n} + ih''_{\eta-n})(x_1, \ldots, x_\ell, z_{\ell+1}, \ldots, z_n) & n+1 \leq \eta \leq N. \end{cases} \tag{2.3}$$

Hence ϕ has the same smoothness as H and we can locally define M' as the graph of the last k components of ϕ over M.

We shall now give one of the many equivalent definitions of an abstract C^s, CR structure with CR dimension K, see [3]. It can be defined by a system of K linearly independent C^s complex vector fields $L_1, \ldots, L_K$, defined in an open neighborhood of the origin, Ω, in $\mathbb{R}^{I+K}$. where $I \geq K$, which satisfies the following two conditions:

(1) the vectors $L_1, \ldots, L_K, \bar{L}_1, \ldots, \bar{L}_K$ are all linearly independent at each point of Ω.

(2) $\{L_1, \ldots, L_K\}$ is closed under Lie bracket, i.e., $[L_i, L_j] = \Sigma_{\nu=1}^{K} c_\nu L_\nu$, $1 \le i, j \le K$.

The $\{\bar{L}_1, \ldots, \bar{L}_K\}$ generates a real subspace of $\mathbb{C}T_p(\Omega)$, the complexified tangent space to Ω at p, which we denote by $HT_0(\Omega)$. A CR embedding of Ω into $\mathbb{C}^I$ is a diffeomorphism

$$\zeta : \Omega \to M \subset \mathbb{C}^I \tag{2.4}$$

whose differential $d\zeta_p$ maps $HT_p(\Omega)$ to $HT_{\zeta(p)}(M)$ isomorphically, where M is an embedded CR $K+I$ dimensional manifold as discussed in Section 1. There is the following well known characterization of embeddability, see [1] Sections 2 and 3.

LEMMA 2.1: *A CR embedding ζ exists if and only if there exist I functionally independent C^s complex value characteristic coordinates $\varkappa_1, \ldots, \varkappa_I$ at each point of Ω, i.e., $L_j \varkappa_k = 0$ for $j = 1, \ldots, k$, $k = 1, \ldots, I$ with $d\varkappa_1, \ldots, d\varkappa_I$ all linearly independent in Ω.*

The proof of the lemma is as follows. Let $\zeta : \Omega \to \mathbb{C}^I$ have components $\varkappa_1, \ldots, \varkappa_I$. For any $p \in \Omega$

$$d\zeta_p(\bar{L}_k) = \sum_{j=1}^{I} \bar{L}_k \varkappa_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{I} \bar{L}_k \bar{\varkappa}_j \frac{\partial}{\partial \bar{z}_j}.$$

Thus, ζ is a CR embedding of Ω into $M = \zeta(\Omega)$ if and only if $L_k \varkappa_j = 0$, $1 \le k \le K$, $1 \le j \le I$, $\Sigma_{j=1}^{I} \bar{L}_k \varkappa_j(\partial/\partial z_j)$, $1 \le k \le K$ spans a K dimensional space space, and $\dim_{\mathbb{R}} M = k + I$, i.e., if and only if $\varkappa_1, \ldots, \varkappa_I$ are a complete set of functionally independent characteristic coordinates.

Thus, see [1], a generic embedded CR manifold, M, of class C^s, $s \ge 1$, can also be characterized by a system of m linearly independent C^s complex vector fields $L_1, \ldots, L_m$ in an open neighborhood of the origin, Ω, of $R^d = \mathbb{R}^{m+n} = \mathbb{R}^\ell \times \mathbb{C}^m$ which satisfy the following conditions:

(2.5)

(1) There exist n functionally independent C^s characteristic coordinates $z_1, \ldots, z_n$ at each point of Ω.
(2) The vectors $L_1, \ldots, L_m, \bar{L}_1, \ldots, \bar{L}_m$ are all linearly independent at each point of Ω.
(3) $\{L_1, \ldots, L_m\}$ is closed under Lie bracket, i.e., $[L_j, L_k] = \Sigma_{\nu=1}^{m} c_\nu L_\nu$, $1 \le j, k \le m$.
(4) $M = \{\zeta(x) \equiv (z_1(x), \ldots, z_n(x)) : x \in \Omega\}$ and without loss of generality we can assume $z(0) = 0$.

Moreover when M is defined as above, one has that $f \in CR(M)$ if and only if $L_j(f \circ \zeta) = 0$, $1 \leq j \leq m$.

Proofs of the Theorems: Throughout this theorem we shall rely heavily on the following two propositions. Their proofs, being technical, have been put in the next section.

Proposition 1: *Let M_1 be a CR manifold $M_1 \subset \mathbb{C}^{N_1}$ of type (m, ℓ) such that M_1 can be written as a graph over its tangent space at some point in M_1. Let $\Psi : M_1 \to \mathbb{C}^k$ and let $M_2 \subset \mathbb{C}^{N_1+k}$ be the graphs of Ψ over M_1. Then M_2 is a CR manifold of the same type, (m, ℓ), if and only if the function $\phi = (I, \Psi) : M_1 \to M_2$ is a CR map, i.e., if and only if each component of Ψ is a CR function on M_1.*

We shall also need the following proposition which is due to Baouendi and Treves (Th. 2.1 [2]). The original statement of the theorem is weaker than the one we state here. However, the proof, with simple modifications, is valid for the form of the theorem we need. For completeness sake we shall include the proof in the following section.

Proposition 2: *Let $L_1, \ldots, L_m$ be a system of $C^s (s \geq 2)$ complex vector fields defined on Ω, a neighborhood of the origin in $\mathbb{R}^{d=m+n}$ which satisfies the following three conditions:*

(1) There exist n functionally independent C^s characteristic coordinates $z_1, \ldots, z_n$ at each point of Ω, i.e., $L_j z_k = 0$, $j = 1, \ldots, m$, $k = 1, \ldots, n$ and $dz_1, \ldots, dz_n$ are all linearly independent in Ω.

(2) The vector space generated by $L_1, \ldots, L_m$ has constant dimension at each point of Ω.

(3) $\{L_1, \ldots, L_m\}$ is closed under Lie bracket, i.e., $[L_j, L_k] = \sum_{\nu=1}^{m} c_\nu L_\nu$, $1 \leq j, k \leq m$.

Then every open neighborhood $\Omega' \subset \Omega$ of the origin contains another open neighborhood of the origin, Ω'', such that every C^s solution of

$$L_j h = 0, \quad j = 1, \ldots, m$$

in Ω' is the limit in the topology of $C^{s-1}(\Omega'')$ of a sequence of polynomials with complex coefficients in $z_1(x), \ldots, z_n(x)$.

We now give the proof of Theorem 1. By hypothesis $M' \subset \mathbb{C}^N$ is a CR manifold of type (m, ℓ) and class C^s, $s \geq 2$. If M' is generic, then the theorem is true and trivial even in the case $s = 1$. In that case, $M' = M$ can be defined as in (2.2) and one merely approximates the ℓ-tuple of C^s functions h by a sequence of real analytic (or, in fact, polynomial) approximations. However, in the non-generic case, one cannot simply approximate the $\ell + 2k$-tuple of C^s functions H and be assured that the approximating real analytic functions H_j give rise to manifolds of the same type (m, ℓ).

Let M' be a non-generic manifold of type (m, ℓ) defined as in (2.1) by H. Let M, ϕ, π, U', U, Ω' be as in Section 2 with M also described as in (2.5) with vector fields L_j, $1 \leq j \leq m$ and characteristic coordinates z_j, $1 \leq j \leq n$ defined on Ω, an open set of the origin in $\mathbb{R}^d = \mathbb{R}^\ell \times C^m$. By Proposition 1, $\phi : M \to M'$ is a *CR* map thus $L_j(\phi_\nu \circ \zeta) = 0$, $\nu = 1, \ldots, N$, $j = 1, \ldots, m$. Applying Proposition 2 we have that there exists $\Omega'' \subset \Omega \cap \Omega'$ such that each $\phi_\nu \circ \zeta$ can be approximated in $C^{s-1}(\Omega'')$ by polynomials in $z_1, \ldots, z_n$. We shall use this approximation for $\nu = n+1, \ldots, N$. Let $\tilde{U} \subset U \subset \mathbb{C}^n$ be a neighborhood of the origin with $\tilde{U} \cap M = \zeta(\Omega'')$. Since $z_1, \ldots, z_n$ are characteristic coordinates for M a polynomial in $z_1, \ldots, z_n$ on Ω'' is merely a holomorphic polynomial on $\zeta(\Omega'')$ and thus extends to be a holomorphic polynomial on $\tilde{U}$. Therefore, for each $\nu = n+1, \ldots, N$ there exists a sequence of holomorphic polynomials $\phi^j_\nu : \tilde{U} \to \mathbb{C}$ such that

$$\phi^j_\nu \to \phi_\nu \quad \text{in } C^{s-1}(\zeta(\Omega'')).$$

Moreover, by (2.3) we see that $\phi_\nu = z_\nu$ for $\nu = 1, \ldots, n$ and for $\nu \geq n+1 \phi_\nu$ vanishes to second order at the origin. Thus, the constant and linear terms associated to each ϕ^j_ν, $\nu = n+1, \ldots, N$ are tending to zero as $j \to \infty$. Therefore, subtracting off these constant and linear terms yields a modified sequence (which we will again call ϕ^j_ν) such that the sequence of N-tuple of functions

$$\{\phi^j\} = \{(z_1, \ldots, z_n, \phi^j_{n+1}, \ldots, \phi^j_N)\} : \tilde{U} \to \mathbb{C}^N$$

converges in $\underbrace{C^{s-1}(\zeta(\Omega'')) \times \cdots \times C^{s-1}(\zeta(\Omega''))}_{N\text{-times}}$ to

$$\phi = (\phi_1, \ldots, \phi_N).$$

Moreover, the last k components of each function vanishes to second order at the origin.

We now state and prove the following general Lemma 3.1.

LEMMA 3.1: *Let $\zeta : \Omega \to M \subset \mathbb{C}^n$ be a generic CR manifold. Let Ψ^j be holomorphic on some fixed neighborhood of M in $\mathbb{C}^n$. Assume $\Psi^j \circ \zeta$ is bounded in $C^k(\Omega)$. Then there exists a bound on $D_z^\alpha \Psi^j(z)$, $z \in M_0$, M_0 any compact subset of M, $|\alpha| \leq k$.*

PROOF: First we do it for $\alpha = 1$. Since Ψ^j is holomorphic and M is generic any vector field, D_{z_k}, is in the complex linear space of vector fields tangential to M. Applying the same argument to $D_{z_k}\Psi^j$ we bound $D_z^\alpha \Psi^j$ for $|\alpha| \leq 2$ and we continue by induction.

Using Lemma 3.1 we can define a sequence of shrinking neighborhoods of the origin, $U_j \subset \tilde{U} \subset \mathbb{C}^n$, with $U_j \supset \zeta(\Omega'')$, such that for $z \in U_j$

$$\sum_{|\alpha| \leq s-1} |D^\alpha \phi_\nu^j(z)| < 2\|\phi_\nu\|_{s-1}^{\zeta(\Omega'')}.$$

Here $\|f\|_{s-1}^B = \Sigma_{|\alpha| \leq s-1} \sup_B |D^\alpha f(z)|$.

Now choose $\Omega^j \subset \Omega''$ so that for $(x, w) = (x_1, \ldots, x_\ell, w_1, \ldots, w_m) \in \Omega^0$ we have

$$(x_1 + ih_1(x, w), \ldots, x_\ell + ih_\ell(x, w), w_1, \ldots, w_m) \in \zeta(\Omega'').$$

Now, one finds a sequence of ℓ-tuples of real analytic functions, or, in fact, polynomials

$$h^j = (h_1^j, \ldots, h_\ell^j) : \Omega^0 \to R^\ell$$

such that (1) each function vanishes to second order at the origin, (2) $h^j \to h$ in $\underbrace{C^s(\Omega^0) \times \cdots \times C^s(\Omega^0)}_{\ell\text{-times}}$ and (3) if we define generic manifolds M_j using (2.2) each $M_j \subset U_j$, for j sufficiently large. Moreover, since $h^j \to h$ in at least C^1, we have, using (1.5) that for some J sufficiently large, each M_j is a generic manifold of type (m, ℓ) for $j \geq J$. We shall now consider our sequence to begin with J and define M'_j to be the image of M_j under ϕ_j. By Proposition 1 we have that each M'_j is a non-generic real analytic CR manifold of type (m, ℓ). Moreover, defining $h_\nu^{j'}, h_\nu^{j''} : \Omega^0 \to \mathbb{R}$, $\nu = 1, \ldots, k$ by

$$h_\nu^{j'}(x, w) \equiv \operatorname{Re} \phi_{\nu+n}^j(x + ih^j(x, w), w)$$

$$h_\nu^{j''}(x, w) \equiv \operatorname{Im} \phi_{\nu+n}^j(x + ih^j(x, w), w).$$

We have that M'_j can also be defined as in (2.1) with $H^j = (h^j, h^{j'}, h^{j''})$. All that remains to be proven is that $M'_j \to M'$ in $C^{s-2}(\Omega^0)$, i.e., that

$H^j \to H$ in $C^{s-2}(\Omega^0)$. Since $h^j \to h$ in $C^{s-1}(\Omega^0)$ and

$$(h'_\nu + ih''_\nu)(x, w) = \phi_{n+\nu}(x + ih(x, w), w)$$

we need only prove that

$$\phi^j(x + ih^j(x, w), w) \to \phi(x + ih(x, w), w) \text{ in } C^{s-2}(\Omega^0).$$

We have

$$\begin{aligned}
&\|\phi(x + ih(x, w), w) - \phi^j(x + ih^j(x, w), w)\|_{s-2} \\
&\quad \le \|\phi(x + ih(x, w), w) - \phi^j(x + ih(x, w), w)\|_{s-2} \\
&\qquad + \|\phi^j(x + ih(x, w), w) - \phi^j(x + ih^j(x, w), w)\|_{s-2} \\
&\quad \le \|\phi - \phi^j\|_{s-2}^{\zeta(\Omega'')} + \|\phi^j\|_{s-1}^{U_j}\|(x + ih(x, w), w) - (x + ih^j(x, w), w)\|_{s-2}^{\Omega_0} \\
&\quad \le \|\phi - \phi^j\|_{s-2}^{\zeta(\Omega'')} + 2\|\phi\|_{s-1}^{M}\|(x + ih(x, w), w) - (x + ih^j(x, w), w)\|_{s-2}^{\Omega}.
\end{aligned}$$

Since the right-hand side converges to zero as $j \to \infty$ we are done.

PROOF OF THEOREM 2: Since $M' \subset \mathbb{C}^N$ is an embedded CR manifold of type (m, ℓ) we can, by the discussion in Section 3, consider M' to be a graph over a generic manifold $M \subset \mathbb{C}^n$ of type (m, ℓ), with $M' = \phi(M)$. Here we may have to shrink the domain of definition of M'. Moreover, by Proposition 1, we have that ϕ is a CR map on M. Also by Proposition 1, if we let M'' be a graph of f over M', then $M'' \subset \mathbb{C}^{N+1}$ is a CR manifold of type (m, ℓ) and thus, $f \circ \phi \in CR(M)$ since M'' is the graph over M with $M'' = (\phi, f \circ \phi)(M)$. As in the proof of Theorem 1, using Proposition 2, there exist a sequence of $N+1$-tuple of holomorphic polynomials (ϕ^j, f^j) defined on some $\tilde{U} \subset C^n$ such that

$$(\phi^j, f^j) \to (\phi, f \circ \phi) \quad \text{in } C^{t-1}(\tilde{U} \cap M).$$

Let $U \subset \mathbb{C}^N$ be any neighborhood of the origin such that $\pi(U) = \tilde{U}$. Define $p^j : U \to \mathbb{C}$ by $p^j(z_1, \ldots, z_N) = f^j(z_1, \ldots, z_n)$. Then for any $(z_1, \ldots, z_N) \in U \cap M'$, we have $(z_1, \ldots, z_N) = \phi(z_1, \ldots, z_n)$. Thus, $p^j(z_1, \ldots, z_N) = f^j(z_1, \ldots, z_n) \to f \circ \phi(z_1, \ldots, z_n) = f(z_1, \ldots, z_N)$. Therefore, $p^j \to f$ on $C^{t-1}(U \cap M')$.

PROOF OF PROPOSITIONS 1 AND 2: We now give the proofs of the propositions.

Proof of Proposition 1: We first note that since $I: M_1 \to M_1$ is the restriction to a CR manifold of a holomorphic map I is a CR map and it is clear that ϕ is CR if and only if each component of Ψ is a CR function. Without loss of generality, we can assume that $0 \in M_1$ and M_1 can be written as a graph by H over some neighborhood $\Omega \subset T_0(M_1) \cong R^{\ell} \times C^m$. Let $d = \ell + 2m$ and $q = 2N - d$ be the codimension of M_1.

We first suppose that M_2 is a CR manifold of type (m, ℓ), i.e., the same type as M_1. Let $\pi : \mathbb{C}^{N_1+k} \to \mathbb{C}^{N_1}$ be the projection map onto the first N_1 components. Then π is holomorphic on $\mathbb{C}^{N_1}$ and therefore, its restriction to M_2, $\pi|_{M_2}$ is CR on M_1. Thus $d(\pi|_{M_2})_p : HT_p(M_2) \to HT_{\pi(p)}(M_1)$, $p \in M_2$. Since $\dim T_p(M_2) = \dim T_{\pi(p)}(M_1)$, $\dim_{\mathbb{C}} HT_p(M_2) = \dim_{\mathbb{C}} HT_{\pi(p)}(M_1)$ and $d(\pi|_{M_2})$ is injective, we have that the transformations $d(\pi|_{M_2})$ and $d(\pi|_{M_2})|_{HT_p(M_2)}$ are surjective. Therefore, the inverse transformation of $d(\pi|_{M_2})$, $d\phi$, is such that

$$d\phi_{\pi(p)} : HT_{\pi(p)}(M_1) \to HT_p(M_2).$$

Hence, ϕ is CR on M_1.

Now suppose ϕ is CR on M_1. Thus, $d\phi_{\pi(p)} : HT_{\pi(p)}(M_1) \to HT_p(M_2)$. From the definitions of M_1, M_2 and π we have that $\dim T_p(M_2) = \dim T_{\pi(p)}(M_1)$ with $d(\pi|_{M_2})$ being bijective. Thus $d\phi$ is bijective and $\dim_{\mathbb{C}} HT_{\pi(p)}(M_1) \leq \dim_{\mathbb{C}} HT_p(M_2)$ for $p \in M_1$. Moreover, since $\dim_{\mathbb{C}} HT_{\pi(p)}(M_1) = m$, we have, from (1.5) that

$$m = N_1 - \operatorname{rank}\left[\frac{\partial \rho_j}{\partial z_\nu}(\pi(p)\right] \quad 1 \leq j \leq q, \quad 1 \leq \nu \leq N_1, \quad p \in M_2$$

where M_1 is defined locally by $p_j(z) = 0$, $z \in \mathbb{C}^{N_1}$, $1 \leq j \leq q$, with $d_{\rho 1} \wedge \cdots \wedge d\rho_q \neq 0$. Thus, $\operatorname{rank}[(\partial \rho / \partial z)(\pi(p))] = N_1 - m$. Let $\tilde{\Psi} = \Psi \circ H : \Omega \to \mathbb{C}^k$. We can of course consider $\tilde{\Psi}$ to be a function on an open set in $\mathbb{C}^{N_1}$ which merely depends on $\ell + 2m$ real variables. We have that M_2 can be defined locally by $\tilde{\rho}_j(z) = 0$, $z \in \mathbb{C}^{N_1+k}$, $1 \leq j \leq q + 2k$ where

$$\tilde{\rho}_j(z_1, \ldots, z_{N_1+k}) = \begin{cases} \rho_j(z_1, \ldots, z_{N_1}), & 1 \leq j \leq q \\ x_{N_1+j-q} - \operatorname{Re} \tilde{\Psi}(z_1, \ldots, z_{N_1}), & q + 1 \leq j \leq q + k \\ y_{N_1+j-q-k} - \operatorname{Im} \tilde{\Psi}(z_1, \ldots, z_{N_1}), & q + k + 1 \leq q \leq q + 2k \end{cases}$$

and $z_j = x_j + iy_j$. From (1.5) we have

$$\dim_{\mathbb{C}} HT_p(M_2) = N_1 + k - \operatorname{rank}\left[\frac{\partial \tilde{\rho}_j}{\partial z_\nu}(p)\right], \; 1 \leq j \leq q + 2k, \; 1 \leq \nu \leq N_1 + k.$$

Clearly, since $[\partial\rho/\rho z]$ has rank $N_1 - m$, we have that $[\partial\tilde{\rho}/\partial z]$ has rank $\geq N_1 - m + k$, and therefore that $\dim_{\mathbb{C}} HT_p(M_2) \leq N_1 + k - (N_1 - m + k) = m$. Thus, $\dim_{\mathbb{C}} HT_p(M_2) \leq \dim_{\mathbb{C}} HT_{\pi(p)}(M_1)$, for $p \in M_2$ and we have $\dim_{\mathbb{C}} HT_p(M_2) = \dim HT_{\pi(p)}(M_1) = m$, $p \in M_2$. Since $\dim T_p(M_2) = \ell + 2m$, this yields that M_2 is of type (m, ℓ) and the lemma is proved.

We now give the proof of Proposition 2.

Proof of Proposition 2: As stated above, the proof, and the notation, is due to [2]. It is a very clever variation of the original proof of the Weierstress Approximation Theorem. We can shrink Ω if necessary and change variables so that, without loss of generality, the L_j's and z_k's can be put in the following form, with new variables $(t_1, \ldots, t_m, x_1, \ldots, x_n)$,

$$\begin{aligned} L_j &= \frac{\partial}{\partial t_j} + \sum_{k=1}^{n} \lambda_k^j(t, x) \frac{\partial}{\partial x_k}, \quad 1 \leq j \leq m \\ z_k &= x_k + i\Phi_k(t, x) \qquad\qquad 1 \leq k \leq n, \end{aligned} \tag{4.1}$$

where $\lambda_k^j \in C^s(\Omega)$, $\Phi = (\Phi_1, \ldots, \Phi_n) \in \mathbb{R}^n$ is of class C^s with

$$\begin{cases} \Phi(0,0) = 0 \\ \dfrac{\partial \Phi_k}{\partial x_\ell}(0,0) = 0 \quad k, \ell = 1, \ldots, n. \end{cases} \tag{4.2}$$

Since the z_k's are characteristic coordinates, we have

$$L_j z_k = 0 \quad \text{on } \Omega, \quad 1 \leq j \leq m, \quad 1 \leq k \leq n. \tag{4.3}$$

Moreover, the Φ's and λ's are related by the following

$$\sum_{\ell=1}^{n} \left(\delta_\ell^k + i \frac{\partial \Phi_k}{\partial x_\ell} \right) \lambda_\ell^j = -i \frac{\partial \Phi_k}{\partial t_j}, \quad 1 \leq j \leq m, \quad 1 \leq k \leq n.$$

We shall use the following notation: for each j, λ^j will denote the n vector $(\lambda_1^j, \ldots, \lambda_n^j)$, $\partial\lambda^j/\partial x$ will denote the $n \times n$ matrix with entries $\partial\lambda_\ell^j/\partial x_k$, and $\partial z/\partial x$ will denote the Jacobian matrix of the z's with respect to the x's. We have the following result, known as Lemma 2.1 in [2].

Lemma 3: *For each j we have*

$$L_j \left[\det \left(\frac{\partial z}{\partial x} \right) \right] + \operatorname{Tr} \left(\frac{\partial \lambda^j}{\partial x} \right) \det \left(\frac{\partial z}{\partial x} \right) = 0.$$

PROOF OF LEMMA: Differentiating the equations

$$L_j z_\ell = 0, \quad \ell = 1, \dots, n$$

with respect to each x_k, $1 \le k \le n$, and using (3.3) we have

$$L_j\left(\frac{\partial z_\ell}{\partial x_k}\right) = -\sum_{\alpha=1}^{n} \frac{\partial \lambda^j_\alpha}{\partial x_k}\frac{\partial z_\ell}{\partial x_\alpha}.$$

Now, letting $dz_\ell = \Sigma_{k=1}^n (\partial z_\ell / \partial x_k)\, dx_k$ we have

$$\begin{aligned} L_j(dz_1 \wedge \cdots \wedge dz_n) &= \sum_{\ell=1}^{n} (dz_1 \wedge \cdots \wedge L_j(dz_\ell) \wedge \cdots \wedge dz_n) \\ &= -\sum_{\ell=1}^{n} dz_1 \wedge \cdots \wedge \sum_{k=1}^{n}\sum_{\alpha=1}^{n} \frac{\partial \lambda^j_\alpha}{\partial x_k}\frac{\partial z_\ell}{\partial x_\alpha}\, dx_k \wedge \cdots \wedge dz_n \end{aligned}$$

which yields, after a bit of linear algebra,

$$= -\mathrm{Tr}\left(\frac{\partial \lambda^j}{\partial x}\right)(dz_1 \wedge \cdots \wedge dz_n).$$

Letting $e_1, \dots, e_n$ be the natural basis in $\mathbb{C}^n$ we have

$$dz_1 \wedge \cdots \wedge dz_n = \det\left(\frac{\partial z}{\partial x}\right) de_1 \wedge \cdots \wedge de_n$$

and we are done. We note that we have used the fact that all functions which arise are at least C^2.

As in [2], we shall, for the sake of simplicity write

$$\Delta(t, x) = \left[\det\left(\frac{\partial z}{\partial x}(t, x)\right)\right]$$

and we note that $\Delta \in C^{s-1}(\Omega)$. Since the transpose of L_j, ${}^tL_j$, is of the form

$${}^tL_j = -L_j - \mathrm{Tr}\left(\frac{\partial \lambda^j}{\partial x}\right)$$

we have from Lemma 3 that

$${}^tL_j\Delta = 0, \quad (t, x) \in \Omega, \quad j = 1, \dots, m.$$

Therefore we have that if h is any complex valued function annihilated by the system of L's in Ω, i.e.,

$$L_j h(t,x) = 0, \quad (t,x) \in \Omega, \quad 1 \leq j \leq m, \tag{4.4}$$

then

$$^tL_j(h\Delta)(t,x) = 0, \quad (t,x) \in \Omega, \quad 1 \leq j \leq m. \tag{4.5}$$

As in [2], we shall assume that Ω' is of the form $U \times V$ with U, V open neighborhoods of the origin in $\mathbb{R}^m$, $\mathbb{R}^n$, respectively, with U connected.

Choosing V' to be a relatively compact open neighborhood of the origin in V, and $g \in C_c^\infty(V)$ such that $g \equiv 1$ on V', we have that if h satisfies (4.4) on Ω', then (4.5) implies

$$\begin{aligned} &\int h(t,x)\Delta(t,x)\left[\sum_{k=1}^n \lambda_k^j(t,x)\frac{\partial g}{\partial x_k}(x)\right] dx \\ &\qquad = \int g(x)\frac{\partial}{\partial t_j}[h(t,x)\Delta(t,x)]\, dx, \qquad 1 \leq j \leq M. \end{aligned}$$

Writing this equation in the notation of differential forms on U, we have

$$\begin{aligned} &\sum_{j=1}^m \left\{\int h(t,x)\Delta(t,x)\left[\sum_{k=1}^n \lambda_j^k(t,x)\frac{\partial g}{\partial x_k}(x)\right] dx\right\} dt_j \\ &\qquad = d_t \int g(x)h(t,x)\Delta(t,x)\, dx. \end{aligned} \tag{4.6}$$

Now, as in [2], we integrate both sides of (4.6) along a smooth curve, $\gamma(t)$, contained in U, which joins 0 to t. To avoid confusion, as in [2], we change the variables of integration from (t,x) to (s,y), respectively. Thus, we have

$$\begin{aligned} \int g(y)h(t,y)\Delta(t,y)\, dy &= \int g(y)h(0,y)\Delta(0,y)\, dy \\ &\quad + \int_{\gamma(t)} \sum_{j=1}^m \left\{\int_{\mathbf{R}^n} h(s,y)\Delta(s,y)\left[\sum_{k=1}^n \lambda_k^j(s,y)\frac{\partial g}{\partial y_k}(y)\right] dy\right\} ds_j. \end{aligned} \tag{4.7}$$

Now let $u \in C^s(\Omega')$ be a solution of

$$L_j u = 0, \quad 1 \leq j \leq m.$$

As in [2], for any fixed $(t, x) \in \Omega'$, we introduce

$$E_\nu(t, x; s, y) = \left(\frac{\nu}{\sqrt{2\pi}}\right)^n e^{-\nu^2[z(t,x)-z(s,y)]^2}, \tag{4.8}$$

where $[\zeta]^2 = \zeta_1^2 + \cdots + \zeta_n^2$, $\zeta \in \mathbb{C}^n$.

Since $h(s, y) = E_\nu(t, x; s, y)u(s, y) \in C^2(\Omega')$ is a solution of (4.4) in Ω, we have

$$\begin{aligned} &\int g(y)E_\nu(t, x; t, y)u(t, y)\Delta(t, y)\, dy \\ &\quad = \int g(y)E_\nu(t, x; 0, y)u(0, y)\Delta(0, y)\, dy \\ &\quad + \int_{\gamma(t)} \sum_{j=1}^{n} \Big\{ \int_{\mathbb{R}^n} E_\nu(t, x; s, y)u(s, y)\Delta(s, y) \\ &\quad \times \Big[\sum_{k=1}^{n} \lambda_k^j(s, y) \frac{\partial g}{\partial y_k}(y)\Big]\, dy \Big\}\, ds_j. \end{aligned} \tag{4.9}$$

Using (4.2) we can assume that U and V are small enough so that for $t \in U$, $x, y \in V$

$$|\Phi(t, x) - \Phi(t, y)| < \tfrac{1}{2}|x - y|. \tag{4.10}$$

Now, letting U'' and V'' be two open neighborhoods of the origin relatively compact in U and V' respectively, then we claim

$$\lim_{\nu\to\infty} \left| u(t, x) - \int g(y)E_\nu(t, x; t, y)u(t, y)\Delta(t, y)\, dy \right|_{s-1} = 0 \tag{4.11}$$

where $|\ |_{s-1}$ is the standard norm on $C^{s-1}(U'' \times V'')$. In [2], (4.11) was proved for the $C^0(U'' \times V'')$-norm, however, the proof is essentially the same. By (4.1) we have

$$z(t, x) - z(t, y) = x - y + i(\Phi(t, x) - \Phi(t, y)).$$

Making the change of variables $y \to x - y/\nu$ in (4.11), we have that the integral in (4.11) becomes

$$\begin{aligned} &(2\pi)^{-(n/2)} \int \exp\{-[y + i\nu(\Phi(t, x) - \Phi(t, x - y/\nu))]^2 \\ &\quad \times g(x - y/\nu)u(t, x - y/\nu)\Delta(t, x - y/\nu)\, dy \end{aligned} \tag{4.12}$$

as $\nu \to \infty$ the integrand in (4.12) converges in $C^{s-1}(V'')$ to

$$\exp\{-[(1+i\Phi_x(t,x))y]^2\Delta(t,x)u(t,x). \tag{4.13}$$

Since $|\Phi_x(t,x)| \leq \frac{1}{2}$ for $(t,x) \in U'' \times V''$, the exponential function in (4.12) is uniformly bounded and we have by the Lebesgue Convergence Theorem, that (4.12) converges to the integral over y of (4.13) in $C^{s-1}(U'' \times V'')$. Now as in [2], one uses the fact that

$$1 + i\Phi_x = \frac{\partial z}{\partial x}$$

and applies the classical formula

$$(2\pi)^{-(n/2)} \int_{\mathbb{R}^n} e^{-[Ay]^2}(\det A)\, dy = 1$$

which is valid for the complex $n \times n$ matrix $A = \partial z/\partial x$.

We now need to restrict U'' even further. We assume U'' is an open ball centered at the origin with radius $r > 0$. We shall show that for r sufficiently small

$$\lim_{\nu\to\infty} \left| \int_{\gamma(t)} \sum_{j=1}^{m} \int_{\mathbb{R}^n} E_\nu(t,x;s,y)u(s,y)\Delta(s,y) \times \left[\sum_{k=1}^{n} \lambda_k^j(s,y)\frac{\partial g}{\partial y_k}(y)\right] dy \right\} ds_j \Bigg|_{s-1} = 0, \tag{4.14}$$

whenever $(t,x) \in U'' \times V''$ with $\gamma(t)$ lying inside U'' and, as above, with this norm taken over $U'' \times V''$. To prove (4.14) we let, as in [2], $d > 0$ be the distance from V'' to the complement of V'. Since $g \equiv 1$ on V' and $g \in C_c^\infty(V)$, the integrand in (4.14) vanishes identically for all $y \in V'$ and all $y \notin V$. Thus, for all $x \in U''$ the integral in (4.14) is only taken over $y \in V$ such that $|x-y| \geq d$. Since Φ is real valued we have

$$\operatorname{Re}[z(t,x)-z(s,y)]^2 \geq |x-y|^2 - |\Phi(t,x)-\Phi(x,y)|^2,$$

and using the triangle inequality, we have

$$|\Phi(t,x)-\Phi(s,y)| \leq |\Phi(t,x)-\Phi(t,y)| + |\Phi(t,y)-\Phi(s,y)| = ① + ②.$$

By (4.10) we have ① $\leq \frac{1}{2}|x-y|$ and by continuity ② $\leq C|t-s| \leq Cr$, for some constant C. Choosing r so small that $Cr \leq d/4$, we have ① + ② $\leq 3d/4$ and

$$\mathrm{Re}[z(t,x)-z(s,y)]^2 \geq d^2 - \frac{9d^2}{16} = \frac{7d^2}{16} \leq \frac{d^2}{4}$$

in (4.14). Now, even if we take $s-1$ derivatives with respect to (t,x) of the integral in (4.14), since each integral is only taken over V (in fact $V-V'$) with respect to the y variable, we have that over the region of integration, each such derivative of the integrand is bounded by

$$\nu^2 C e^{-\nu^2(d^2/\nu)},$$

for some constant C. Thus, letting $\nu \to \infty$, we have (4.14) $\to 0$ in $C^{s-1}(U'' \times V'')$.

Now, returning to (4.9) and using (4.11) and (4.14) that

$$\text{(4.15)}\quad \lim_{\nu\to 0}\left| u(t,x) - \int g(y)E_\nu(t,x;0,y)u(0,y)\Delta(0,y)\,dy\right|_{s-1} = 0$$

again with convergence in the usual norm on $C^{s-1}(\Omega'')$, $\Omega'' = U'' \times V''$. Since the integral in (4.15) is an entire function of $z(t,x)$, the proof of Proposition 2 is complete.

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