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ON THE LOGARITHMIC PLURIGENERA OF ALGEBRAIC SURFACES

Yoshiyuki Kuramoto

Introduction

Let S be a nonsingular algebraic surfaces over the complex number field $\mathbb{C}$. We study the logarithmic plurigenera $\bar{P}_m(S)$ under the assumption that there is a surjective morphism $f: S \rightarrow \Delta$ to a nonsingular curve Δ whose general fiber C_w is an irreducible curve such that the logarithmic Kodaira dimension $\bar{\kappa}(\Delta)$ is non-negative. Such a situation can arise from the quasi-Albanese mapping α_S of S . If $\bar{\kappa}(C_w) = -\infty$, i.e. $C_w \cong \mathbb{P}^1$ or $\mathbb{A}^1$, then $\bar{P}_m(S) = 0$ for all $m > 0$. Thus we assume that $\bar{\kappa}(C_w) \geq 0$. Then by the addition formula for logarithmic Kodaira dimension ([6]), we have some $m > 0$ such that $\bar{P}_m(S) \geq 1$. In this paper we shall look for integers m such that $\bar{P}_m(S) \geq 1$.

We use the following notations:

- $\bar{S}$: a nonsingular complete algebraic surface which contains S as a Zariski open set,
- $D = \bar{S} - S$: the complement of S in $\bar{S}$. We assume that D has only normal crossings and every irreducible component of D is nonsingular.
- Δ : a nonsingular complete algebraic curve which contains Δ as a Zariski open set,
- $\bar{f}: \bar{S} \rightarrow \bar{\Delta}$: a rational mapping defined by f . We can assume that $\bar{f}$ is a morphism by taking a suitable $\bar{S}$.
- $\bar{C}_w$: a general fiber of $\bar{f}$,
- $K_{\bar{S}}$: the canonical divisor of $\bar{S}$,
- $p_g(\bar{S})$: the geometric genus of $\bar{S}$,

- $q(\bar{S})$: the irregularity of $\bar{S}$,
- $P_m(\bar{S})$: the m -genus of $\bar{S}$,
- $\bar{p}_g(S)$: the logarithmic geometric genus of S ,
- $\bar{q}(S)$: the logarithmic irregularity of S ,
- $\bar{P}_m(S)$: the logarithmic m -genus of S ,
- $\kappa(\bar{V})$: the Kodaira dimension of a nonsingular complete algebraic variety $\bar{V}$,
- $\bar{\kappa}(V)$: the logarithmic Kodaira dimension of a nonsingular algebraic variety V ,
- $g(C)$: the genus of a nonsingular complete algebraic curve C ,
 $\pi(C) = (1/2)(K_{\bar{S}} + C, C) + 1$, if C is a divisor on $\bar{S}$,
- $\sim$: the linear equivalence of divisors,
- $D_1 \geq D_2$: means $D_1 - D_2$ is effective or 0 if D_1 and D_2 are divisors.

The following are our results.

THEOREM 1: *Under the above notations if $\bar{\kappa}(\Delta) \geq 0$ and $\bar{\kappa}(C_w) \geq 0$, then $\bar{P}_4(S) \geq 1$ or $\bar{P}_6(S) \geq 1$.*

COROLLARY: *Let $S = \mathbb{A}^2 - V(\varphi)$ where $\varphi \in \mathbb{C}[x, y]$ and φ is irreducible. If $\bar{P}_4(S) = \bar{P}_6 = 0$, then $S \cong \mathbb{A}^1 \times G_m$.*

THEOREM 2: *Let S be a nonsingular algebraic surface over $\mathbb{C}$. If $\bar{\kappa}(S) \geq 0$ and $\bar{q}(S) \geq 1$, then $\bar{P}_4(S) \geq 1$ or $\bar{P}_6(S) \geq 1$.*

If $\bar{S}$ is neither ruled nor rational, then $P_4(\bar{S}) \geq 1$ or $P_6(\bar{S}) \geq 1$ from the theory of complete algebraic surfaces. Thus to prove Theorem 1, it suffices to treat the case that $\bar{S}$ is ruled or rational. In §1, we give preliminary results. In §2–§7, we prove Theorem 1. The following theorem due to Tsunoda plays an essential role in §7.

THEOREM 3: (Tsunoda) *Let $S, \bar{S}$ and D be as above. If $\bar{\kappa}(S) = 2$ and the intersection matrix of D is not negative definite, then $\bar{P}_4(S) \geq 1$ or $\bar{P}_6(S) \geq 1$.*

In §8 we prove Theorem 3. In §9 we prove Corollary and Theorem 2.

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§1. Preliminaries

LEMMA 1: Let S , $\bar{S}$ and D be as in the introduction. Suppose that there is an exceptional curve of the first kind E on $\bar{S}$ such that $(E, K_{\bar{S}} + D) \leq 0$. Let $\mu: \bar{S} \rightarrow \bar{S}^b$ be the contraction of E . Then $\mu(D)$ is a divisor with only normal crossings and $\bar{P}_m(S) = \bar{P}_m(\bar{S}^b - \mu(D))$.

PROOF: Easy and omitted. Especially if $D \not\supset E$ and $(D, E) = 1$, then $\bar{S}^b - \mu(D)$ is called a half point detachment from S , and if $D \supset E$ and $(D - E, E) = 2$, then μ is called a canonical blowing down. (c.f. [5])

LEMMA 2: Let S , $\bar{S}$ and D be as in the introduction. Let $D = \sum D_i$ be the irreducible decomposition of D . Let $h(\Gamma(D))$ be the cyclotomic number of the dual graph $\Gamma(D)$ of D , i.e.

$$\begin{aligned} h(\Gamma(D)) = & (\text{number of connected components of } \Gamma(D)) \\ & - (\text{number of vertices of } \Gamma(D)) \\ & + (\text{number of 1-simplices of } \Gamma(D)). \end{aligned}$$

Then we have

$$\bar{p}_g(S) = \sum \pi(D_i) + p_g(\bar{S}) - q(\bar{S}) + h(\Gamma(D)) + t.$$

Where t is the dimension of the kernel of the canonical homomorphism $H^1(\bar{S}, \mathcal{O}_{\bar{S}}) \rightarrow H^1(D, \mathcal{O}_D)$. (c.f. [9; Theorem 2.2]).

LEMMA 3: Let $\pi: V \rightarrow W$ be an r -sheeted Galois covering, where V and W are complete normal algebraic varieties over $\mathbb{C}$. Let D be a Cartier divisor on W . Then we have

- a) $\dim H^0(V, \mathcal{O}_V(\pi^*D)) \geq 1 \Rightarrow \dim H^0(W, \mathcal{O}_W(rD)) \geq 1$,
- b) $\dim H^0(V, \mathcal{O}_V(\pi^*D)) \geq 2 \Rightarrow \dim H^0(W, \mathcal{O}_W(rD)) \geq 2$.

PROOF: Using the same argument as [2; §6] or [4; §10.11], we can prove the above lemma. Details are omitted.

In that follows we tacitly use the notation in the introduction.

§2. The case $\kappa(\bar{C}_w) = -\infty$ and $\kappa(\bar{\Delta}) = -\infty$

PROPOSITION 1: *If $\bar{\kappa}(C_w) \geq 0$, $\bar{\kappa}(\Delta) \geq 0$, $\kappa(\bar{C}_w) = -\infty$ and $\kappa(\bar{\Delta}) = -\infty$, then $\bar{P}_2(S) \geq 1$.*

PROOF: By the assumption, $\bar{S}$ is rational and $\bar{\Delta} \cong \mathbb{P}^1$. Hence there exists a composition of a finite sequence of quadratic transformations $\mu: \bar{S} \rightarrow \hat{S}$ where $\hat{f}: \hat{S} \rightarrow \bar{\Delta}$ is a $\mathbb{P}^1$ -bundle over $\bar{\Delta}$ and $\bar{f} = \hat{f} \circ \mu$. Since $\bar{\kappa}(\Delta) \geq 0$, $\Delta \subset \mathbb{C}^*$ and so D contains two reduced fibers of $\bar{f}$ which we denote by F_1 and F_2 . If D contains two horizontal components with respect to $\bar{f}$, then by Lemma 2 $\bar{p}_g(S) \geq 1$. Thus we assume that D contains only one horizontal component which we denote by H . If $(H, F_1) \geq 2$, then by Lemma 2 $\bar{p}_g(S) \geq 1$. Hence we assume $(H, F_1) = 1$. Let $\bar{C}_w$ be a general fiber of $\bar{f}$ and $\tilde{F}_1$ be the fiber such that $F_1 = \text{supp}(\tilde{F}_1)$. Since $\bar{\kappa}(C_w) \geq 0$, it follows that $(\bar{C}_w, H) = (F_1, H) \geq 2$. Hence $F_1 \neq \tilde{F}_1$ and F_1 contains an exceptional curve E_1 . If $E_1 \cap H = \emptyset$, we can contract E_1 . Thus we may assume $(E_1, H) = 1$. If E_1 is an edge component of F_1 (i.e. $(E_1, F_1 - E_1) = 1$), then $(E_1, D - E_1) = 2$ and so by Lemma 1 we can contract E_1 . Thus we may assume that F_1 contains two components C_1 and C_2 such that $(C_1, E_1) = (C_2, E_1) = 1$. Let $\mu': \bar{S} \rightarrow \bar{S}'$ be the contraction of E_1 . Put $\mu'_*(C_1) = C'_1$, $\mu'_*(C_2) = C'_2$ and $\mu'_*(H) = H'$. Since $2F_1 \geq C_1 + C_2 + 2E_1$, we have

$$K_{\bar{S}} + H + 2F_1 \geq K_{\bar{S}} - E_1 + H + C_1 + C_2 + 3E_1 = \mu'^*(K_{\bar{S}'} + H' + C'_1 + C'_2).$$

On the other hand by the Riemann-Roch theorem we have

$$\begin{aligned} \dim H^0(\bar{S}', \mathcal{O}_{\bar{S}'}(K_{\bar{S}'} + H' + C'_1 + C'_2)) \\ \geq (1/2)(K_{\bar{S}'} + H' + C'_1 + C'_2, H' + C'_1 + C'_2) + 1 = 1. \end{aligned}$$

Thus we have

$$\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(K_{\bar{S}} + H + 2F_1)) \geq 1.$$

Similarly, $\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(K_{\bar{S}} + H + 2F_2)) \geq 1$.

Hence, $\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(2K_{\bar{S}} + 2H + 2F_1 + 2F_2)) \geq 1$.

Therefore, $\bar{P}_2(S) = \dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(2(K_{\bar{S}} + D))) \geq 1$.

Q.E.D.

§3. The case $\kappa a(\bar{C}_w) = -\infty$ and $\kappa(\bar{\Delta}) = 0$

PROPOSITION 2: *If $\bar{\kappa}(C_w) \geq 0$, $\bar{\kappa}(\Delta) \geq 0$, $\kappa(\bar{C}_w) = -\infty$ and $\kappa(\bar{\Delta}) = 0$, then $\bar{P}_2(S) \geq 1$.*

PROOF: By the assumption we have the following commutative diagram:

$$\begin{array}{ccccc} S & \hookrightarrow & \bar{S} & \xrightarrow{\mu} & \hat{S} \\ & \searrow f & & \searrow \bar{f} & \downarrow \hat{f} \\ & & \Delta & \hookrightarrow & \bar{\Delta} \end{array}$$

Where $\bar{\Delta}$ is a nonsingular elliptic curve, $\hat{f}: \hat{S} \rightarrow \bar{\Delta}$ is a $\mathbb{P}^1$ -bundle, $\mu: \bar{S} \rightarrow \hat{S}$ is a composition of a finite number of quadratic transformations and $\bar{S} - S = D$ is a divisor with normal crossings. We may assume that all irreducible components of D are horizontal with respect to $\bar{f}$. Put $D = \sum H_j$, where the H_j are irreducible components. Since the H_j are horizontal $g(H_j) \geq g(\bar{\Delta}) = 1$. Hence if $\sum g(H_j) \geq 2$, then by Lemma 2 we have $\bar{p}_g(S) \geq 1$. Thus we assume that D is an irreducible horizontal curve such that $g(D) = 1$. We put $\mu = \mu_1 \circ \mu_2 \circ \cdots \circ \mu_r$, where $\mu_i: S_i \rightarrow \bar{S}_{i-1}$ is a quadratic transformation with center p_i , $\bar{S}_0 = \hat{S}$ and $\bar{S}_r = \bar{S}$. Let $E_i = \mu_i^{-1}(p_i)$. For the sake of simplicity we use the same letter E_i for $(\mu_{i+1} \circ \cdots \circ \mu_r)^* E_i$ also. Put $\mu(D) = \hat{D}$. Let ν_i be the multiplicity of the proper transform of $\hat{D}$ to $\bar{S}_{i-1}$ at p_i . Then we have

$$\mu^* \hat{D} = D + \sum_{i=1}^r \nu_i E_i, \quad \mu^* K_{\hat{S}} \sim K_{\bar{S}} - \sum_{i=1}^r E_i.$$

Let $\hat{F}$ be a fiber of $\hat{f}: \hat{S} \rightarrow \bar{\Delta}$. We put $(\hat{F}, \hat{D}) = d$. Since every exceptional curve on $\bar{S}$ is contained in some fiber of $\bar{f}$, it follows that $(\hat{F}, \hat{D}) = (\bar{C}_w, D)$. By the assumption that $\bar{\kappa}(C_w) \geq 0$, we have $(\bar{C}_w, D) \geq 2$. Hence $d \geq 2$. Since $\hat{S}$ is a ruled surface of genus 1, there is a non-negative integer b such that

$$\hat{D} \equiv -(d/2)K_{\hat{S}} + b\hat{F},$$

where $\equiv$ means numerical equivalence. Since $g(D) = 1$, we have

$$0 = (D, D + K_{\bar{S}}) = \left(\hat{D} - \sum_{i=1}^r \nu_i E_i, \hat{D} + K_{\hat{S}} - \sum_{i=1}^r (\nu_i - 1) E_i \right)$$

$$\begin{aligned}
&= (\hat{D}, \hat{D} + K_{\hat{S}}) - \sum_{i=1}^r \nu_i(\nu_i - 1) \\
&= (-(d/2)K_{\hat{S}} + b\hat{F}, (1 - (d/2))K_{\hat{S}} + b\hat{F}) - \sum_{i=1}^r \nu_i(\nu_i - 1) \\
&= 2b(d - 1) - \sum_{i=1}^r \nu_i(\nu_i - 1).
\end{aligned}$$

Thus we have

$$(1) \quad \sum_{i=1}^r (1/2)\nu_i(\nu_i - 1) = b(d - 1).$$

First we treat the case that $\nu_i = 1$ for all i . In this case $b = 0$ and $\hat{D} + (d/2)K_{\hat{S}} \equiv 0$. We may assume $\bar{S} = \hat{S}$. Then we have

$$(\hat{D} + K_{\hat{S}}, \hat{F}) = ((1 - (d/2))K_{\hat{S}}, \hat{F}) = d - 2 \geq 0.$$

If $d > 2$, then $H^0(\hat{S}, \mathcal{O}_{\hat{S}}(-(m - 1)\hat{D} - (m - 1)K_{\hat{S}})) = 0$ for $m \geq 2$, because $\hat{F}^2 = 0$. Thus from the exact sequence

$$\begin{aligned}
\mathcal{O}_{\hat{S}}(-m\hat{D} - (m - 1)K_{\hat{S}}) &\rightarrow \mathcal{O}_{\hat{S}}(-(m - 1)\hat{D} - (m - 1)K_{\hat{S}}) \\
&\rightarrow \mathcal{O}_{\hat{D}}(-(m - 1)(\hat{D} + K_{\hat{S}})|_{\hat{D}}) \cong \mathcal{O}_{\hat{D}},
\end{aligned}$$

we get the exact sequence

$$0 \rightarrow H^0(\hat{D}, \mathcal{O}_{\hat{D}}) \rightarrow H^1(\hat{S}, \mathcal{O}_{\hat{S}}(-m\hat{D} - (m - 1)K_{\hat{S}})).$$

Hence, $\dim H^1(\hat{S}, \mathcal{O}_{\hat{S}}(m(\hat{D} + K_{\hat{S}}))) = \dim H^1(\hat{S}, \mathcal{O}_{\hat{S}}(-m\hat{D} - (m - 1)K_{\hat{S}})) \geq 1$.

Since $H^2(\hat{S}, \mathcal{O}_{\hat{S}}(m(\hat{D} + K_{\hat{S}}))) \cong H^0(\hat{S}, \mathcal{O}_{\hat{S}}(-m\hat{D} - (m - 1)K_{\hat{S}})) = 0$, applying the Riemann-Roch theorem, we obtain

$$\dim H^0(\hat{S}, \mathcal{O}_{\hat{S}}(m(\hat{D} + K_{\hat{S}}))) \geq 1 \text{ for } m \geq 2.$$

Especially, we know $\bar{P}_2(S) \geq 1$ in this case.

Now we assume that $d = 2$. Put $D = \bar{\Delta}$. Corresponding to the morphism $\bar{f}|_D: D \rightarrow \bar{\Delta}$, we get the homomorphism $\psi: \bar{\Delta} \rightarrow \bar{\Delta}$ of 1-dimensional Abelian varieties. We denote the kernel of ψ by G . Let $\tilde{S} = \bar{S} \times_{\bar{\Delta}} \bar{\Delta}$ be the fiber product. Then we have the following com-

mutative diagram:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\Psi} & \bar{S} \\ \hat{f} \downarrow & & \downarrow \bar{f} \\ \tilde{D} & \xrightarrow{\psi} & \bar{D} \end{array}$$

Where $\Psi: \tilde{S} \rightarrow \bar{S}$ and $\bar{f}: \bar{S} \rightarrow \bar{D}$ are projections. Since $\psi: \tilde{D} \rightarrow \bar{D}$ is a 2-sheeted unramified Galois covering, so is $\Psi: \tilde{S} \rightarrow \bar{S}$. And we know that $\bar{f}: \bar{S} \rightarrow \bar{D}$ is also a ruled surface of genus 1. Let $\tilde{D}$ be a cross-section of $\bar{f}$ defined by $\tilde{D} = \{(a, a) | a \in D\}$. Then we have

$$\Psi^*D = \Psi^{-1}(D) = \{(a, a + \sigma) | \sigma \in G, a \in D\} = \sum_{\sigma \in G} \sigma(\tilde{D}),$$

and $\sigma(D) \cap \sigma'(D) = \emptyset$ if $\sigma \neq \sigma'$. Hence $\Psi^{-1}(D)$ consists of 2 connected components and each component is a nonsingular elliptic curve. Thus applying Lemma 2, we obtain

$$\dim H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\Psi^*D + K_{\tilde{S}})) = 1.$$

Since $\Psi^*D + K_{\tilde{S}} = \Psi^*(D + K_{\bar{S}})$, we can apply Lemma 3a) and obtain

$$\dim H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2(D + K_{\bar{S}}))) \geq 1.$$

Thus we complete the case that $\nu_i = 1$ for all i .

Now we assume that $\sum_{i=1}^r (\nu_i - 1) > 0$. It is obvious that $\nu_i \leq d$ for all i .

Suppose that there is ν_i such that $\nu_i = d$. Then $\hat{D}$ has a d -ple point p_0 . Let $\hat{F}_0$ be the fiber of $\hat{f}$ which contains p_0 . Since $(\hat{D}, \hat{F}_0) = d$, we know $\hat{D} \cap \hat{F}_0 = \{p_0\}$, and $\hat{D}$ and $\hat{F}_0$ have no common tangent line at p_0 . The quadratic transformation with center p_0 appears in $\mu = \mu_1 \circ \cdots \circ \mu_r$. We denote it by μ_0 . Let F'_0 and D' be proper transforms of $\hat{F}_0$ and $\hat{D}$ by μ_0 , respectively. Then we have $F'_0 \cap D' = \emptyset$ and $(F'_0)^2 = -1$. And the proper transform of F'_0 to $\bar{S}$ is also an exceptional curve which does not intersect with D . By Lemma 1, we may assume that there is not such a curve on $\bar{S}$. Thus we assume that $\nu_i < d$ for all i .

If $d = 2$, then we have $\nu_i = 1$ for all i . But this contradicts the assumption. Therefore we assume that $d \geq 3$.

Suppose that there is ν_i such that $\nu_i = d - 1$. Then $\hat{D}$ has a $(d - 1)$ -ple point p_0 . Let $\hat{F}_0$ be the fiber of $\hat{f}$ which contains p_0 . Let μ_0 be the

quadratic transformation with center p_0 , F'_0 and D' be the proper transforms of $\hat{F}_0$ and $\hat{D}$ by μ_0 , respectively. If $\hat{D} \cap \hat{F}_0 - \{p_0\} = \phi$, then $\hat{D}$ and $\hat{F}_0$ have no common tangent line at p_0 and intersect simply at some other point. If $\hat{D} \cap \hat{F}_0 = \{p_0\}$, then D' and F'_0 intersect simply at one point of $\mu_0^{-1}(p_0)$. In each case, we have $(F'_0, D') = 1$ and $(F'_0)^2 = -1$. Thus, there is an exceptional curve E on $\bar{S}$ such that $(E, D) = 1$. By Lemma 1, we may assume that there is no such curves on $\bar{S}$. Hence we reduce the problem to the case that $d \geq 3$ and $\nu_i \leq d - 2$ for all i .

Since $D + K_{\bar{S}} \sim \mu^*(\hat{D} + K_{\hat{S}}) - \sum_{i=1}^r (\nu_i - 1)E_i$, we have

$$(2) \quad \dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(m(D + K_{\bar{S}}))) \\ \cong \dim H^0(\hat{S}, \mathcal{O}_{\hat{S}}(m(\hat{D} + K_{\hat{S}}))) - \sum_{i=1}^r (1/2)m(\nu_i - 1)(m(\nu_i - 1) + 1).$$

On the other hand, by the Riemann-Roch theorem we have

$$(3) \quad \dim H^0(\hat{S}, \mathcal{O}_{\hat{S}}(m(D + K_{\hat{S}}))) \\ \cong (1/2)(m(\hat{D} + K_{\hat{S}}), m\hat{D} + (m - 1)K_{\hat{S}}) = m^2(d - 1)b - m(m - 1)b.$$

Combining (2) and (3), we calculate

$$\begin{aligned} & \dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(m(D + K_{\bar{S}}))) \\ & \cong m^2(d - 1)b - m(m - 1)b - m^2 \sum_{i=1}^r (1/2)\nu_i(\nu_i - 1) \\ & \quad + m(m - 1) \sum_{i=1}^r (1/2) \times (\nu_i - 1) \\ & = m(m - 1) \left(\sum_{i=1}^r (1/2)(\nu_i - 1) - b \right) \\ & = (1/2)m(m - 1) \left(\sum_{i=1}^r (\nu_i - 1) - (d - 1)^{-1} \sum_{i=1}^r \nu_i(\nu_i - 1) \right) \\ & = (1/2)m(m - 1) \sum_{i=1}^r (1 - (\nu_i/(d - 1)))(\nu_i - 1) \\ & \cong (1/2)m(m - 1)(d - 1)^{-1} \sum_{i=1}^r (\nu_i - 1) \\ & \cong (1/2)m(m - 1)(d - 1)^{-1}. \end{aligned}$$

Especially we get $\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(2(D + K_{\bar{S}}))) \cong (d - 1)^{-1} > 0$. Therefore we have $\bar{P}_2(S) \geq 1$. Q.E.D.

§4. The case $\kappa(\bar{C}_w) = -\infty$ and $\kappa(\bar{\Delta}) = 1$

PROPOSITION 3: *If $\bar{\kappa}(C_w) \geq 0$, $\bar{\kappa}(\bar{\Delta}) \geq 0$, $\kappa(\bar{C}_w) = -\infty$ and $\kappa(\bar{\Delta}) = 1$, then $\bar{p}_g(S) \geq 1$ and $\bar{P}_2 \geq 2$.*

PROOF: Let $D = \sum_{i=1}^r C_i$ be the irreducible decomposition. From the assumption we know $(\bar{C}_w, D) \geq 2$. We may assume that all C_i are horizontal with respect to $\bar{f}$. Since $g(C_i) \geq g(\bar{\Delta}) = q(\bar{S}) \geq 2$, by virtue of Lemma 2 we have $\bar{p}_g(S) \geq \sum_{i=1}^r g(C_i) - q(\bar{S}) \geq (r-1)q(\bar{S})$. Thus if $r \geq 2$, we have $\bar{p}_g(S) \geq 2$. Hence we assume that $r = 1$. Put $\bar{f}|_{C_1} = \psi$. Applying Hurwitz' formula to $\psi: C_1 \rightarrow \bar{\Delta}$, we get

$$g(C_1) = (\deg \psi)(q(\bar{S}) - 1) + 1 + (1/2) \sum_{p \in C_1} (e(p) - 1),$$

where $e(p)$ is the ramification index of ψ at p . Since $\deg \psi = (C_1, \bar{C}_w) = (D, \bar{C}_w) \geq 2$, we see

$$g(C_1) \geq 2(q(\bar{S}) - 1) + 1 \geq q(\bar{S}) + 1.$$

Hence $\bar{p}_g(S) \geq g(C_1) - q(\bar{S}) \geq 1$, and if $\bar{p}_g(S) = 1$, then we have $q(\bar{S}) = 2$, $\deg \psi = 2$, $\sum_{p \in C_1} (e(p) - 1) = 0$ and $g(C_1) = 3$. Thus if $\bar{p}_g(S) = 1$, then $\psi: C_1 \rightarrow \bar{\Delta}$ is a 2-sheeted unramified covering. Let $\tilde{S} = \bar{S} \times_{\bar{\Delta}} C_1$ be the fiber product. Let $\Psi: \tilde{S} \rightarrow \bar{S}$ and $\tilde{f}: \tilde{S} \rightarrow C_1$ denote the first and the second projections respectively. Then Ψ is a 2-sheeted unramified covering and $q(\tilde{S}) = g(C_1) = 3$. Put $\Psi^{-1}(C_1) = \sum_{i=1}^s C'_i$ where the C'_i are irreducible components. Applying Hurwitz' formula to $\Psi|_{C'_i}: C'_i \rightarrow C_1$, we get

$$g(C'_i) \geq (\deg(\Psi|_{C'_i}))(3 - 1) + 1 = 2(\deg(\Psi|_{C'_i})) + 1.$$

Thus by Lemma 2 we have

$$\begin{aligned} \dim H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\Psi^{-1}(C_1) + K_{\tilde{S}})) &\geq \sum_{i=1}^s g(C'_i) - q(\tilde{S}) \\ &\geq 2 \sum_{i=1}^s \deg(\Psi|_{C'_i}) + s - 3 = 2 \deg \psi + s - 3 = s + 1 \geq 2. \end{aligned}$$

Since $\Psi^*(D + K_{\bar{S}}) \geq \Psi^{-1}(C_1) + \Psi^*K_{\bar{S}} = \Psi^{-1}C_1 + K_{\tilde{S}}$, applying Lemma 3b), we infer that $\bar{P}_2(S) \geq 2$. Q.E.D.

REMARK: $\bar{p}_g(S) \geq 1$ was proved by Miyanishi-Sugie in [12; §2].

§5. The case $\kappa(\bar{C}_w) \geq 0$, $\kappa(\bar{\Delta}) = -\infty$ and $\bar{S}$ is an irrational ruled surface

PROPOSITION 4: *If $\bar{\kappa}(C_w) \geq 0$, $\bar{\kappa}(\Delta) \geq 0$, $\kappa(\bar{C}_w) \geq 0$, $\kappa(\bar{\Delta}) = -\infty$, $\bar{S}$ is ruled and $q(\bar{S}) \geq 1$, then $\bar{p}_g(S) \geq 1$.*

PROOF: Let $\alpha: \bar{S} \rightarrow B: = \alpha(\bar{S}) \hookrightarrow \text{Alb}(\bar{S})$ be the Albanese mapping of $\bar{S}$, then B is a nonsingular curve of genus $q(\bar{S})$, and the ruling of $\bar{S}$ is given by α . By the assumption, we know that $D = \bar{S} - S$ contains two reduced fibers of $\bar{f}$ which we denote by F_1 and F_2 . For a general point p of $\bar{S}$, the fiber of $\bar{f}$ which contains p is a nonsingular curve of genus greater than 0 which we denote by C_1 , and the fiber of α which contains p is a nonsingular rational curve which we denote by C_2 . Then $C_1 \neq C_2$ and $C_1 \cap C_2 \neq \emptyset$, hence $(C_1, C_2) \geq 1$. Therefore for any $u \in B$ and $w \in \bar{\Delta}$, we have $(\bar{f}^*(w), \alpha^*(u)) \geq 1$. Especially, F_1 and F_2 contain horizontal components with respect to α which we denote by H_1 and H_2 respectively. Then from Lemma 2 we infer that $\bar{p}_g(S) \geq g(H_1) + g(H_2) - q(\bar{S}) \geq q(\bar{S}) \geq 1$. Q.E.D.

§6. The case $\kappa(\bar{C}_w) = 0$, $\kappa(\bar{\Delta}) = -\infty$, and $\bar{S}$ is rational

PROPOSITION 5: *If $\bar{\kappa}(C_w) \geq 0$, $\bar{\kappa}(\Delta) \geq 0$, $\kappa(\bar{C}_w) = 0$, $\kappa(\bar{\Delta}) = -\infty$ and $\bar{S}$ is rational, then $\bar{P}_3(S) \geq 1$ or $\bar{P}_4(S) \geq 1$.*

PROOF: By assumption, D contains two reduced fibers of $\bar{f}$ which we denote by F_1 and F_2 . We may assume $D = F_1 + F_2$. We contract all exceptional curves contained in fibers of $\bar{f}$ and then we have the following commutative diagram:

$$\begin{array}{ccc} \bar{S} & \xrightarrow{\mu} & \hat{S} \\ \bar{f} \searrow & & \downarrow \hat{f} \\ & & \bar{\Delta} \end{array}$$

where $\hat{f}: \hat{S} \rightarrow \bar{\Delta}$ is a rational elliptic surface which has no exceptional curves contained in fibers and $\mu: \bar{S} \rightarrow \hat{S}$ is composed of a finite number of quadratic transformations. Let $\hat{F}_1$ and $\hat{F}_2$ be fibers of $\hat{f}$ such that $F_1 = \text{supp } \mu^* \hat{F}_1$ and $F_2 = \text{supp } \mu^* \hat{F}_2$. Then by the canonical bundle formula [10], we know

$$K_{\hat{S}} \sim -\hat{F} + \sum_{\nu} (m_{\nu} - 1)P_{\nu},$$

where the $m_\nu P_\nu$'s are all multiple fibers of $\hat{f}$ and $\hat{F}$ is a general fiber of $\hat{f}$. Since $\hat{S}$ has an exceptional curve E which is horizontal with respect to $\hat{f}$, we have

$$\begin{aligned} -1 &= (E, K_{\hat{S}}) = -(E, \hat{F}) + \sum_{\nu} (m_{\nu} - 1)(E, P_{\nu}) \\ &= \left(-1 + \sum_{\nu} (1 - m_{\nu}^{-1}) \right) (E, \hat{F}). \end{aligned}$$

Hence $\hat{f}: \hat{S} \rightarrow \bar{\Delta}$ has at most one multiple fiber which we denote by $m_0 P_0$. Putting $m_0 = 1$ if $\hat{f}$ is free from multiple fibers, we have

$$K_{\hat{S}} \sim -P_0.$$

Now we use the classification of singular fibers of elliptic surfaces by Kodaira [11]. Since $p_g(\hat{S}) = q(\hat{S}) = 0$, we have

$$\begin{aligned} (4) \quad 12 &= \sum_b b\nu(I_b) + \sum_b (6+b)\nu(I_b^*) + 2\nu(II) + 10\nu(II^*) \\ &\quad + 9\nu(III^*) + 4\nu(IV) + 8\nu(IV^*), \end{aligned}$$

where $\nu(T)$ is the number of the singular fibers of $\hat{S}$ of type T .

LEMMA 4: If $\hat{F}_i$ is a singular fiber of type I_b or I_b^* or II or III or IV , then $|K_{\bar{S}} + 2F_i| \neq \phi$.

PROOF: If $\hat{F}_i$ is of type I_b , by Lemma 2 we have $|K_{\bar{S}} + F_i| \neq \phi$. If $\hat{F}_i$ is of type I_b^* , then $2 \operatorname{supp} \hat{F}_i \cong \hat{F}_i$ and we may assume $\mu^* \hat{F}_i$ contains no exceptional curves. Hence we have

$$K_{\bar{S}} + 2F_i \cong \mu^* K_{\hat{S}} + \mu^* \hat{F}_i \cong 0.$$

If $\hat{F}_i$ is of type II i.e. a rational curve with one cusp, then F_i contains an exceptional curve E and nonsingular rational curves C_1, C_2 and C_3 such that $(C_1, E) = (C_2, E) = (C_3, E) = 1$. Let $\mu': \bar{S} \rightarrow \bar{S}'$ be the contraction of E and denote $\mu'_*(C_j)$ by C'_j for $j = 1, 2, 3$. Then we infer that

$$K_{\bar{S}} + 2F_i \cong K_{\bar{S}} - E + C_1 + C_2 + C_3 + 3E = \mu'^*(K_{\bar{S}'} + C'_1 + C'_2 + C'_3).$$

And by the Riemann-Roch theorem we have

$$\begin{aligned} & \dim H^0(\bar{S}', \mathcal{O}_{\bar{S}'}(K_{\bar{S}'} + C'_1 + C'_2 + C'_3)) \\ & \geq (1/2)(K_{\bar{S}'} + C'_1 + C'_2 + C'_3, C'_1 + C'_2 + C'_3) + 1 = 1. \end{aligned}$$

Thus we have $\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(K_{\bar{S}} + 2F_i)) \geq 1$. If $\hat{F}_i$ is of type III i.e. two nonsingular rational curves intersecting at one point with multiplicity two, or type IV i.e. three nonsingular rational curves intersecting one point, then F_i contains also an exceptional curve E and nonsingular rational curves C_1 , C_2 and C_3 such that $(C_1, E) = (C_2, E) = (C_3, E) = 1$. Hence similarly we have

$$\dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(K_{\bar{S}} + 2F_i)) \geq 1. \quad \text{Q.E.D.}$$

By Lemma 4 we infer that if both $\hat{F}_1$ and $\hat{F}_2$ are of type I_b or I_b^* or II or IV, then $|2(K_{\bar{S}} + F_1 + F_2)| \supset |K_{\bar{S}} + 2F_1| + |K_{\bar{S}} + 2F_2| \neq \phi$, and hence $\bar{P}_2(S) \geq 1$.

Now we assume that one of $\hat{F}_i$ is of type II^* or III^* or IV^* . By the assumption, the functional invariant of $\hat{S}$ (if $\hat{S}$ has a multiple fiber, the functional invariant of the corresponding elliptic surface free from multiple fibers) is not constant. Therefore we know that $\sum_b (\nu(I_b) + \nu(I_b^*)) \geq 1$. Hence if $\nu(II^*) \geq 1$, then we infer from (4) that $\sum_b b\nu(I_b) = 2$ and remaining $\nu(T)$ are 0. Thus one of $\hat{F}_i$ is of type I_b and therefore by Lemma 2 we have $\bar{p}_g(S) \geq 1$. Thus we assume that one of $\hat{F}_i$ is of type III^* or IV^* .

Case 1. First we consider the case that one of $\hat{F}_i$ is of type III^* . We assume that $\hat{F}_1$ is of type III^* . We may assume that $\hat{F}_2$ is not of type I_b by Lemma 2. Then from (4) we know that F_2 is of type II. We put

$$\begin{aligned} \hat{F}_1 &= \Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 3\Theta_4 + 2\Theta_5 + 2\Theta_6 + \Theta_7, (\Theta_0, \Theta_1) = (\Theta_1, \Theta_2) \\ &= (\Theta_2, \Theta_3) = (\Theta_3, \Theta_5) = (\Theta_3, \Theta_4) = (\Theta_4, \Theta_6) = (\Theta_6, \Theta_7) = 1, \\ \hat{F}_2 &= C, \end{aligned}$$

where the Θ_i are nonsingular rational curves with self-intersection number -2 and C is a rational curve with one cusp. Since $\mu: \bar{S} \rightarrow \hat{S}$ gives rise to an isomorphism on a neighborhood of F_1 , we have

$$F_1 = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 + \Theta_6 + \Theta_7,$$

where we denote $\mu^*\Theta_i$ by the same letter Θ_i . On the other hand we

have

$$\mu^* \hat{F}_2 = C''' + 2E_1'' + 3E_2' + 6E_3,$$

where $\mu = \mu_1 \circ \mu_2 \circ \mu_3$ and C''' is the proper transform of C by μ and E_i are the exceptional curves of the quadratic transformations μ_i and E_1'' and E_2' are the proper transforms of E_1 and E_2 to $\bar{S}$, respectively. We denote the total transform of E_i by the same letter E_i . Then we have

$$F_2 = C''' + E_1'' + E_2' + E_3.$$

Since $m_0 P_0 \sim C$, we infer that

$$K_{\bar{S}} + F_2 + E_3 \sim \mu^* K_{\bar{S}} + E_1 + E_2 + E_3 + F_2 + E_3 \sim \mu^* (K_{\bar{S}} + C) \geq 0.$$

On the other hand,

$$\begin{aligned} 4F_1 &= \mu^* F_1 + 3\Theta_0 + 2\Theta_1 + \Theta_2 + \Theta_4 + 2\Theta_5 + 2\Theta_6 + 3\Theta_7 \\ &\sim \mu^* C + 3\Theta_0 + 2\Theta_1 + \Theta_2 + \Theta_4 + 2\Theta_5 + 2\Theta_6 + 3\Theta_7. \end{aligned}$$

Hence we have

$$\begin{aligned} 4(K_{\bar{S}} + F_1 + F_2) &= 4(K_{\bar{S}} + F_2) + 4F_1 \geq -4E_3 + \mu^* C \\ &= C''' + 2E_1'' + 3E_2' + 2E_3 \geq 0. \end{aligned}$$

Therefore we get $\bar{P}_4(S) \geq 1$.

Case 2. Now we consider the case that one of F_i is of type IV*. We assume F_1 is of type IV*. We may assume F_2 is not of type I_b. Then from (4) we know that F_2 is of type II or III. We put

$$\begin{aligned} \hat{F}_1 &= \Theta_0 + 2\Theta_1 + 3\Theta_3 + 2\Theta_4 + \Theta_5 + \Theta_6, \\ (\Theta_0, \Theta_1) &= (\Theta_1, \Theta_2) = (\Theta_2, \Theta_3) \\ &= (\Theta_2, \Theta_4) = (\Theta_3, \Theta_5) = (\Theta_4, \Theta_6) = 1, \end{aligned}$$

where the Θ_i are nonsingular rational curves with self-intersection number -2 . Since μ gives rise to an isomorphism on a neighborhood of F_1 , we denote $\mu^* \Theta_i$ by the same letter Θ_i .

Case 2.1: First we consider the case that $\hat{F}_2$ is of type II. Then $\hat{F}_2$ and F_2 are the same as in the case 1. Quite similarly we have

$K_{\bar{S}} + F_2 + E_3 \geq 0$. On the other hand,

$$\begin{aligned} 3F_1 &= \mu^* \hat{F}_1 + 2\Theta_0 + \Theta_1 + \Theta_3 + \Theta_4 + 2\Theta_5 + 2\Theta_6 \\ &\sim \mu^* C + 2\Theta_0 + \Theta_1 + \Theta_3 + \Theta_4 + 2\Theta_5 + 2\Theta_6. \end{aligned}$$

Hence we have

$$\begin{aligned} 3(K_{\bar{S}} + F_1 + F_2) &= 3(K_{\bar{S}} + F_2) + 3F_1 \geq -3E_3 + \mu^* C \\ &= C''' + 2E_1'' + 3E_2' + 3E_3 \geq 0. \end{aligned}$$

Therefore we get $\bar{P}_3(S) \geq 1$.

Case 2.2: Now we consider the case where $\hat{F}_2$ is of type III. Then we have $\hat{F}_2 = C_1 + C_2$, where C_1 and C_2 are nonsingular rational curves, $C_1 \cap C_2$ is one point and $(C_1, C_2) = 2$. We put $C_1 \cap C_2 = \{p\}$. Let $\mu_1: \bar{S}_1 \rightarrow \hat{S}$ be the quadratic transformation with center p . We denote the exceptional curve of μ_1 by E_1 and the proper transforms of C_1 and C_2 by C_1' and C_2' respectively. Let $\mu_2: \bar{S}_2 \rightarrow \bar{S}_1$ be the quadratic transformation with center $C_1' \cap C_2'$. We denote the exceptional curve of μ_2 by E_2 and the proper transforms of C_1' , C_2' and E_1 by C_1'' , C_2'' and E_1' respectively. Then we may assume that $\bar{S} = \bar{S}_2$ and $\mu = \mu_1 \circ \mu_2$. And we have

$$\mu^* \hat{F}_2 = C_1'' + C_2'' + 2E_1' + 4E_2.$$

Hence

$$F_2 = C_1'' + C_2'' + E_1' + E_2.$$

Thus we infer that

$$\begin{aligned} K_{\bar{S}} + F_2 &= \mu^* K_{\bar{S}} + E_1 + E_2 + C_1'' + C_2'' + E_1' + E_2 \\ &= \mu^* K_{\bar{S}} + \mu^* \hat{F}_2 - E_2 \geq -E_2. \end{aligned}$$

Hence we have

$$\begin{aligned} 3(K_{\bar{S}} + F_1 + F_2) &= 3(K_{\bar{S}} + F_2) + 3F_1 \geq -3E_2 + C_1'' + C_2'' + 2E_1' + 4E_2 \\ &= C_1'' + C_2'' + 2E_1' + E_2 \geq 0. \end{aligned}$$

Therefore we obtain $\bar{P}_3(S) \geq 1$.

Q.E.D.

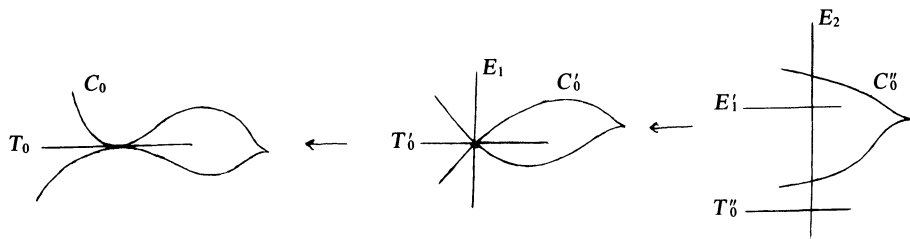


Figure 1.

REMARK: We can construct rational elliptic surfaces free from multiple fiber of case 1, case 2.1 and case 2.2 as follows.

Construction of case 1

Let C_0 be a quartic curve in $\mathbb{P}^2$ defined by

$$Z^2 Y^2 - X^4 = X^3 Y,$$

where $(X : Y : Z)$ is a homogenous coordinate in $\mathbb{P}^2$. Then C_0 has a tacnode at $(0:0:1)$ and a cusp at $(0:1:0)$. Let T_0 be the tangent line of C_0 at $(0:0:1)$. We can resolve the singularity of C_0 at $(0:0:1)$ by the quadratic transformations as Figure 1.

Then we know T'_0 is an exceptional curve and $T'_0 \cap C'_0 = \phi$. We contract T'_0 and denote the resulting surface by $\hat{S}_0$. Let $\hat{E}_2$ be the direct image of E_2 . Then we have $(\hat{E}_2)^2 = 0$, $(E'_1)^2 = -2$ and $C''_0 \sim 4\hat{E}_2 + 2E'_1$. Hence $\hat{S}_0 \cong \Sigma_2$ and $C''_0 \in |2M + 4l|$, where M is the minimal section and l is a fiber. We put $\hat{E}_2 = l_0$ and $C''_0 \cap l_0 = \{p, q\}$. We perform quadratic transformations at p and q as Figure 2.

Let $\mu : \hat{S} \rightarrow \hat{S}_0$ be the composition of these quadratic transformations. Let C and Θ_3 be the proper transform of C''_0 and l_0 by μ , respectively. Put $E_{13} - E_{14} = \Theta_0$, $E_{23} - E_{24} = \Theta_7$, $E_{12} - E_{13} = \Theta_1$, $E_{22} - E_{23} = \Theta_6$, $E_{11} - E_{12} = \Theta_2$, $E_{21} - E_{22} = \Theta_4$ and $M = \Theta_5$, then the Θ_i are nonsingular rational curves with self-intersection number -2 . Put

$$\hat{F}_1 = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 3\Theta_4 + 2\Theta_5 + 2\Theta_6 + \Theta_7, \hat{F}_2 = C.$$

Then $\hat{F}_1 \sim \hat{F}_2$ and $\Phi_{|C|} : \hat{S} \rightarrow \mathbb{P}^1$ is a rational elliptic surface with singular fibers of type III* and of type II. Since E_{14} and E_{24} are cross-sections of $\Phi_{|C|}$, $\hat{S}$ is free from multiple fibers. We put $S = \hat{S} - \text{supp } \hat{F}_1 - \hat{F}_2$. Then we infer that $\bar{P}_2(S) = \bar{P}_3(S) = 0$, $\bar{P}_4(S) = 1$ and $\bar{P}_{12}(S) = 2$.

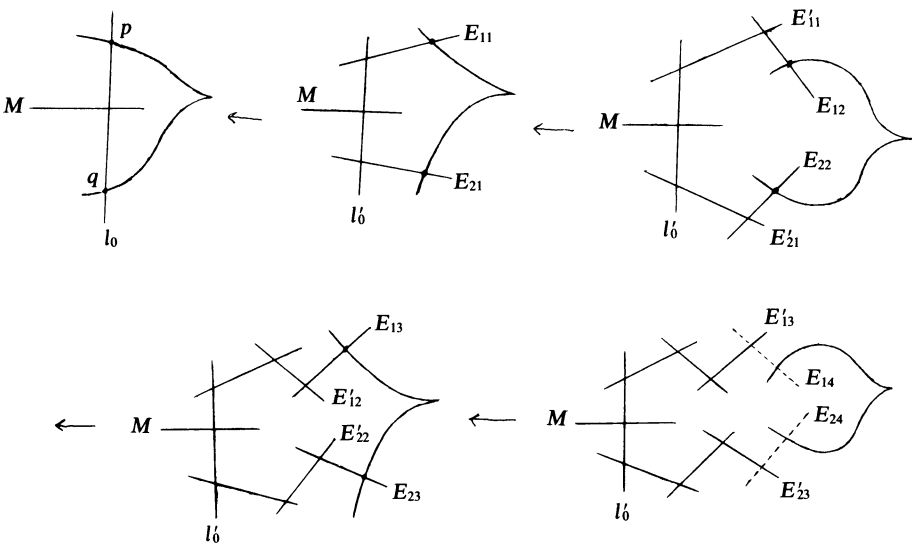


Figure 2.

Construction of case 2.2

Let C be a conic and l_1 a line in $\mathbb{P}^2$ such that C and l_1 intersect at one point q with multiplicity 2. Let l_0 be a line which does not contain q and intersects with C simply at two point p_2 and p_3 . Put $l_0 \cap l_1 = p_1$. We perform quadratic transformations at p_1 , p_2 and p_3 as follows:

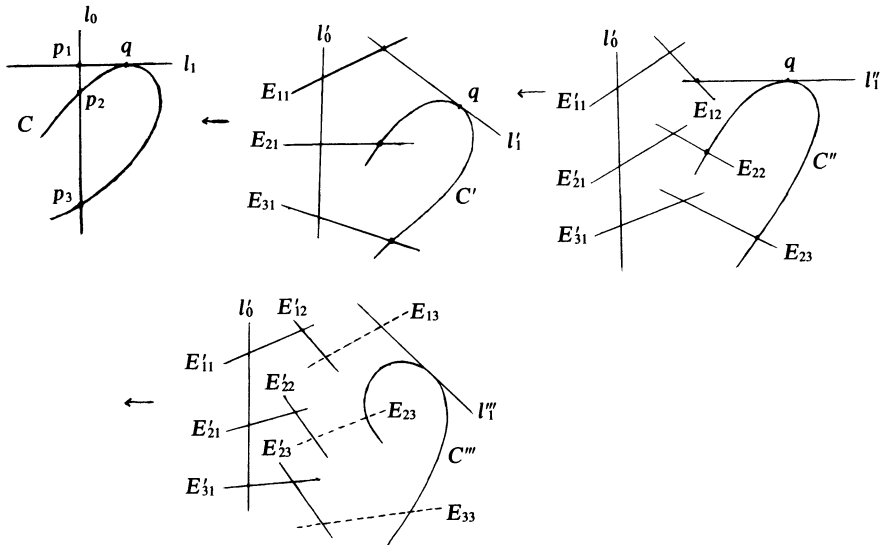


Figure 3.

Let $\mu: \hat{S} \rightarrow \mathbb{P}^2$ be the composition of these quadratic transformations. We denote the proper transforms of l_0 , l_1 and C by l'_0 , l'_1 and C''' . We put $l'_0 = \Theta_2$, $E_{12} - E_{13} = \Theta_0$, $E_{11} - E_{12} = \Theta_1$, $E_{22} - E_{23} = \Theta_5$, $E_{21} - E_{22} = \Theta_3$, $E_{32} - E_{33} = \Theta_6$ and $E_{31} - E_{32} = \Theta_4$. And we put

$$\hat{F}_1 = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 2\Theta_3 + 2\Theta_4 + \Theta_5 + \Theta_6, \hat{F}_2 = l'_1 + C'''.$$

Then $\hat{F}_1 \sim \hat{F}_2$ and $\Phi_{|F_1|}: \hat{S} \rightarrow \mathbb{P}^1$ is a rational elliptic surface free from multiple fibers and with singular fibers of type IV* and of type III. Put $S = \hat{S} - \text{supp } \hat{F}_1 - \hat{F}_2$. Then we infer that $\bar{P}_2(S) = 0$, $\bar{P}_3(S) = 1$ and $\bar{P}_{12}(S) = 2$.

Construction of case 2.1

In the above construction of case 2.2, we replace $C + l_1$ by a cubic curve with one cusp and the remaining part goes quite similarly. Then $\bar{P}_2(S) = 0$, $\bar{P}_3(S) = 1$ and $\bar{P}_6(S) = 2$.

§7. The case $\kappa(\bar{C}_w) = 1$, $\kappa(\bar{\Delta}) = -\infty$ and $\bar{S}$ is rational

PROPOSITION 6: If $\kappa(\bar{C}_w) = 1$, $\bar{\kappa}(\Delta) \geq 0$, $\kappa(\bar{\Delta}) = -\infty$ and $\bar{S}$ is rational, then $\bar{P}_4(S) \geq 1$ or $\bar{P}_6(S) \geq 1$.

PROOF: From the addition formula ([6]), we know $\bar{\kappa}(S) \geq 1$. By assumption D contains two reduced fibers of $\bar{f}$ which we denote by F_1 and F_2 . Hence the intersection matrix of D is not negative definite. Therefore if $\bar{\kappa}(S) = 2$, then by Theorem 3 $\bar{P}_4(S) \geq 1$ or $\bar{P}_6(S) \geq 1$. Thus we assume that $\bar{\kappa}(S) = 1$. Then by [7; (2.3)–(2.8)] we have a fibering $g: \bar{S} \rightarrow \mathbb{P}^1$ such that a general fiber of $g|_S$ is a C^* or an elliptic curve. Since the genus of $\bar{C}_w$ is greater than 1, F_1 and F_2 contain horizontal components with respect to g . Therefore a general fiber of $g|_S$ is a C^* . Hence using [7; (2.6) and (2.8)], we reduce the problem to the following:

Under the condition that $-2 + \sum_i (1 - m_i^{-1}) > 0$, look for the integer m such that $-2m + \sum_i [m(1 - m_i^{-1})] \geq 0$, where m_i are integers or ∞ and $[\]$ is the integral part.

We infer that for any such m_i , $m = 4$ or 6 is sufficient.

Q.E.D.

§8. Proof of Theorem 3

First we note that when we contract an exceptional curve E on $\bar{S}$ such that $(E, K_{\bar{S}} + D) \leq 0$, the condition on the intersection matrix of D is also preserved. Hence by Lemma 1 we may assume that there is no such exceptional curve on $\bar{S}$. By Proposition 2, 3, 4 we may assume that $\bar{S}$ is rational and by Lemma 2 we may assume that each connected component of D is a tree of nonsingular rational curves.

Let $K_{\bar{S}} + D = (K_{\bar{S}} + D)^+ + (K_{\bar{S}} + D)^-$ be the Zariski decomposition of $\mathbb{Q}$ -divisor (c.f. [7]). We put $(K_{\bar{S}} + D)^- = \sum_{i=1}^t a_i C_i$ where the a_i are rational numbers such that $0 < a_i \leq 1$ and the C_i are irreducible curves. Suppose that $C_i \not\subset D$ for $1 \leq i \leq t$ and $C_i \subset D$ for $t+1 \leq i \leq r$. If $(K_{\bar{S}}, C_i) \geq 0$ for $1 \leq i \leq t$, then

$$\begin{aligned} \left(\sum_{i=1}^t a_i C_i, \sum_{i=1}^t a_i C_i \right) &= (K_{\bar{S}} + D - (K_{\bar{S}} + D)^+ - \sum_{i=t+1}^r a_i C_i, \sum_{i=1}^t a_i C_i) \\ &= \left(K_{\bar{S}} + D - \sum_{i=t+1}^r a_i C_i, \sum_{i=1}^t a_i C_i \right) \geq \left(K_{\bar{S}}, \sum_{i=1}^t a_i C_i \right) \geq 0. \end{aligned}$$

This contradicts to the negative definiteness of $(K_{\bar{S}} + D)^-$. Hence $(K_{\bar{S}}, C_{i_0}) < 0$ for some i_0 such that $1 \leq i_0 \leq t$. Since $(C_{i_0}, C_{i_0}) < 0$, C_{i_0} is an exceptional curve such that $(K_{\bar{S}} + D, C_{i_0}) \leq 0$, which is a contradiction. Therefore $C_i \subset D$ for $1 \leq i \leq r$. Thus we have $(K_{\bar{S}} + D)^+ = K_{\bar{S}} + D_m$, where $D_m = D - (K_{\bar{S}} + D)^-$ is an effective $\mathbb{Q}$ -divisor. Since the intersection matrix of D is not negative definite, there are some irreducible components of D which don't occur in $(K_{\bar{S}} + D)^-$. Hence the part of D_m with coefficient 1 is an effective integral divisor which we denote by D_0 . Then we have

$$D_m = D_0 + \sum_{i=1}^s d_i C_i,$$

where the C_i are irreducible curves and the d_i are rational number such that $0 < d_i < 1$. It is easy to see that if C_i intersects with D_0 , $d_i = 1 - m_i^{-1}$ where m_i is a positive integer. Since $n(K_{\bar{S}} + D) \geq nK_{\bar{S}} - [-(n-1)D_m] + D_0 \geq nK_{\bar{S}} + [nD_m]$, we have $\bar{P}_n(S) = \dim H^0(\bar{S}, \mathcal{O}_{\bar{S}}(nK_{\bar{S}} - [-(n-1)D_m] + D_0))$ for $n \geq 2$, where $[\]$ is the integral part of a $\mathbb{Q}$ -divisor. Since $\dim H^2(\bar{S}, \mathcal{O}_{\bar{S}}(nK_{\bar{S}} - [-(n-1)D_m] + D_0)) = 0$, by the Riemann-Roch theorem we have

$$\begin{aligned} \bar{P}_n(S) &\geq (1/2)(nK_{\bar{S}} - [-(n-1)D_m], (n-1)K_{\bar{S}} - [-(n-1)D_m]) + 1 \\ &\quad + (1/2)(D_0, (2n-1)K_{\bar{S}} - 2[-(n-1)D_m] + D_0) \text{ for } n \geq 2. \end{aligned}$$

By Kawamata's vanishing theorem [8] we have

$$\begin{aligned} & H^1(\bar{S}, \mathcal{O}_{\bar{S}}(nK_{\bar{S}} - [-(n-1)D_m])) \\ & \cong H^1(\bar{S}, \mathcal{O}_{\bar{S}}(-(n-1)K_{\bar{S}} + [-(n-1)D_m])) \cong 0 \end{aligned}$$

for $n \geq 2$. Hence we have

$$(5) \quad \bar{P}_n(S) \geq (1/2)(D_0, (2n-1)K_{\bar{S}} - 2[-(n-1)D_m] + D_0) \text{ for } n \geq 2.$$

Suppose that $\bar{P}_2(S) = 0$. Then we have $(D_0, 3K_{\bar{S}} - 2[-D_m] + D_0) \leq 0$. Hence $3(D_0, K_{\bar{S}} + D_0) + 2(D_0, \sum_{i=1}^s C_i) \leq 0$. Let D_{0j} , $j = 1, \dots, u$ be connected components of D_0 . Then we have $(D_0, K_{\bar{S}} + D_0) = -2u$. Thus we have

$$(6) \quad \left(D_0, \sum_{i=1}^s C_i \right) \leq 3u.$$

On the other hand, from $(K_{\bar{S}} + D_m, D_{0j}) \geq 0$ we know

$$(K_{\bar{S}} + D_{0j}, D_{0j}) + \left(\sum_{i=1}^s d_i C_i, D_{0j} \right) + \left(\sum_{i \neq j} D_{0i}, D_{0j} \right) \geq 0.$$

Therefore we have

$$(7) \quad \left(\sum_{i=1}^s d_i C_i, D_{0j} \right) \geq 2 \text{ for } 1 \leq j \leq u.$$

Since $0 < d_i < 1$, we obtain

$$(8) \quad \left(\sum_{i=1}^s C_i, D_{0j} \right) \geq 3 \text{ for } 1 \leq j \leq u.$$

From (6) and (8) we have

$$(9) \quad \left(\sum_{i=1}^s C_i, D_{0j} \right) = 3 \text{ for } 1 \leq j \leq u.$$

Since every connected components of D is a tree of nonsingular rational curves, we know from (9) that 3 irreducible components of $\sum_{i=1}^s C_i$ intersect with D_{0j} for $1 \leq j \leq u$. We denote the coefficients of these 3 components in D_m by $1 - a_j^{-1}$, $1 - b_j^{-1}$ and $1 - c_j^{-1}$, where a_j , b_j and c_j are positive integers such that $a_j \leq b_j \leq c_j$.

Suppose moreover that $\bar{P}_3(S) = 0$. Then from (5) we infer that

$$-\sum_{j=1}^u ([-2(1 - a_j^{-1})] + [-2(1 - b_j^{-1})] + [-2(1 - c_j^{-1})]) \leq 5u.$$

Hence for some j_0 we have

$$-[-2(1 - a_{j_0}^{-1})] - [-2(1 - b_{j_0}^{-1})] - [-2(1 - c_{j_0}^{-1})] \leq 5$$

Since we have $a_j^{-1} + b_j^{-1} + c_j^{-1} \leq 1$ from (7), we get $a_{j_0} = 2$, $b_{j_0} \geq 3$ and $c_{j_0} \geq 3$, and therefore we have

$$-[-2(1 - a_{j_0}^{-1})] - [-2(1 - b_{j_0}^{-1})] - [-2(1 - c_{j_0}^{-1})] = 5.$$

Hence we have

$$-\sum_{j \neq j_0} ([-2(1 - a_j^{-1})] + [-2(1 - b_j^{-1})] + [-2(1 - c_j^{-1})]) \leq 5(u - 1).$$

Thus we deduce that

$$a_j = 2, b_j \geq 3 \text{ and } c_j \geq 3 \text{ for } 1 \leq j \leq u.$$

Suppose moreover that $\bar{P}_4(S) = 0$. Then we can deduce by similar way that $b_j = 3$ and $c_j \geq 6$ for $1 \leq j \leq u$. Then we have

$$\begin{aligned} \bar{P}_6(S) &\geq (1/2)(D_0, 11K_{\bar{S}} - 2[-D_m] + D_0) \\ &= (11/2)(D_0, K_{\bar{S}} + D_0) + \sum_{j=1}^u (3 + 4 - [-5(1 - c_j^{-1})]) \\ &= -11u + \sum_{j=1}^u (3 + 4 + 5) = u \geq 1. \end{aligned} \quad \text{Q.E.D.}$$

§9. Conclusion

By Proposition 1 ~ 6, we complete the proof of Theorem 1.

PROOF OF THEOREM 2: Let $\alpha_S: S \rightarrow B \subset \mathcal{A}_S$ be the quasi-Albanese mapping where B is the closure of $\alpha_S(S)$ in the quasi-Albanese variety $\mathcal{A}_S$. Then by the assumption we may assume that $\dim B = 1$. Then by the property of the quasi-Albanese mapping ([3]), $\bar{\kappa}(B) \geq 0$

and a general fiber of α_S is irreducible. Since $\bar{\kappa}(S) \geq 0$, we have $\bar{\kappa}$ (a general fiber of α_S) ≥ 0 . Thus we can apply Theorem 1 to $\alpha_S: S \rightarrow B$.

Q.E.D.

PROOF OF COROLLARY TO THEOREM 1: We have a fibering $\varphi: S \rightarrow \mathbb{C}^*$. Let C_u be the affine curve defined by $\varphi = u$. Then by Theorem 1 we get $\bar{\kappa}(C_u) = -\infty$ for a general $u \in \mathbb{C}^*$. Since C_u is affine, we have $C_u \cong \mathbb{A}^1$. Thus by [4; Theorem 9.7] we complete the proof.

Q.E.D.

REMARK: We can prove more sharpened result apart from Theorem 1 as follows. But to do this the deeper result [1] is necessary.

PROPOSITION 7: Let $S = \mathbb{A}^2 - V(\varphi)$ where $\varphi \in \mathbb{C}[x, y]$ and φ is irreducible. If $\bar{P}_2(S) = 0$, then $S \cong \mathbb{A}^1 \times \mathbb{C}^*$.

PROOF: Let C be the closure of $V(\varphi)$ in $\mathbb{P}^2$ and $H_\infty = \mathbb{P}^2 - \mathbb{A}^2$. Then $S = \mathbb{P}^2 - (C \cup H_\infty)$. We put $C \cup H_\infty = D$. If there are at least two points at which D is not normal crossing, it is easy to see that $\bar{P}_2(S) \geq 1$. Hence C has at most one cusp. We may assume $\deg C \geq 2$. Since $\bar{p}_g(S) = 0$, $C \cap H_\infty$ is one point, C is rational and C has only cusp singularity. Put $C \cap H_\infty = \{p\}$. If p is not the cusp of C , then p and the cusp of C are two points at which D is not normal crossing. Thus C has one cusp at p and so $V(\varphi) = C - \{p\} \cong \mathbb{A}^1$. Therefore we can apply [1] and complete the proof.

Q.E.D.

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