HANS KLEPPE
STEVEN L. KLEIMAN

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# REDUCIBILITY OF THE COMPACTIFIED JACOBIAN

Hans Kleppe* and Steven L. Kleiman**

Let $X$ be an integral, projective curve of arithmetic genus $p$ over an algebraically closed ground field $k$. Denote by $P$ the compactified jacobian of $X$, defined as the moduli space of torsion-free sheaves on $X$ with rank 1 and Euler characteristic $1-p$. Altman, Iarrobino and Kleiman proved an irreducibility theorem [1, Theorem (9)]: $P$ is irreducible if $X$ lies on a smooth surface, or equivalently, if the embedding dimension at each point of $X$ is at most two [3, Corollary (9)]. They also constructed an example [1, Example (13)] of an $X$ which is a complete intersection in $\mathbb{P}^3$ and for which $P$ is reducible. The example suggests that the converse of the theorem holds. In the present article, we prove the converse in the following form.

THEOREM (1): *If $X$ does not lie on a smooth surface, then the compactified jacobian $P$ is reducible.*

Rego [5] asserted Theorem (1) and offered a sketchy proof, which runs as follows. First he showed that $\mathrm{Hilb}^2(X)$ is reducible if $X$ does not lie on a smooth surface. Then, if $X$ is also Gorenstein, he concluded that $P$ is reducible from the fact that the Abel map, $\mathrm{Hilb}^n(X) \to P$, is smooth for large $n$. This map is no longer smooth if $X$ is not Gorenstein, and so Rego devised other methods to obtain reducibility in general.

However, Altman and Kleiman [2] developed a theory in which $\mathrm{Quot}^n(\omega/X)$, where $\omega$ is the dualizing sheaf on $X$, replaces $\mathrm{Hilb}^n(X)$ as the source of an Abel map,

$$A^n_\omega : \mathrm{Quot}^n(\omega/X) \to P.$$

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Whether or not $X$ is Gorenstein, $A_\omega^n$ is smooth and its fibers are projective spaces for all $n \geq 2p-1$ [2, Theorem (8.4) (v), Lemma (5.17) (ii) and Theorem (4.2)]. Hence, $P$ will be reducible if $\mathrm{Quot}^n(\omega/X)$ is reducible for large $n$.

This reducibility is proved below in two steps. First, we show that, if $\mathrm{Quot}^m(\omega/X)$ is reducible, then $\mathrm{Quot}^n(\omega/X)$ is reducible for $n \geq m$ (Proposition (3)). Secondly, we show that, if $X$ does not lie on a smooth surface, then $\mathrm{Quot}^d(\omega/X)$ is reducible, for small $d$, in fact, for $d = 2$ if $X$ is Gorenstein, and for $d = 1$ if $X$ is not Gorenstein (Proposition (4)). Thus, by a natural adaptation of part of Rego's method, Theorem (1) is proved.

Fix a torsion-free, rank-1 sheaf $\mathcal{F}$ on $X$. Denote by $U$ the open subscheme of $X$ consisting of nonsingular points.

The open subscheme $Q^nU$ of $\mathrm{Quot}^n(\mathcal{F}/X)$ parameterizing quotients of $\mathcal{F}$ with support contained in $U$ is isomorphic to $\mathrm{Hilb}^n(U)$, because $\mathcal{F}$ restricted to $U$ is invertible; so $Q^nU$ is irreducible of dimension $n$ [1, Lemma (1)]. Hence, $\mathrm{Quot}^n(\mathcal{F}/X)$ is irreducible if and only if $Q^nU$ is dense in $\mathrm{Quot}^n(\mathcal{F}/X)$. Using the valuative criterion [4, Ch. II, Prop. 7.1.4 (i)], we therefore get Lemma (2) below.

LEMMA (2): $\mathrm{Quot}^n(\mathcal{F}/X)$ *is irreducible if and only if, for all quotients $F$ of $\mathcal{F}$ of length $n$, there exists a scheme $T = \mathrm{Spec}(A)$, where $A$ is a complete, discrete valuation ring, and a $T$-flat quotient $\bar{F}$ of $\mathcal{F}_T$ such that $\bar{F}(t) \simeq F$ and $\mathrm{Supp}\, \bar{F}(g) \subseteq U_T(g)$, where $t$ and $g$ denote the closed and generic points of $T$.*

PROPOSITION (3): *If* $\mathrm{Quot}^n(\mathcal{F}/X)$ *is irreducible, then* $\mathrm{Quot}^m(\mathcal{F}/X)$ *is irreducible for all $m < n$.*

PROOF: Let $F$ be a quotient of $\mathcal{F}$ of length $m$. Let $I$ denote the kernel of the natural map $\mathcal{F} \to F$ and let $x_1, \ldots, x_{n-m}$ be different nonsingular points on $X$ such that $x_i \notin \mathrm{Supp}\, F$ for $i = 1, \ldots, n-m$. Then

$$F' = \mathcal{F}/M_1 \ldots M_{n-m}I,$$

where $M_i$ denotes the ideal of $x_i$, is a quotient of $\mathcal{F}$ of length $n$. By Lemma (2) there exists a complete, discrete valuation ring $A$ and a quotient $\bar{F}'$ of $\mathcal{F}_T$, $T = \mathrm{Spec}(A)$, with all the properties listed in that lemma and such that $\bar{F}'(t) \simeq F'$.

Let $W$ be the closed subscheme of $X_T$ defined by the annihilator of $\bar{F}'$. It is easy to see that we have an inclusion

$$\{x_1\} \cup \cdots \cup \{x_{n-m}\} \cup V \subseteq W(t),$$

where $V$ is the closed subscheme of $X$ defined by the annihilator of $F$. Hence, since $A$ is a henselian ring [4, Ch. IV, Prop. 18.5.14], $W$ may be written in the form,

$$W = W_1 \oplus \cdots \oplus W_{n-m} \oplus W',$$

where $\{x_i\} \subseteq W_i(t)$ and $V \subseteq W'(t)$ [4, Ch. IV, Théorème 18.5.11(c)].

Denote by $i$ the inclusion $W' \subseteq X_T$ and put

$$\bar{F} = i_* i^* \bar{F}'.$$

Then $\bar{F}$ is a flat quotient of $\bar{F}'$ and $\bar{F}(t) \simeq F$. Hence, by Lemma (2) the proposition is proved.

PROPOSITION (4): *Let $x$ be a point of $X$ and denote by $M$ the ideal defining $x$.*
(a) *If* $\dim_k(\omega/M\omega) \geq 2$, *then* $\mathrm{Quot}^1(\omega/X)$ *is reducible.*
(b) *If* $\dim_k(\omega/M\omega) = 1$ *and if* $\dim_k(M/M^2) \geq 3$, *then* $\mathrm{Quot}^2(\omega/X)$ *is reducible.*

PROOF: (a). Set $\omega_1 = \omega/M\omega$. Obviously the functors $\underline{\mathrm{Quot}}^1(\omega_1/X)$ and $\underline{\mathrm{Grass}}_1(\omega_1/k)$ are isomorphic. Since $\dim_k(\omega_1) \geq 2$, $\mathrm{Grass}_1(\omega_1/k)$ has dimension at least 1. Hence, since $\mathrm{Quot}^1(\omega_1/X)$ is a closed subscheme of $\mathrm{Quot}^1(\omega/X)$, we therefore get

$$\dim \mathrm{Quot}^1(\omega/X) \geq 1.$$

If equality holds, $\mathrm{Quot}^1(\omega/X)$ is reducible since $\mathrm{Quot}^1(\omega_1/X)$ is a closed 1-dimensional subscheme which is obviously different from $\mathrm{Quot}^1(\omega/X)$. If equality fails, then the closure of $Q^1U$ is a component of dimension 1, and so $\mathrm{Quot}^1(\omega/X)$ is reducible.
(b) $\omega$ is torsion-free [2, 6.5], so $\omega$ is invertible at $x$ because $\dim_k(\omega/M\omega) = 1$. Since $\dim_k(M/M^2) \geq 3$, we get that

$$\dim_k(M\omega/M^2\omega) \geq 3.$$

Set $\omega_2 = \omega/M^2\omega$. A vector subspace of $M\omega/M^2\omega$ of codimension 1 corresponds to a quotient of $\omega_2$ of length 2. It is not hard to see that this correspondence extends to families of quotients and vector subspaces, so that $\underline{\mathrm{Grass}}_1(M\omega/M^2\omega)$ can be considered as a subfunctor of $\underline{\mathrm{Quot}}^2(\omega_2/X)$. Hence, since a proper monomorphism is a closed embedding [4, Ch. IV, Prop. 8.11.5], $\mathrm{Quot}^2(\omega_2/X)$ contains

$\mathrm{Grass}_1(M\omega/M^2\omega)$. Since the latter has dimension at least 2, reasoning as in the proof of (a) we conclude that $\mathrm{Quot}^2(\omega/X)$ is reducible.

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M.I.T. 2-278
Cambridge, MA 02139
USA