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CHARACTERIZATION OF ABELIAN VARIETIES*

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§0. Introduction

Let X be an algebraic variety. S. Iitaka [5] defined the Kodaira dimension $\kappa(X)$ as a fundamental birational invariant of X. $\kappa(X)$ takes a value among $-\infty, 0, 1, \ldots, \dim X$. In this paper we shall study the structure of X such that $\kappa(X) = 0$. *Everything in this paper is assumed to be defined over the complex number field* $\mathbb{C}$.

THEOREM 1 = MAIN THEOREM: *Let X be a non-singular and projective algebraic variety and assume that $\kappa(X) = 0$. Then the Albanese map $\alpha : X \to A(X)$ is an algebraic fiber space.*

An *algebraic fiber space* is a morphism of non-singular projective algebraic varieties which is surjective and has connected fibers.

COROLLARY 2: *If $\kappa(X) = 0$, then the irregularity $q(X) =_{\mathrm{def}} \dim H^0(X, \Omega_X^1) \leq \dim X$. Moreover, if the equality holds, then α is a birational morphism. In other words, $\kappa(X) = 0$ and $q(X) = \dim X$ give a characterization of an abelian variety up to birational equivalences.*

Theorem 1 follows from the following two theorems by a standard argument in the classification theory of algebraic varieties by Iitaka and Ueno.

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THEOREM 3: *Let $f: X \to Y$ be an algebraic fiber space. Assume that $\kappa(X) \geq 0$ and $\kappa(Y) = \dim Y$. Then $\kappa(X) = \kappa(Y) + \kappa(F)$, where F is a general fiber of f.*

THEOREM 4 ([9]): *Let $f: X \to A$ be a finite and surjective morphism from a complete normal algebraic variety X to an abelian variety A. If $\kappa(X) = 0$, then f is an etale morphism.*

Theorem 3 follows from the following Theorem 5, which is a generalization of the main result of Fujita [2].

THEOREM 5 = MAIN LEMMA: *Let $f: X \to Y$ be an algebraic fiber space which satisfies the following conditions:*

(i) *There is a Zariski open dense subset Y_0 of Y such that $D =_{\text{def}} Y - Y_0$ is a divisor of normal crossing on Y.*

(ii) *Put $X_0 = f^{-1}(Y_0)$ and $f_0 = f|_{X_0}$. Then f_0 is smooth.*

(iii) *The local monodromies of $R^n f_{0*}\mathbb{C}_{X_0}$ around D are unipotent, where $n = \dim X - \dim Y$.*

Then $f_* K_{X/Y}$ is a locally free sheaf and semi-positive, where $K_{X/Y}$ denotes the relative canonical sheaf.

A locally free sheaf V on a complete normal algebraic variety X is said to be *semi-positive* if for any non-singular projective curve C, for any morphism $\varphi: C \to X$ and for any quotient invertible sheaf Q of $\varphi^* V$, we have $\deg_C Q \geq 0$. Note that in Theorem 5 only the special hypothesis is the unipotence in (iii). For a proof of Theorem 5 we use the theory of variations of Hodge structures and the theory of mixed Hodge structures by P. Griffiths, P. Deligne and W. Schmid.

In §1 we shall recall some results by Iitaka and then prove that Theorems 3 and 4 imply Theorem 1. In §2 the problem will be reduced to the case where the local monodromies are unipotent. This step is similar to a "stable reduction". In §3 we shall prove that Theorem 5 implies Theorem 3. In §4 we shall prove Theorem 5. In §5 we shall extend our results to compact Kaehler manifolds and to non-complete algebraic varieties.

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advisor, for the discussions during the preparation of the present version of this paper.

§1. Classification theory

Let us recall some definitions and some fundamental results in the classification theory of algebraic varieties by Iitaka. We refer the reader to [13].

DEFINITION: Let X be a complete normal algebraic variety and let L be a line bundle on X. The *L-dimension* $\kappa(L, X)$ of the pair (L, X) is defined as follows. Let $N(L, X) = \{m \in \mathbb{N};\ H^0(X, mL) \neq 0\}$, where mL is the m-th tensor power which is usually denoted by $L^{\otimes m}$. For $m \in N(L, X)$ let $\Phi_{mL} : X \dashrightarrow \mathbb{P}^{l(mL)-1}$ be the rational map associated to the line bundle mL, where $l(mL) = \dim H^0(X, mL)$. If $N(L, X) = \emptyset$, then we put $\kappa(L, X) = -\infty$. If $N(L, X) \neq \emptyset$, then $\kappa(L, X) = \max_{m \in N(L,X)}(\dim \Phi_{mL}(X))$. Note that $\kappa(L, X) \leq \dim X$. Let D be a Cartier divisor on X. We define the *D-dimension* $\kappa(D, X)$ to be $\kappa(\mathcal{O}(D), X)$.

If X is non-singular, the *Kodaira dimension* $\kappa(X)$ of X is defined by $\kappa(X) = \kappa(K_X, X)$, where K_X is the canonical line bundle. For $m \in \mathbb{N}$ we call $P_m(X) =_{\text{def}} l(mK_X) = \dim H^0(X, mK_X)$ the *m-genus* of X. It is easy to see that if X and X' are birational, then $\kappa(X) = \kappa(X')$ and $P_m(X) = P_m(X')$ for all $m \in \mathbb{N}$. Thus in general for any algebraic variety X, we can define $\kappa(X) =_{\text{def}} \kappa(X^*)$ and $P_m(X) =_{\text{def}} P_m(X^*)$, where X^* is a complete non-singular algebraic variety which is birationally equivalent to X. It is important to note that they do not depend on the choice of the birational model X^*.

THEOREM 6 (Theorem 8.1 of [13]): *Let X be a complete normal algebraic variety and let L be a line bundle on X. Then there exist positive numbers α, β and a positive integer m_0 such that the following inequalities hold for any integer $m \geq m_0$:*

$$\alpha m^{\kappa(L,X)} \leq l(mdL) \leq \beta m^{\kappa(L,X)},$$

where d is the greatest common divisor of $N(L, X)$.

PROPOSITION 7: *Let L_1 and L_2 be two line bundles on X and assume that there is a non-zero homomorphism $L_1 \to L_2$. Then $\kappa(L_1, X) \leq \kappa(L_2, X)$.*

PROPOSITION 8 (Theorem 5.13 of [13]): *Let* $f: X \to Y$ *be a surjective morphism of complete normal algebraic varieties and let* L *be a line bundle on* Y. *Then* $\kappa(f^*L, X) = \kappa(L, Y)$.

COROLLARY 9: *Let* $f: X \to Y$ *be a generically surjective and generically finite morphism of algebraic varieties. Then* $\kappa(X) \geq \kappa(Y)$. *Moreover if* X *and* Y *are both complete and normal and* f *is etale, then* $\kappa(X) = \kappa(Y)$.

THEOREM 10 (Theorem 5.10 of [13]): *Let* X *be a complete normal algebraic variety and let* L *be a line bundle on* X. *We assume that* $\kappa(L, X) \geq 0$. *Then there exist non-singular projective algebraic varieties* X^* *and* Y^* *and a surjective morphism* $\Phi: X^* \to Y^*$ *which satisfy the following conditions*:

(1) *There is a birational morphism* $\pi: X^* \to X$.
(2) $\dim Y^* = \kappa(L, X)$.
(3) *There is a subset* U *of* Y^* *which is a complement of countable union of proper algebraic subsets of* Y^*, *such that each fiber* $X^*_y =_{\text{def}} \Phi^{-1}(y)$ *is irreducible and non-singular for* $y \in U$.
(4) $\kappa(\pi^*L \otimes \mathcal{O}_{X^*_y}, X^*_y) = 0$ *for* $y \in U$.
(5) $\Phi: X^* \to Y^*$ *is birationally equivalent to the map* $\Phi_{mL}: X \cdot\cdot\to \Phi_{mL}(X)$ *for some* m.
(6) *The triple* (X^*, Y^*, f) *is uniquely determined up to birational equivalences by the properties* (1) *through* (4).

DEFINITION: The morphism $\Phi: X^* \to Y^*$ is called the *Iitaka fibering* associated to the pair (L, X). When X is non-singular and $L = K_X$, then we call it the *Iitaka fibering* of X. A fiber X^*_y for $y \in U$ is called a *general fiber* of Φ. An *algebraic fiber space* is a surjective morphism $f: X \to Y$ of non-singular projective algebraic varieties which has connected fibers. In general, a geometric fiber of an algebraic fiber space which is situated on a generic point of Y is called a *general fiber.* This concept is not so clear because we use both algebraic and analytic methods. But we can define the Kodaira dimension of a general fiber as an invariant of an algebraic fiber space and in this sense no problem will occur.

THEOREM 11 (Theorem 5.11 of [13]): *Let* $f: X \to Y$ *be an algebraic fiber space and let* L *be a line bundle on* X. *Then there is an open dense subset* U *of* Y *such that for any fiber* $X_y =_{\text{def}} f^{-1}(y)$ *for* $y \in U$, *we have* $\kappa(L, X) \leq \kappa(L \otimes \mathcal{O}_{X_y}, X_y) + \dim Y$.

We have to know something about abelian varieties. Let us begin with the following theorem of Ueno.

THEOREM 12 (Theorem 10.3 of [13]): *Let X be a subvariety of an abelian variety A. Then $\kappa(X) \geq 0$ and there exist an abelian subvariety B of A and a subvariety Y of the abelian variety A/B such that:*
(1) *X is an analytic fiber bundle over Y whose fiber is B.*
(2) $\kappa(Y) = \dim Y = \kappa(X)$.
(3) *There are etale covers $\tilde{X}$ and $\tilde{Y}$ of X and Y, respectively, such that $\tilde{X} = B \times \tilde{Y}$.*

We extend Theorem 12 making use of Theorem 4.

THEOREM 13: *Let $f: X \to A$ be a finite morphism from a complete normal algebraic variety to an abelian variety. Then $\kappa(X) \geq 0$ and there are an abelian subvariety B of A, etale covers $\tilde{X}$ and $\tilde{B}$ of X and B, respectively, and a complete normal algebraic variety $\tilde{Y}$ such that:*
(1) *$\tilde{Y}$ is finite over A/B.*
(2) *$\tilde{X}$ is isomorphic to $\tilde{B} \times \tilde{Y}$.*
(3) $\kappa(\tilde{Y}) = \dim \tilde{Y} = \kappa(X)$.

PROOF: The first assertion follows from Theorem 12 and Corollary 9. Let $\Phi: X^* \to Y^*$ be the Iitaka fibering of X and let B_y be the image of a general fiber $X_y^* = \Phi^{-1}(y)$ by f for $y \in U$. By Theorem 12 $\kappa(B_y) \geq 0$. By Corollary 9 $\kappa(B_y) = 0$. By Theorem 12 B_y is a translation of an abelian subvariety of A. f induces a morphism $X_y^* \to B_y$ which is birationally equivalent to an etale cover by Theorem 4. Since there are at most countably many abelian subvarieties in A, there is a Zariski dense subset U' of U such that the B_y for $y \in U'$ are parallel to an abelian subvariety B of A. By the following Lemma 14, there is a rational map $g^*: Y^* \to A/B$ such that the composition of the morphisms $X^* \to A \to A/B$ is birationally equivalent to $g^* \circ \Phi$. Since A/B is an abelian variety, g^* is a morphism. By the consideration of the dimensions, g^* is generically finite. Let X_0 and Y_0 be the images of X and Y^* by f and g^*, respectively, and let Y be the normalization of Y_0 in the field $\mathbb{C}(Y^*)$. Let $g: Y \to Y_0$ be the projection. By Zariski's main theorem, Φ induces a morphism $\Psi: X \to Y$.

By the Poincare's theorem, there is an etale cover $C \to A/B$ such that $A \times_{A/B} C$ is isomorphic to $B \times C$. Thus we may assume that X_0 is isomorphic to $B \times Y_0$ when we take etale covers. There is an open subset U of Y such that U is non-singular and Ψ is smooth over U. The fibers $\Psi^{-1}(y)$ for $y \in U$ are etale over B and hence isomorphic to each other. Let us denote it by $\tilde{B}$ and let G be the kernel of the projection $\tilde{B} \to B$ which is induced by f. Let $\tilde{Y}$ be the normalization of the inverse image of $\{0\} \times Y_0 \subset X_0$ in X by f. B acts on X_0 as birational automorphisms. Since Lie $\tilde{B}$ = Lie B, $\tilde{B}$ acts on X as birational automorphisms by the Zariski's main theorem. Therefore, $\tilde{Y}$ is a Galois cover of the normalization of Y_0 with the Galois group G. The group G acts also on $\tilde{X} =_{\text{def}} \tilde{B} \times \tilde{Y}$ and $\tilde{X}/G$ is isomorphic to X. By the construction the action of G on $\tilde{X}$ is fixed point free and $\tilde{X}$ is etale over X. Thus $\kappa(X) = \kappa(\tilde{X}) = \kappa(\tilde{Y})$ and $\dim \tilde{Y} = \dim Y = \kappa(X)$. Therefore we complete the proof. Q.E.D.

LEMMA 14: *Let $f: X \to Y$ and $g: X \to Z$ be two morphisms of algebraic varieties and assume that there is a Zariski dense subset U of Y such that $g(f^{-1}(y))$ is a point for $y \in U$. Then there is a rational map $h: Y \cdots\to Z$ such that $h \circ f$ is birationally equivalent to g.*

PROOF: Let $G \subset Y \times Z$ be the image of X by the map $(f \times g) \circ \Delta_X$, where Δ_X is the diagonal map, and let $p_Y: G \to Y$ and $p_Z: G \to Z$ be the projections. Since for $y \in U$ $p_Y^{-1}(y)$ is a point, p_Y is a birational morphism. The composition $h = p_Z \circ p_Y^{-1}$ satisfies our lemma. Q.E.D.

CLAIM 1: *Theorem 3 implies Theorem 1.*

PROOF: In this proof and also in other parts of this paper we shall change freely by birational models of algebraic varieties. Let us consider the following commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dd, "h"] & \\ Y \arrow[dr, "g"'] & & \\ & Z \arrow[r, "p"'] & A(X) \end{tikzcd}$$

where Z is the image of α and $g \circ f$ is the Stein factorization of h which is induced by α. By Theorem 13 there is an etale cover $\tilde{Y}$ of Y such that $\tilde{Y} = B \times W$, where B is an abelian variety and W is an algebraic variety of general type, i.e., $\kappa(W) = \dim W$. Let $\tilde{X} = X \times_Y \tilde{Y}$. Then $\kappa(\tilde{X}) = \kappa(X)$ by Corollary 9. Since the composition of the morphisms $\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y} \xrightarrow{pr} W$ satisfies the hypothesis of Theorem 3, we have $\kappa(\tilde{X}) \geq \dim W$ (see also Theorem 11). Hence W is a point and $\tilde{Y} = B$. Therefore, Y is also an abelian variety. By the universality of the Albanese map, $p \circ g$ is an isomorphism. Thus we prove Theorem 1, assuming Theorem 3. Q.E.D.

REMARK: The Iitaka fibering decomposes the study of algebraic varieties into two parts: (1) the study of an algebraic fiber space with a general fiber F such that $\kappa(F)=0$, and (2) the study of a variety such that $\kappa(X)=-\infty, 0$ or $\dim X$. This paper gives a partial affirmative answer to the following conjecture of Ueno about the structure of an algebraic variety with $\kappa(X)=0$.

CONJECTURE K: *Let X be a non-singular projective algebraic variety such that $\kappa(X)=0$ and let $\alpha: X \to A(X)$ be the Albanese map. Then*

(1) *α is surjective and has connected fibers, i.e., α is an algebraic fiber space.*
(2) *Let F be a general fiber of α. Then $\kappa(F)=0$.*
(3) *There is an etale covering $B \to A(X)$ such that $X \times_{A(X)} B$ is birationally equivalent to $F \times B$ over B.*

By Theorem 3 Conjecture K is true in case $q(X)=\dim X$. We can also prove the following theorem when we assume Theorem 3.

THEOREM 15: *If $q(X)=\dim X-1$, then Conjecture K is true. Moreover, in this case $p_g(X)=_{\mathrm{def}} P_1(X)=0$.*

PROOF: In this case a general fiber of $\alpha: X \to A(X)$ is a curve. By the addition theorem of Viehweg [16], it is an elliptic curve and moreover, there is an open dense subset U of $A(X)$ such that all the fibers $\alpha^{-1}(y)$ for $y \in U$ are isomorphic to an elliptic curve E. Therefore, there is a Galois cover Y_0 of $A(X)$ such that $X \times_{A(X)} Y_0$ is birationally equivalent to $E \times Y_0$. Let G be the Galois group. Let $D_0 \subset A(X)$ be the reduced discriminant divisor of $Y_0 \to A(X)$ and let $\mu: Z \to A(X)$ be a birational morphism such that Z is a non-singular projective algebraic variety and the reduced total transform D of D_0 is a divisor of normal crossing on Z. Let Y be the normalization of Z in $\mathbb{C}(Y_0)$. Let $g: Y \to Z$ be the projection. We may assume that we have a morphism $f: X \to Z$. G acts also on $E \times Y$ and $X' =_{\mathrm{def}} (E \times Y)/G$ is birationally equivalent to X. Let $f': X' \to Z$ be the projection.

Fix a holomorphic 1-form on E and an origin of E. They define a relative 1-form ω and a section s on the fiber space $E \times Y \to Y$. The action of G on $E \times Y$ induces contravariant actions of G on ω and s and gives two representations of G by abelian groups $\mathbb{C}^\times$ and E. $a: G \to \mathbb{C}^\times$ is defined by $g^*\omega = a(g)^{-1}\omega$ for $g \in G$ and $b: G \to E$ is defined by $g^*s = s - b(g)$ for $g \in G$. Since G is a quotient group of

$\pi_1(Z-D)$ and the abelianization of the latter is $H_1(Z-D,\mathbb{Z})$, we get two representations $a: H_1(Z-D,\mathbb{Z})\to\mathbb{C}^\times$ and $b: H_1(Z-D,\mathbb{Z})\to E$, which we denote by the same letters. Consider an exact sequence

$$0\to L\to H_1(Z-D,\mathbb{Z})\to H_1(Z,\mathbb{Z})\to 0,$$

where L is the subgroup generated by the lassoes around D. There is a positive integer m such that $a(g)^m=1$ for any $g\in G$. Then ω^m defines a holomorphic section of $mK_{X/Z}$. Pick a point $z\in Z$ and a small coordinate neighborhood $\{U; t_1,\dots,t_d\}$ around z such that the local equation of D is given by $t_1\dots t_e=0$. Put $U'=U-\bigcup_{i=1}^e\{t_i=0\}$. Let $l_1,\dots,l_e$ be the corresponding lassoes. Then there are numbers $0\le r_i<1$ for $i=1,\dots,e$, such that $\prod_{i=1}^e t_i^{-r_i}\omega$ gives a simple valued section of $K_{X/Z}$ over U'. $r_i=0$ if and only if $a(l_i)=1$. Since $\prod_{i=1}^e(t_i\bar{t}_i)^{-r_i}\,dt_i\,d\bar{t}_i$ is integrable on U', this gives a section over U by the following Proposition 16. If $a(L)\not\equiv 1$, then there exists an irreducible component $D_{0,1}$ of D_0 such that for the lasso l_1 around $D_{0,1}$ we have $a(l_1)\neq 1$. Let D_1 be the strict transform of $D_{0,1}$ in Z. Then by the argument above, $mK_{X/Z}\ge f^*D_1$. Since $\kappa(K_Z+D_1,Z)=\kappa(D_{0,1},A(X))>0$, we get $\kappa(X)>0$, which is a contradiction. Thus, $a(L)\equiv 1$.

PROPOSITION 16 (Theorem 2.1 of [11]): *Let X be a complex manifold of dimension n and let D be a divisor on X. A holomorphic n-form ω on $X-D$ can be extended to a holomorphic n-form on X if $|\int_{X-D}\omega\wedge\bar{\omega}|<+\infty$.*

Continue the proof of Theorem 15. The representation a induces a representation of $\pi_1(Z)=\pi_1(A(X))$. Let $\tilde{Z}$ be the corresponding etale cover of Z and let $\tilde{X}=X\times_Z\tilde{Z}$. To prove the theorem we may replace X and Z by $\tilde{X}$ and $\tilde{Z}$, respectively. Thus we may assume that $a\equiv 1$. This means that there is an action of E on X' which is induced by the trivial action of E on $Y\times E$. Let j be a positive integer such that $jb\equiv 0$. Then the multiplication $j: E\to E$ induces a finite and surjective morphism $X'\to Z\times E$. Hence by Theorem 13, there is an etale cover $\tilde{X}'\to X'$ such that $\tilde{X}'=\tilde{Z}\times E$, where $\tilde{Z}$ is finite over Z. Since $\kappa(\tilde{Z})=\kappa(\tilde{X}')=0$, $\tilde{Z}$ is etale over Z by Theorem 4. Thus we proved the first part of the theorem.

Let $\tilde{X}=X\times_{A(X)}B$. There is an element $g\in G=_{\mathrm{def}}\mathrm{Gal}(\tilde{X}/X)$ such that $g\neq 1$. Since $H^0(X,\Omega_X^1)=(H^0(\tilde{X},\Omega_{\tilde{X}}^1))^G$ and $q(X)=q(\tilde{X})-1$, there is a base $\omega_1,\dots,\omega_d$ of $H^0(\tilde{X},\Omega_{\tilde{X}}^1)$ such that $g^*\omega_1=c\omega_1$ for $c\neq 1$

and $g^*\omega_i = \omega_i$ for $i \neq 1$. Since $\omega_1 \wedge \cdots \wedge \omega_d$ is a generator of $H^0(\tilde{X}, K_{\tilde{X}})$ and $H^0(X, K_X) = (H^0(\tilde{X}, K_{\tilde{X}}))^G$, the latter is zero. Q.E.D.

REMARK: The above proof can be extended to the case where $q(X)$ is general, except the first step where we used the addition theorem of Viehweg. A modification is as follows. We assume that there is a non-singular projective algebraic varity F and an open subset U of $A(X)$ such that $\kappa(F) = 0$ and that fibers $\alpha^{-1}(y)$ for $y \in U$ are isomorphic to F. *Under this assumption we shall prove Conjecture K*. There is a finite Galois cover $Y \to A(X)$ such that $X_0 =_{\text{def}} (F \times Y)/G$ is birationally equivalent to X, where $G = \mathrm{Gal}(Y/A(X))$. This trivialization induces a representation $b : \pi_1(U) \to \mathrm{Aut}\ F$. Replace G by the image of b and replace Y by the one corresponding to the new G. By the Fujita's reduction at the beginning of § 3, a non-zero section ω of mK_X (Where m is a positive integer such that $P_m(X) \neq 0$) gives a generically finite and surjective morphism $X^* \to X$ such that $\kappa(X^*) = 0$ and $p_g(X^*) = 1$. Let ω^* be a non-zero section of K_{X^*}. Let $X^* \to B \to A(X)$ be the Stein factorization of the composition of $X^* \to X \to A(X)$. Since $\kappa(X^*) = 0$, we have $\kappa(B) = 0$ and hence $B \to A(X)$ is etale. Replace X and X^* by $X \times_{A(X)} B$ and an irreducible component of $X^* \times_{A(X)} B$, respectively. Then $X^* \to A(X)$ becomes also an algebraic fiber space. Let X_0^* be the normalization of X_0 in $\mathbb{C}\ (X^*)$. Fix a general point $y \in A(X)$ and fix isomorphisms $X_y \xrightarrow{\sim} F$ and $X_y^* \xrightarrow{\sim} F^*$, where $F^* \to F$ is a generically finite and surjective morphism obtained by a non-zero section ω_F of mK_F which is induced by ω. Let ω_F^* be a non-zero section of K_{F^*}. Since $X_0^* \to X_0$ is finite, the representation b induces a representation $b^* : \pi_1(U) \to \mathrm{Aut}\ F_0^*$, where F_0^* is the normalization of F in $\mathbb{C}\ (F^*)$. Let G^* be the image of b^* and let Y^* be a Galois cover of $A(X)$ corresponding to G^*. G^* acts on $F^* \times Y^*$ and when we put $X_1^* = (F^* \times Y^*)/G^*$, there is a birational morphism $\pi : X^* \to X_1^*$. Since there is a homomorphism $c : G^* \to G$ such that $b = c \circ b^*$, we have only to show that Y^* is etale over $A(X)$. By the argument in the proof of the theorem, we can prove that b^* induces a trivial representation on $H^0(F^*, K_{F^*})$. Hence there is a subset E_1 of F^* such that $\mathrm{cod}_{F^*} E_1 \geq 2$ and that the action of G^* on $(F^* - E) \times Y^*$ is fixed point free, where $E = E_1 \bigcup \mathrm{div}\ \omega_F^*$. Therefore, $\bar{\kappa}((F^* - E) \times Y^*) = \bar{\kappa}(X_1^* - p(E \times Y^*))$, where $\bar{\kappa}$ denotes the logarithmic Kodaira dimension (see § 5 and [6]) and $p : F^* \times Y^* \to X_1^*$ is the projection. Since $\pi^{-1}p(E \times Y^*)$ is a sum of div ω^* and an algebraic subset of codimension greater than 1, we have $\bar{\kappa}(X^* - \pi^{-1}p(E \times Y^*)) = \kappa(X^*) = 0$. Hence $\kappa(Y^*) = \bar{\kappa}((F^* - E) \times Y^*) = 0$. Thus $Y^* \to A(X)$ is etale and we complete the proof. Q.E.D.

§2. Unipotent reduction

THEOREM 17: *Let X be a non-singular projective algebraic variety and let D be a divisor of normal crossing on X, i.e., D is a reduced effective divisor and if $D = \Sigma_{i=1}^N D_i$ is the decomposition to irreducible components, then the D_i are non-singular and cross normally. Let m_i be positive integers for $i = 1, \ldots, N$. Then there exists a finite surjective morphism $p : \tilde{X} \to X$ satisfying the following conditions:*

(1) *$\tilde{X}$ is non-singular.*
(2) *$\tilde{D} =_{\text{def}} (p^*D)_{\text{red}}$ is a divisor of normal crossing on $\tilde{X}$.*
(3) *Let $p^*D_i = \Sigma_j m_{ij}\tilde{D}_j$ be the decomposition to irreducible components, where $\tilde{D}_j \neq \tilde{D}_{j'}$ for $j \neq j'$. Then $m_i \mid m_{ij}$ for any i and j.*

PROOF: We construct inductively a sequence of finite surjective morphisms $X_N \xrightarrow{p_N} X_{N-1} \xrightarrow{p_{N-1}} \cdots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X$ as follows. X_N is the desired $\tilde{X}$. Let M be an ample line bundle on X and let m be a positive multiple of m_1 such that $mM - D_1$ is very ample. Put $d = \dim X$. Let H_k $(1 \leq k \leq d)$ be general members of the linear system $|mM - D_1|$ such that $\Sigma_{k=1}^d H_k + D$ is a divisor of normal crossing. Let $\mathcal{U} = \{U_s\}$ be a covering of X by affine open subsets and let a_{st} be transition functions of M with respect to $\mathcal{U}$. Let $\varphi_{k,s}$ be local equations of $H_k + D_1$ in U_s such that $\varphi_{k,s} = a_{st}^m \varphi_{k,t}$ for $1 \leq k \leq d$. Then the fields $\mathbb{C}\,(X)(\sqrt[m]{\varphi_{1,s}}, \ldots, \sqrt[m]{\varphi_{d,s}})$ are the same for all s, which we denote by L. Let X_1 be the normalization of X in L. We shall show that X_1 is non-singular. Pick a point $x \in U_s$. Since $\bigcap_{k=1}^d H_k \cap D_1 = \emptyset$, at least one of the H_k or D_1 does not pass through x. If $x \notin D_1$, $x \in H_k$ for $1 \leq k \leq e$ and $x \notin H_k$ for $e+1 \leq k \leq d$, then the $\varphi_{k,s}$ for $1 \leq k \leq e$ make a part of a regular system of parameters at x and the $\varphi_{k,s}$ for $e+1 \leq k \leq d$ are units at x. Hence $X_1 \times_X \operatorname{Spec} \mathcal{O}_{X,x}$ is non-singular. If $x \in D_1$, $x \notin H_k$ for $1 \leq k \leq e$ (where $e \neq 0$) and $x \in H_k$ for $e+1 \leq k \leq d$, then $\varphi_{1,s}$ and the $\varphi_{k,s}/\varphi_{1,s}$ for $e+1 \leq k \leq d$ make a part of a regular system of parameters at x and the $\varphi_{k,s}/\varphi_{1,s}$ for $2 \leq k \leq e$ are units at x. Thus $X_1 \times_X \operatorname{Spec} \mathcal{O}_{X,x}$ is non-singular also in this case. Since $\Sigma_{k=1}^d H_k + D$ is a divisor of normal crossing, by applying the above argument to the D_i instead of X, we prove that the $D_i^1 =_{\text{def}} (p_1^* D_i)_{\text{red}}$ are non-singular and cross normally, where $p_1 : X_1 \to X$ is the projection. Since X_1 is ramified along D_1 with index m, $p_1^* D_1 = mD_1^1$. Although D_2^1 may be reducible, it is non-singular and we can make X_2 from X_1 when we apply the above process for D_2^1 instead of D_1. Thus repeating the above procedures, we complete the proof. Q.E.D.

The monodromy theorem says that the local monodromies are quasi-unipotent, i.e., some positive powers are unipotent. Therefore:

COROLLARY 18: *Let $f: X \to Y$ be an algebraic fiber space which satisfies the conditions (i) and (ii) of Theorem 5. Then there exists a finite surjective morphism $q: \tilde{Y} \to Y$ from a non-singular projective algebraic variety $\tilde{Y}$ such that for a desingularization $\tilde{X}$ of $X \times_Y \tilde{Y}$, the morphism $\tilde{f}: \tilde{X} \to \tilde{Y}$ induced from f satisfies the conditions (i) through (iii) in Theorem 5.*

COROLLARY 19: *Let K_1/K be a finite extension of algebraic function fields. Then there exist a finite algebraic extension L/K_1 and a non-singular projective model X of K such that the normalization of X in L is also non-singular.*

PROOF: We may assume that K_1/K is a Galois extension. Let X be a non-singular projective model of K and let X_1 be the normalization of X in K_1. By the resolution of singularities we may assume that the reduced discriminant $D = \Delta(X_1/X)$ is a divisor of normal crossing on X. Let $D = \Sigma_{i=1}^{N} D_i$ be the irreducible decomposition and let m_i be the ramification indices of the D_i. Then apply the theorem to X, D and the m_i and get a finite surjective morphism $p: \tilde{X} \to X$. Let Z be the normalization of an irreducible component of the fiber product $X_1 \times_X \tilde{X}$. Then Z is etale over $\tilde{X}$ and hence non-singular. $L = \mathbb{C}(Z)$ is the desired field. Q.E.D.

§3. Addition theorem

CLAIM 2: *Theorem 5 implies Theorem 3.*

PROOF: First we reduce the problem to the case where $p_g(X) =_{\text{def}} P_1(X) \neq 0$ by the following argument which is due to T. Fujita (lemma 1.8 of [15]). Let m be a positive integer such that $P_m(X) \neq 0$. Let $\mathcal{U} = \{U_i\}$ be a covering of X by affine open subsets and let φ_{ij} be transition functions of K_X with respect to $\mathcal{U}$. Let f_i be functions on the U_i which represent a non-zero section of mK_X and such that $f_i = \varphi_{ij}^m f_j$ on $U_i \cap U_j$. Let $U_i' = \{(x, t) \in U_i \times \mathbb{C};\ t^m = f_i(x)\}$. Then for $x \in U_i \cap U_j$, $(x, t) \in U_j'$ if and only if $(x, \varphi_{ij}(x)t) \in U_i'$. Therefore, the U_i' can be patched together to make an algebraic subset X' of the total space of K_X. Let X^* be a desingularization of an irreducible component of X' and let $\pi: X^* \to X$ be the projection. Let $R =_{\text{def}} K_{X^*} - \pi^* K_X$ be the ramification divisor. Then by the construction, there is a positive integer N such that $0 \leq R \leq N\pi^* K_X$. Hence $\kappa(X^*) = \kappa(X)$ by Proposition 8 (Note that in general

$\kappa(mL, X) = \kappa(L, X)$ for a positive integer m.) The functions $U_i' \ni (x, t) \mapsto t \in \mathbb{C}$ define a section of $\pi^* K_X$ on X^*. Thus $p_g(X^*) \neq 0$. Let $X^* \xrightarrow{f^*} Y^* \to Y$ be the Stein factorization of the composition of morphisms $X^* \xrightarrow{\pi} X \xrightarrow{f} Y$. Replace Y^* by its desingularization and let F^* be a general fiber of f^* which is mapped onto the general fiber F by π. Then $\kappa(Y^*) \geq \kappa(Y)$ and $\kappa(F^*) \geq \kappa(F)$. Therefore, it is enough to prove the theorem for the algebraic fiber space $f^*: X^* \to Y^*$, that is, we may assume that $p_g(X) \neq 0$.

Let $F = f^{-1}(y)$ for a general $y \in Y$ and let $t_1, \ldots, t_d$ be a local parameter system of Y centered at y. For a non-zero section ω of K_X, $\mathrm{residue}_F \left(\frac{\omega}{t_1 \ldots t_d}\right)$ defines a non-zero section of K_F if y is general. Thus $p_g(F) \neq 0$. If we change birational models, the morphism $f: X \to Y$ satisfies the assumption of Corollary 18. We get a commutative diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\mu"] \arrow[dr, "\tilde{f}"'] & X \times_Y \tilde{Y} \arrow[r, "p"] \arrow[d, "f_1"] & X \arrow[d, "f"] \\
 & \tilde{Y} \arrow[r, "q"] & Y
\end{tikzcd}$$

where q is finite and surjective. By Theorem 5, $\tilde{f}_* K_{\tilde{X}/\tilde{Y}}$ is a non-zero locally free sheaf and semi-positive.

LEMMA 20 (Kodaira [10]): *Let L be a very ample line bundle on $\tilde{Y}$. Then there exists a positive integer m such that* $\mathrm{Hom}_{\tilde{Y}}(L, mq^* K_Y) \neq 0$.

PROOF: We use the fact that $\kappa(Y) = \dim Y$. There is an exact sequence:

$$0 \to H^0(\tilde{Y}, mq^* K_Y - L) \to H^0(\tilde{Y}, mq^* K_Y) \to H^0(L, mq^* K_Y \otimes \mathcal{O}_L),$$

where L denotes also a general member of the linear system determined by $H^0(\tilde{Y}, L)$. Since $\kappa(\tilde{Y}, q^* K_Y) = \dim Y$, there is a positive number α and a positive integer m_0 such that $l(mdq^* K_Y) \geq \alpha m^{\dim Y}$, for $m \geq m_0$, where d is the largest common divisor of $N(q^* K_Y, \tilde{Y})$ by Theorem 6. On the other hand, there is a positive number β such that $l(mq^* K_Y \otimes \mathcal{O}_L) \leq \beta m^{\dim Y - 1}$ by the consideration of the dimension. Thus for a large multiple m of d, the first term of the exact sequence does not vanish. Q.E.D.

Continue the proof of Claim 2.

Let $P = \mathbb{P}(\tilde{f}_* K_{\tilde{X}/\tilde{Y}})$, let $\pi : P \to \tilde{Y}$ be the projection and let H be the tautological line bundle on P. We shall show that H is semi-positive, i.e., for any curve C on P, we have $H \cdot C \geq 0$. Let $\nu : C^* \to C$ be a resolution. π induces a morphism $\varphi : C^* \to \tilde{Y}$ and a quotient invertible sheaf Q of $\varphi^* \tilde{f}_* K_{\tilde{X}/\tilde{Y}}$, which is just $\nu^* H$. Thus $\deg_{C^*} \nu^* H \geq 0$.

Seshadri's criterion (p. 37 of [4]) says: Let S be a complete algebraic variety and let D be a Cartier divisor on S. Then D is ample if and only if there is a positive number ϵ such that for every integral curve C on S, we have $D \cdot C \geq \epsilon m(C)$, where $m(C) =_{\mathrm{def}} \max_{p \in C} \mathrm{mult}_p(C)$. Since L is ample, there is a positive number $\epsilon \leq 1$ such that for any curve C_0 on $\tilde{Y}$, we have $L \cdot C_0 \geq \epsilon m(C_0)$. Let C be a curve on P and let $C_0 = \pi(C)$. If C_0 is a curve on $\tilde{Y}$, then $(mH + \pi^* L) \cdot C \geq \pi^* L \cdot C = (\deg \pi) L \cdot C_0 \geq (\deg \pi)$ $\epsilon m(C_0) \geq \epsilon m(C)$. If C is in a fiber of π, then $(mH + \pi^* L) \cdot C = mH \cdot C \geq m \cdot m(C)$. Thus $mH + \pi^* L$ is ample on P.

Since q is flat, $K_{X \times_Y \tilde{Y}/\tilde{Y}} = p^* K_{X/Y}$ and hence $X \times_Y \tilde{Y}$ is Gorenstein. Since μ is birational, $\mathrm{Hom}(\mu_* K_{\tilde{X}}, K_{X \times_Y \tilde{Y}}) \neq 0$. Therefore, a composition of homomorphisms $\mathrm{Sym}_{\tilde{Y}} \tilde{f}_* K_{\tilde{X}/\tilde{Y}} \to \mathrm{Sym}_{\tilde{Y}} f_{1*} K_{X \times_Y \tilde{Y}/\tilde{Y}} \to \Sigma_{n \geq 0} f_{1*}(nK_{X \times_Y \tilde{Y}/\tilde{Y}})$ defines a subscheme R of P and π induces a surjective morphism $R \to \tilde{Y}$. Assume that $\dim Y > 0$; otherwise there is nothing to prove. Pick two distinct points y_1 and y_2 of $\tilde{Y}$ and let r_1 and r_2 be points of R which lie above y_1 and y_2, respectively. Since $mH + \pi^* L$ is ample, there is a positive integer n and sections s_1 and s_2 of $n(mH + \pi^* L)$ such that $s_1 |_{\pi^{-1}(y_1)} = 0$, $s_1(r_2) \neq 0$, $s_2 |_{\pi^{-1}(y_2)} = 0$ and $s_2(r_1) \neq 0$. Since $H^0(P, n(mH + \pi^* L)) = H^0(\tilde{Y}, \mathrm{sym}^{nm}_{\tilde{Y}} \tilde{f}_* K_{\tilde{X}/\tilde{Y}} \otimes nL)$, s_1 and s_2 induce sections ω_1 and ω_2 of $f_{1*}(nmK_{X \times_Y \tilde{Y}/\tilde{Y}}) \otimes nL$, respectively, such that $\omega_1(y_1) = 0$, $\omega_1(y_2) \neq 0$, $\omega_2(y_2) = 0$ and $\omega_2(y_1) \neq 0$. Hence $\dim H^0(\tilde{Y}, f_{1*}(nmK_{X \times_Y \tilde{Y}/\tilde{Y}}) \otimes nL) \geq 2$. Therefore, $\dim H^0(X \times_Y \tilde{Y}, nmp^* K_X) = \dim H^0(X \times_Y \tilde{Y}, nm(K_{X \times_Y \tilde{Y}/\tilde{Y}} \otimes f_1^* q^* K_Y)) \geq 2$. Hence $\kappa(X) > 0$.

Let $\Phi : X \to Z$ be the Iitaka fibering and let $G = \{(y, z) \in Y \times Z;$ $\exists x \in X$ *s.t.* $y = f(x)$ and $z = \Phi(x)\}$. Let Z' be an intersection of general hyperplane sections of Z such that $Y^* =_{\mathrm{def}} G \times_Z Z'$ is generically finite and dominating over Y. Since $\kappa(Y) = \dim Y$, $\kappa(Y^*) = \dim Y^*$. A general fiber of the algebraic fiber space $Y^* \to Z'$ is $f(X_z)$ for $z \in Z'$. By Theorem 11, $\kappa(f(X_z)) = \dim f(X_z)$ for general z. Apply the result in the first part of this proof to the algebraic fiber space $f |_{X_z} : X_z \to f(X_z)$. Since $\kappa(X_z) = 0$, we conclude that $f(X_z)$ is a point. By Lemma 14, there is a morphism $g : Z \to Y$ such that $f = g \circ \Phi$. By Theorem 11, $\kappa(F) \leq \dim \Phi(F)$. Then $\kappa(X) = \dim Z =$

$\dim \Phi(F)+\dim Y \geq \kappa(F)+\kappa(Y)$. On the other hand, $\kappa(X) \leq \kappa(Y)+\kappa(F)$ by Theorem 11. Thus we complete the proof of Claim 2. Q.E.D.

REMARK: Theorem 3 is an affirmative answer for a special case of the following conjecture of Iitaka:

CONJECTURE C: *Let $f: X \to Y$ be an algebraic fiber space. Then $\kappa(X) \geq \kappa(Y)+\kappa(F)$, where F is a general fiber.*

The conjecture is also true in the following cases: when
(1) F is a curve ([16]),
(2) F is a surface such that $\kappa(F) \neq 2$, or
(3) F is an abelian variety ([14]).
The case (2) will be treated in a forthcoming paper.

§4. Semi-positivity

In this section we shall prove Theorem 5. First we recall some results in [12].

(1) (p. 234 of [12]) Put $\mathscr{H}_0 = (R^n f_{0*}\mathbb{C}_{X_0})_{\mathrm{prim}} \otimes \mathcal{O}_{Y_0}$ and $\mathscr{F}_0 = f_{0*}K_{X_0/Y_0}$, where prim denotes the primitive part with respect to the polarization and $n = \dim X - \dim Y$. $\mathscr{H}_0$ has the so-called Hodge filtration $\{F^p\}_{0 \leq p \leq n}$ and $\mathscr{F}_0 = F^n(\mathscr{H}_0)$. $\mathscr{H}_0$ has a canonical extension $\mathscr{H}$ on Y which is locally free. This extension is locally described as follows: Let $s = \Sigma f_i s_i$ be a single-valued local section of $\mathscr{H}_0$, where the f_i are multi-valued holomorphic functions and the s_i are multi-valued flat sections of $\mathscr{H}_0$ with respect to the Gauss-Manin connection. s can be extended over the boundary D to a local section of $\mathscr{H}$ if and only if every f_i has at most logarithmic singularities along the boundaries, when the s_i are chosen to be linearly independent. Moreover, the filtration $\{F^p\}$ can be extended over $\mathscr{H}$. (Note that this is not the so-called limit filtration F_∞ on the fiber $\mathscr{H}_y$ for a base point $y \in Y_0$, because there is no connection on $\mathscr{H}$ which is an extension of the Gauss-Manin connection on $\mathscr{H}_0$). Put $\mathscr{F} = F^n(\mathscr{H})$. $\mathscr{F}$ is a locally free sheaf on Y and an extension of $\mathscr{F}_0$. Let y be a point of D and let $\{U; t_1, \ldots, t_d\}$ be a local coordinate system of Y centered at y such that the local equation of D is $t_1 \ldots t_e = 0$. Let $s = \Sigma f_i s_i$ be a rational section of $\mathscr{F}_0$ on $U - D$. Then $\int_{X_t} s \wedge \bar{s} = \Sigma a_{ij} f_i \bar{f}_j$, where the $a_{ij} =_{\mathrm{def}} \int_{X_t} s_i \wedge \bar{s}_j$ are constants. Hence by Proposition 16, s can be extended to a section of $\mathscr{F}$ on U if and only if it defines a section of $f_* K_{X/Y}$ on U. Thus $\mathscr{F} = f_* K_{X/Y}$.

(2) (§6 of [12]) Let D_1 be an irreducible component of D and put $D_1^0 = D_1 - (D - D_1)^{\mathrm{cl}}$, where cl denotes the closure. Let U denote a small open neighborhood of D_1 and put $U^0 = U - (D - D_1)^{\mathrm{cl}}$. Then the local monodromy γ_1 around D_1 can be defined up to conjugations on $\mathscr{H}|_{U-D}$. There is an ascending filtration $\{W_l\}_{0 \le l \le 2n}$ on $\mathscr{H}|_{U-D}$, which is called a weight filtration with respect to γ_1, such that (i) $N(W_l) \subset W_{l-2}$, where $N = \log \gamma_1$, and (ii) $N^l : \mathrm{Gr}^W_{n+l}(\mathscr{H}|_{U-D}) \to \mathrm{Gr}^W_{n-l}(\mathscr{H}|_{U-D})$ is an isomorphism for $l \ge 0$. Since $\gamma_1 \equiv 1$ on $\mathrm{Gr}^W(\mathscr{H}|_{U-D})$, the filtration $\{W_l\}$ can be extended on $\mathscr{H}|_{U^0}$ and $\mathrm{Gr}^W(\mathscr{H}|_{U^0})$ has a flat connection which is induced by the Gauss-Manin connection. The two filtrations $\{W_l\}$ and $\{F^p\}$ define a variation of mixed Hodge structures on $\mathscr{H}|_{D_1^0}$, that is, $\{F^p\}$ defines a variation of Hodge structures without polarization on $\mathrm{Gr}^W(\mathscr{H}|_{D_1^0})$. Denote by $\mathscr{P}_l^0$ the kernel of $N^{l-n+1} : \mathrm{Gr}_l^W(\mathscr{H}|_{D_1^0}) \to \mathrm{Gr}^W_{2n-l-2}(\mathscr{H}|_{D_1^0})$ if $l \ge n$ and put $\mathscr{P}_l^0 = 0$ if $l < n$. Then for $n \le l \le 2n$, $\mathscr{P}_l^0$ turns out to be a variation of Hodge structures with a polarization S_l, which is defined as follows: Let $\tilde{u}, \tilde{v} \in \mathscr{P}^0_{l,y}$ for $y \in D_1^0$ and let u and v be multi-valued flat sections of $W_l(\mathscr{H}|_{U-D})$ which induce flat sections of $\mathrm{Gr}_l^W(\mathscr{H}|_{U^0})$ passing through $\tilde{u}$ and $\tilde{v}$, respectively. Then put $S_l(\tilde{u}, \tilde{v}) = S(u, v)$, where S is the original polarization on $\mathscr{H}|_{U-D}$. The $\mathscr{P}_l^0$ can be extended to locally free sheaves $\mathscr{P}_l$ on D_1 canonically by (1). On the other hand, also by (1), the $W_l(\mathscr{H}|_{D_1^0})$ can be extended to locally free subsheaves $W_l(\mathscr{H}|_{D_1})$ of $\mathscr{H}|_{D_1}$ on D_1. The induced extensions $\mathrm{Gr}_l^W(\mathscr{H}|_{D_1})$ are compatible with the extensions $\mathscr{P}_l$. By the comparison of the types, we get $\mathrm{Gr}_l^W(\mathscr{F}|_{D_1}) = F^n(\mathscr{P}_l) \subset \mathscr{P}_l$.

Now let C, φ and Q be as in the definition after Theorem 5. We shall prove that $\deg_C Q \ge 0$ inductively as follows.

(i) We assume that $\varphi(C) \cap Y_0 \ne \emptyset$. Define a positive definite hermitian metric h on $\mathscr{F}_0$ by $h(u, v) = S(u, \bar{v})$, where $u, v \in \mathscr{F}_{0,y}$ for $y \in Y_0$ and S is the polarization of $\mathscr{H}_0$. Put $C_0 = \varphi^{-1}(Y_0)$. Then $Q|_{C_0}$ has an induced hermitian metric h_Q. P. Griffiths shows that its curvature Θ is non-negative (Theorem 5.2 of [3] or §7 of [12]). On the other hand, the α_p in the following lemma are all zero for $p \in C - C_0$ by the argument at the end of (1). Thus $\deg_C Q \ge 0$ in this case.

In general, let L be an invertible sheaf over a non-singular projective curve C, let C_0 be a Zariski open dense subset of C and let h be a hermitian metric on $L|_{C_0}$. The metric connection of $L|_{C_0}$ with respect to h defines a curvature form Θ on C_0, which is locally described as follows: $\Theta = \bar{\partial}\partial \log h(v, v)$, where v is a non-zero holomorphic local section of L. Let p be a point of $C - C_0$ and let t_p be a local parameter of C centered at p. We assume that for a uniformizing section v_p of L near p, $h(v_p, v_p) = 0(|t|^{-2\alpha_p}|\log t|^{\beta_p})$ for some real numbers α_p and β_p.

LEMMA 21: $\deg_C L = \frac{i}{2\pi}\int_{C_0} \Theta + \Sigma_{p\in C-C_0}\alpha_p$, *where the first integral is an improper Riemann integral.*

PROOF: This is essentially due to Fujita [2]. Let us take small open neighborhoods $U_p = \{x; |t_p(x)| < \epsilon\}$ of the $p \in C - C_0$ and modify the metric h inside of the U_p into the h_p which is C^∞ to make a global hermitian metric of L. Let Θ' be the curvature of the new metric connection. Then

$$\deg_C L = \frac{i}{2\pi}\int_C \Theta' = \frac{i}{2\pi}\int_{C-\cup_p U_p} \Theta + \sum_{p\in C-C_0} \frac{i}{2\pi}\int_{U_p} \Theta'.$$

By Stokes' theorem,

$$\frac{i}{2\pi}\int_{U_p} \Theta' = \frac{i}{2\pi}\int_{\partial U_p} \partial \log h(v_p, v_p) = -\frac{1}{2}\int_{\partial U_p} \frac{\partial \log h}{\partial \log r}\, d\theta,$$

where

$$t_p = re^{2\pi i\theta}.$$

Since

$$\lim_{\epsilon\to 0}\int_{\partial U_p} \frac{\partial \log h}{\partial \log r}\, d\theta = -2\alpha_p,$$

we get the desired formula. Q.E.D.

(ii) We assume that $\varphi(C) \subset D_1$ and $\varphi(C) \cap D_1^0 \neq \emptyset$. Then the map $\varphi^*\mathcal{F} \to Q$ induces a non-zero homomorphism $\varphi^*\mathrm{Gr}_l^W(\mathcal{F}|_{D_1}) \to Q$ for some l. Let Q' be the image of this map. Apply the argument in (i) to $\mathrm{Gr}_l^W(\mathcal{F}|_{D_1}) = F^n(\mathcal{P}_l) \subset \mathcal{P}_l$ instead of $\mathcal{F} = F^n(\mathcal{H}) \subset \mathcal{H}$. Then we conclude that $\deg_C Q' \geq 0$. Since $\deg_C Q \geq \deg_C Q'$, we get the desired result in this case.

(iii) Let D_2 be another irreducible component of D and put $D_{12} = D_1 \cap D_2$ and $D_{12}^0 = D_{12} - (D - D_1 - D_2)^{\mathrm{cl}}$. We assume that $\varphi(C) \subset D_{12}$ and $\varphi(C) \cap D_{12}^0 \neq \emptyset$. Then we have a non-zero homomorphism $\varphi^*\mathrm{Gr}_l^W(\mathcal{F}|_{D_1}) \to Q$ for some l. Let Q' be the image. Apply the argument in (ii) to $\mathcal{P}_l$, $\mathrm{Gr}_l^W(\mathcal{F}|_{D_1})$ and γ_2 instead of $\mathcal{H}$, $\mathcal{F}$ and γ_1, where γ_2 is the local monodromy around D_{12}. Then we get $\deg_C Q' \geq 0$ and the desired result.

(iv) We complete the proof inductively by the classification of images of the φ as in (i) through (iii). Q.E.D.

§5. Extensions

First we shall extend our theory to compact Kaehler manifolds. We can define the Kodaira dimensions also for compact complex manifolds just as in § 1 (cf. Definition 6.2 of [13]).

THEOREM 22: *Let X be a compact normal complex space, let A be a complex torus and let $f: X \to A$ be a finite surjective morphism. If $\kappa(X) = 0$, then f is an etale morphism.*

PROOF: We shall only show how to modify the proof of Theorem 4 (main theorem of [9]). Let D be the reduced discriminant divisor of f in A. Let $B = \{a \in A;\ a + D \subset D\}^0$, where 0 denotes the connected component of the origin, and let $A' = A/B$. Let a' be a general point of A' and let $\tilde{B}$ be the inverse image of a' in X. Then $\tilde{B}$ is etale over B. A is a fiber bundle over A' with a fiber B. Let $\tilde{A} = A \times_B \tilde{B}$ and $\tilde{X} = X \times_A \tilde{A}$. Then $\tilde{X}$ is a fiber bundle over a compact normal complex space X' with a fiber $\tilde{B}$, and there is a finite and surjective morphism $f': X' \to A'$. Let X'^* be a resolution of $X'$and let $\tilde{X}^* = \tilde{X} \times_{X'} X'^*$. The pull back of a general d-form on A for $d = \dim B$ to $\tilde{X}^*$ induces non-zero homomorphisms $H^0(X'^*,\ mK_{X'^*}) \to H^0(\tilde{X}^*, mK_{\tilde{X}^*})$ for $m \in \mathbb{N}$. Hence $\kappa(X') \leq \kappa(X)$, and thus $\kappa(X') = 0$. On the other hand, if D' is the reduced discriminant divisor of f', then $D = p^{-1}(D')$, where $p: A \to A'$ is the projection. Therefore, we may replace X and A by X' and A', respectively, that is, we may assume that $B = \{0\}$.

Let $C = \Sigma\, C_i$ be the irreducible decomposition of the reduced ramification divisor of f and let $\mu: X^* \to X$ be a birational morphism from a compact complex manifold X^* such that the strict transforms C_i^* of the C_i are non-singular and mutually disjoint. Since $K_{X^*} \geq \Sigma_i\, C_i^*$ and $\kappa(X) = 0$, we conclude that $\Sigma_i\, p_g(C_i^*) \leq n =_{\text{def}} \dim A$. Put $D_i = f(C_i)$. Then $\Sigma_i\, p_g(D_i) \leq n$. Let $A_i = \{a \in A;\ a + D_i \subset D_i\}^0$ and let $E_i = D_i/A_i$. We know that $\kappa(E_i) = \dim E_i$ and hence the E_i are algebraic. Then $p_g(D_i) \geq p_g(E_i) \geq \dim E_i + 1 = \operatorname{cod}_A A_i$. Since $\bigcap_i A_i = B = \{0\}$, we get the equalities $p_g(D_i) = p_g(E_i) = \dim E_i + 1$. Hence by Theorem 1 of [9], $|\chi(\mathcal{O}_{E_i})| = 1$. When we take etale covers $m: A \to A$ for $m \in \mathbb{N}$ and make base changes $X_m = X \times_A A$, we shall get a contradiction. Then we shall conclude that $D = \emptyset$. Q.E.D.

THEOREM 23: *Let X be a compact normal complex space, let A be a complex torus and let $f: X \to A$ be a finite morphism. Then $\kappa(X) \geq 0$ and there are a complex subtorus B of A, etale covers $\tilde{X}$ and $\tilde{B}$ of X*

and B, respectively, and a compact normal complex space $\tilde{Y}$ such that:

(1) *$\tilde{Y}$ is finite over A/B.*

(2) *$\tilde{X}$ is an analytic fiber bundle over $\tilde{Y}$ with a fiber $\tilde{B}$ and translations by $\tilde{B}$ as a structure group.*

(3) $\kappa(\tilde{Y}) = \dim \tilde{Y} = \kappa(X)$.

PROOF: We shall only show how to modify the proof of Theorem 13. The point is that we have no Poincaré's reducibility theorem. Use the same notation as in Theorem 13. Let $\tilde{B}$ be a general fiber X_y of Ψ which is etale over B. A is an analytic fiber bundle over A/B with a fiber B. Let $\tilde{A} = A \times_B \tilde{B}$ and let $\tilde{X} = X \times_A \tilde{A}$. Then $\tilde{X}$ is etale over X and when we replace X by $\tilde{X}$, then $\Psi : X \to Y$ becomes a fiber bundle with a fiber $\tilde{B}$. Now we have to show that $\kappa(Y) = \dim Y$. Assume the contrary and apply the above process to Y instead of X. Then we get the following situation: There is a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A/B & \longrightarrow & A/C, \end{array}$$

where C is a complex subtorus of A which contains B and Y is a fiber bundle over Z with a fiber C/B. Then X is a fiber bundle over Z with a fiber C. By Theorem 11, we have $\kappa(X) \leq \dim Z$, which is a contradiction. Thus $\kappa(Y) = \dim Y$. Q.E.D.

THEOREM 24: *Let X be a compact Kaehler manifold and let $\alpha : X \to A(X)$ be the Albanese map to a complex torus $A(X)$. We assume that $\kappa(X) = 0$. Then α is surjective and has connected fibers.*

COROLLARY 25: *If $\kappa(X) = 0$, then the irregularity $q(X) =_{\text{def}} \dim H^0(X, \Omega^1_X) \leq \dim X$. Moreover, if the equality holds, then α is a bimeromorphic morphism. In other words, $\kappa(X) = 0$ and $q(X) = \dim X$ give a criterion for a compact Kaehler manifold to be bimeromorphic to a complex torus.*

PROOF: Let Z_0 be the image of α and let $X \to Z \to Z_0$ be the Stein factorization. Apply Theorem 23 to $Z \to A(X)$. Then there are an etale cover $\tilde{Z}$ of Z and a morphism $\tilde{Z} \to Y$ such that $\kappa(Y) = \dim Y$. Then since Y is Moishezon, there is a resolution Y^* of Y which is projective. Let us study the morphism $X \to Y^*$. Since the monodromy theorem is also true in this case, we can reduce it to the case where local monodromies are unipotent by Theorem 17. Since the analytic method in §4 can be applied without modification also in this case, we get the semi-positivity of $f_* K_{X/Y^*}$, and hence $\kappa(X) > 0$ if $\dim Y > 0$. Thus we conclude that Y is a point. This means that $Z = A(X)$ and we complete the proof. Q.E.D.

Next we shall extend our theory to non-complete algebraic varieties. General references for the following paragraphs are [6] and [7]. Iitaka defined the logarithmic Kodaira dimensions of algebraic varieties for finer classification of algebraic varieties.

DEFINITION: Let X be a non-singular algebraic variety. The *logarithmic Kodaira dimension* $\bar{\kappa}(X)$ and the *logarithmic m-genus* $\bar{P}_m(X)$ for $m \in \mathbb{N}$ are defined as follows: Find a non-singular and complete algebraic variety $\bar{X}$ which contains X as a Zariski open dense subset and such that the complement $D =_{\text{def}} \bar{X} - X$ is a divisor of normal crossing on $\bar{X}$. Then define $\bar{\kappa}(X) = \kappa(K_{\bar{X}} + D, \bar{X})$ and $\bar{P}_m(X) = l(m(K_{\bar{X}} + D))$. It is easy to check that they do not depend on the choice of $\bar{X}$ and D. If $f: X_1 \to X_2$ is a proper and birational morphism of non-singular algebraic varieties, then $\bar{\kappa}(X_1) = \bar{\kappa}(X_2)$ and $\bar{P}_m(X_1) = \bar{P}_m(X_2)$. Thus we can define the *logarithmic Kodaira dimension* $\bar{\kappa}(X)$ and the *logarithmic m-genus* $\bar{P}_m(X)$ of a general algebraic variety X as follows: Find a non-singular algebraic variety X^* such that there is a proper and birational morphism $X^* \to X$. Then put $\bar{\kappa}(X) = \bar{\kappa}(X^*)$ and $\bar{P}_m(X) = \bar{P}_m(X^*)$ for any $m \in \mathbb{N}$.

DEFINITION: A *quasi-abelian variety* A is a quasi-projective commutative algebraic group variety which is an extension of an abelian variety A_0 by an algebraic torus $\mathbb{G}_m^d$ of dimension d:

$$0 \to \mathbb{G}_m^d \to A \to A_0 \to 0.$$

Let X be a non-singular algebraic variety. The *quasi-Albanese map* $\alpha: X \to A$ is a morphism to a quasi-abelian variety A such that

(1) for any other morphism $\beta: X \to B$ to a quasi-abelian variety B, there is a morphism $f: A \to B$ such that $\beta = f \circ \alpha$, and

(2) f is uniquely determined up to translations. The quasi-Albanese map $\alpha: X \to A$ can be constructed using the space of *logarithmic 1-forms* $H^0(\bar{X}, \Omega^1_{\bar{X}}(\log D))$, where $\bar{X}$ and D are as in the preceding definition. We call $\bar{q}(X) =_{\text{def}} \dim H^0(\bar{X}, \Omega^1_{\bar{X}}(\log D))$ the *logarithmic irregularity* of X. We know that $\dim A = \bar{q}(X)$.

DEFINITION: An *open algebraic fiber space* $f: X \to Y$ is a morphism of non-singular quasi-projective algebraic varieties which is generically surjective and has irreducible general fibers.

THEOREM 26: *Let X be a normal algebraic variety, let A be a quasi-abelian variety and let $f: X \to A$ be a surjective and finite morphism. If $\bar{\kappa}(X) = 0$, then f is an etale morphism.*

PROOF: We shall prove the theorem by induction on d in the definition. If $d = 0$, then this is just Theorem 4. Assume that $d > 0$ and pick a subgroup B of A which is isomorphic to $\mathbb{G}_m$. Put $A' = A/B$. Then A is a $\mathbb{G}_m$-bundle over A'. The compactification $\mathbb{G}_m \subset \mathbb{P}^1$ induces a natural partial compactification $\bar{A}$ of A, which is a $\mathbb{P}^1$-bundle over A'. Let $\bar{X}$ be the normalization of $\bar{A}$ in the field $\mathbb{C}(X)$. Let $\bar{f}: \bar{X} \to \bar{A}$ be the projection and let $\bar{X} \to X' \xrightarrow{f'} A'$ be the Stein factorization of the composition of the morphisms $\bar{X} \to \bar{A} \to A'$. Let $x \in X'$ and $y \in A'$ be general points such that $f'(x) = y$. First $\bar{\kappa}(X') \geq 0$. Apply the addition theorem ([8]) to the morphism $X \to X'$, where a general fiber X_x is a curve. Since $\bar{\kappa}(X) = 0$, we conclude that $\bar{\kappa}(X') = 0$ and $\bar{\kappa}(X_x) = 0$. Therefore, $f|_{X_x}: X_x \to A_y$ is an etale morphism. Let $\tilde{A} = A \times_{A_y} X_x$, let $\tilde{X} = X \times_A \tilde{A}$ and let $\tilde{X}'$ be the normalization of X' in the field $\mathbb{C}(\tilde{X})$. Then $\tilde{X}$ is a fiber bundle over $\tilde{X}'$ with a fiber X_x. Then since $\bar{\kappa}(\tilde{X}') = 0$, $\tilde{X}'$ is a quasi-abelian variety by induction. Hence $\tilde{X}$ is also a quasi-abelian variety and thus f is etale. Q.E.D.

Note that we have no Poincaré's reducibility theorem in this case, either. By a similar argument as in Theorem 23, we get the following, where an etale cover means a finite etale morphism:

THEOREM 27: *Let X be a normal algebraic variety, let A be a quasi-abelian variety and let $f: X \to A$ be a finite morphism. Then $\bar{\kappa}(X) \geq 0$ and there are a quasi-abelian subvariety B of A, etale covers $\tilde{X}$ and $\tilde{B}$ of X and B, respectively, and a normal algebraic variety $\tilde{Y}$ such that:*

(1) $\tilde{Y}$ is a finite over A/B.

(2) $\tilde{X}$ is a fiber bundle over $\tilde{Y}$ with a fiber $\tilde{B}$ and translations by $\tilde{B}$ as a structure group.

(3) $\bar{\kappa}(\tilde{Y}) = \dim \tilde{Y} = \bar{\kappa}(X)$.

The main theorem in this case is the following:

THEOREM 28: *Let X be a non-singular and quasi-projective algebraic variety such that $\bar{\kappa}(X) = 0$. Then the quasi-Albanese map $\alpha: X \to A(X)$ is an open algebraic fiber space.*

COROLLARY 29: *In the situation above, we have* $\bar{q}(X) \leq \dim X$. *Moreover, the equality holds if and only if* α *is a birational morphism.*

We can reduce Theorem 28 to the following "addition theorem":

THEOREM 30: *Let* $f: X \to Y$ *be an open algebraic fiber space. If* $\bar{\kappa}(X) \geq 0$ *and* $\bar{\kappa}(Y) = \dim Y$, *then* $\bar{\kappa}(X) = \bar{\kappa}(Y) + \bar{\kappa}(F)$, *where* F *is a general fiber of* f.

We shall need the following lemma.

LEMMA 31: *Let* $f: X \to Y$ *be an algebraic fiber space, let* $q: \tilde{Y} \to Y$ *be a finite and surjective morphism from a non-singular projective algebraic variety* $\tilde{Y}$ *and let* $\mu: \tilde{X} \to X \times_Y \tilde{Y}$ *be a desingularization. Let* $p: X \times_Y \tilde{Y} \to X$ *and* $\tilde{f}: \tilde{X} \to \tilde{Y}$ *be projections. Let* C *and* D *be reduced effective divisors on* X *and* Y, *respectively, such that* $f^{-1}(D) \subset C$ *and that for any irreducible component* C_0 *of* C, *we have either* $f(C_0) \subset D$ *or* $f(C_0) = Y$. *Put* $\tilde{C} = p^{-1}(C) =_{\text{def}} (p^*(C))_{\text{red}}$ *and* $\tilde{D} = q^{-1}(D)$. *Then* $\text{Hom}(\mu_*(K_{\tilde{X}/\tilde{Y}} + \tilde{C} - \tilde{f}^*\tilde{D}), p^*(K_{X/Y} + C - f^*D)) \neq 0$.

PROOF: We have $K_{\tilde{X}/X} = \mu^*p^*C - \tilde{C} + A$ and $K_{\tilde{Y}/Y} = q^*D - \tilde{D} + B$, where A and B are effective divisors on $\tilde{X}$ and $\tilde{Y}$ which have no mutual irreducible components with $\tilde{C}$ and $\tilde{D}$, respectively. By the assumption on C and D, we deduce that $\tilde{f}^*B$ has no mutual irreducible components with $\tilde{C}$. Since q is flat, $K_{X \times_Y \tilde{Y}/\tilde{Y}} = p^*K_{X/Y}$. Since μ is a birational morphism, $\text{Hom}(\mu_*K_{\tilde{X}}, K_{X \times_Y \tilde{Y}}) \neq 0$. Hence $\text{Hom}(\mu_*K_{\tilde{X}/X}, f_1^*K_{\tilde{Y}/Y}) \neq 0$, where f_1 is induced by f. Thus, $\text{Hom}(\mu_*A, f_1^*B) \neq 0$, which shows our conclusion. Q.E.D.

Now we shall prove Theorem 30. Let $\bar{X}$ and $\bar{Y}$ be non-singular projective algebraic varieties which contain X and Y as Zariski open dense subsets and such that $C =_{\text{def}} \bar{X} - X$ and $D =_{\text{def}} \bar{Y} - Y$ are divisors of normal crossing on $\bar{X}$ and $\bar{Y}$, respectively. Let $\bar{f}: \bar{X} \to \bar{Y}$ be a morphism which is an extension of f. Then $\bar{f}^{-1}(D) \subset C$. Note that in general if $X_1 \to X_2$ is a birational morphism, then $\bar{\kappa}(X_1) \geq \bar{\kappa}(X_2)$. Thus we may assume that for any irreducible component C_0 of C, we have either $\bar{f}(C_0) \subset D$ or $\bar{f}(C_0) = \bar{Y}$. Now apply Corollary 19 to every irreducible component of C which is mapped onto $\bar{Y}$. We make a base change $\bar{Y}' \to \bar{Y}$ such that $C \times_{\bar{Y}} \bar{Y}'$ decomposes into components which are degree one over $\bar{Y}'$. After that apply Theorem 17 to local monodromies. Thus when we take into account Lemma 31, we can reduce Theorem 30 to the following Theorem 32 by the argument in §3.

THEOREM 32: *Let us consider the following commutative diagram which consists of non-singular and quasi-projective algebraic varieties and morphisms among them*:

$$\begin{array}{ccccc} X & \hookrightarrow & \bar{X} & \hookleftarrow & X_0 \\ {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & {\scriptstyle f_0}\downarrow \\ Y & \hookrightarrow & \bar{Y} & \hookleftarrow & Y_0. \end{array}$$

We assume the following conditions:

(i) *$\bar{f}$ is an algebraic fiber space of relative dimension n.*

(ii) *$Y \supset Y_0$. $C =_{\mathrm{def}} \bar{X} - X \cap X_0$ and $D =_{\mathrm{def}} \bar{Y} - Y_0$ are divisors of normal crossing on $\bar{X}$ and $\bar{Y}$, respectively.*

(iii) *$C = \bar{f}^*(D)_{\mathrm{red}} + C^h$ and $C^h = \Sigma_{i=1}^N C_i^h$, where the C_i^h are irreducible components of C which are mapped onto $\bar{Y}$ by $\bar{f}$.*

(iv) *Let d be any non-negative integer and let Z be an intersection of any d distinct members of the C_i^h. If $d = 0$, we put $Z = \bar{X}$. Then $\bar{f}\,|_Z : Z \to \bar{Y}$ is an algebraic fiber space. Put $Z_0 = Z \cap X_0$. Then $f_0\,|_{Z_0} : Z_0 \to Y_0$ is proper and smooth.*

(v) *The local monodromies of $R^{n-d} f_{0*}\mathbb{C}_{Z_0}$ around D are unipotent.*

Then $\bar{f}_*((K_{\bar{X}} + C_1) - \bar{f}^*(K_{\bar{Y}} + D_1))$ is a locally free sheaf and semi-positive, where $C_1 = \bar{X} - X$ and $D_1 = \bar{Y} - Y$.

PROOF: Let X^d and X_0^d be the disjoint unions of all the distinct Z and Z_0 for a fixed d in (iv), respectively. By (3.2.13) of [1], there are the following two spectral sequences which degenerate at E_2:

(a) $$E_1^{-d,n+d} = R^{n-d} f_{0*}\mathbb{C}_{X_0^d} \Rightarrow R^n f_{0*}\mathbb{C}_{X \cap X_0}$$

(b) $$E_1^{-d,q+d} = R^q f_{0*}\Omega^{p-d}_{X_0^d/Y_0} \Rightarrow R^q f_{0*}\Omega^p_{X_0/Y_0}(\log C^h).$$

The two spectral sequences are related by the usual Hodge's spectral sequences. From (a) we deduce that the local monodromies of $R^n f_{0*}\mathbb{C}_{X \cap X_0}$ around D are also unipotent. By Schmid's method (1) in §4, we can extend the sheaves $(R^{n-d} f_{0*}\mathbb{C}_{X_0^d})_{\mathrm{prim}} \otimes \mathcal{O}_{Y_0}$ and $(R^n f_{0*}\mathbb{C}_{X \cap X_0})_{\mathrm{prim}} \otimes \mathcal{O}_{Y_0}$ to locally free sheaves $\mathcal{H}^d$ and $\mathcal{H}$, respectively, canonically and compatibly. Since the Hodge filtrations can be extended on $\mathcal{H}^d$, we deduce that $\mathcal{H}$ has also an extended Hodge filtration when we take into account the sequence (b). Put $\mathcal{F} = F^n\mathcal{H}$. Then by the sequence (b), $\mathrm{Gr}^W\mathcal{F} = \Sigma_{d=1}^n \ker(F^{n-d}\mathcal{H}^d \to F^{n-d+1}\mathcal{H}^{d-1}/F^{n-d+2}\mathcal{H}^{d-1})$, where W denotes the weight filtration with respect to the boundary C^h. The d-th direct summand of the right hand side is an intersection of $F^{n-d}\mathcal{H}^d$ with a flat subbundle of $\mathcal{H}^d$ and hence semi-positive by the arguments in §7 of [12] and Theorem 5. Thus $\mathrm{Gr}^W\mathcal{F}$ and hence also $\mathcal{F}$ are also semi-positive.

We shall prove that $\mathcal{F} = \bar{f}_*((K_{\bar{X}} + C_1) - \bar{f}^*(K_{\bar{Y}} + D_1))$. Denote by $\mathcal{F}'$ the right hand side. Note that both sides are torsion free extensions of $\mathcal{F}|_{Y_0}$ on $\bar{Y}$. Let $x \in \bar{X}$ and $y = \bar{f}(x) \in \bar{Y}$. Let $(U; x_1, \ldots, x_p)$ and $(V; y_1, \ldots, y_r)$ be small coordinate neighborhoods of x and y, such that C_1 and D_1 are defined in U and V by equations $x_1 \ldots x_q = 0$ and $y_1 \ldots y_s = 0$, for some $q \leq p$ and $s \leq r$, respectively. We assume that $\bar{f}(U) \subset V$. Let ω be a section of $\mathcal{F}|_{V \cap Y_0}$. Write $\omega = \Sigma f_i s_i$, where the f_i are multi-valued holomorphic functions on $V \cap Y_0$ and the s_i are multi-valued flat sections of $\mathcal{H}|_{V \cap Y_0}$ which are linearly independent. Put

$$\theta = \omega \wedge \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_s}{y_s} \wedge dy_{s+1} \wedge \cdots \wedge dy_r.$$

θ induces a homomorphic section of $K_{\bar{X}} + C_1$ on U if and only if for a fixed small positive number ϵ_0, we have an estimation

$$\int_{\epsilon \leq |x_1| \leq \epsilon_0} \cdots \int_{\epsilon \leq |x_q| \leq \epsilon_0} \int_{|x_{q+1}| \leq \epsilon_0} \cdots \int_{|x_p| \leq \epsilon} \theta \wedge \bar{\theta} = 0((\log \epsilon)^u).$$

Let us cover $\bar{f}^{-1}(V)$ by a finite set of the U and sum up the above estimations. Then we deduce that this is equivalent to saying that the f_i have at most logarithmic singularities along the boundary $V \cap D_1$, that is, ω defines a holomorphic section of $\mathcal{H}|_V$. Thus we complete the proof. Q.E.D.

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