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CANONICAL CURVES AND QUADRICS OF RANK 4

Enrico Arbarello and Joseph Harris

1. Introduction

Throughout this paper C will denote a *compact non-hyperelliptic Riemann surface* of genus $g > 2$.

Let $H^0(C, K_C)$ be the vector space of holomorphic differentials on C. The image of C under the canonical map

(1.1) $$\varphi_K : C \to \mathbb{P}H^0(C, K_C)^* \cong \mathbb{P}^{g-1}$$

is a non-degenerate, non-singular curve of degree $2g-2$. By a fundamental result due to Max Noether, *the canonical curve* $\varphi_K(C)$ *is projectively normal.* In particular the homomorphism

$$S^2H^0(C, K_C) \xrightarrow{v} H^0(C, 2K_C)$$

is surjective. From this one deduces that there are exactly

$$\left\{\begin{matrix} g-2 \\ 2 \end{matrix}\right\}$$

linearly independent quadrics through $\varphi_K(C)$.

Set

$$I_2 = \text{Ker } v.$$

Another beautiful and classical result due to K. Petri (see [8] and [10]) is the following

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(1.2) THEOREM: *The homogeneous ideal of $\varphi_K(C)$ is generated by I_2 with only two exceptions*

a) *when C is trigonal,* b) *when C is isomorphic to a plane smooth quintic. (In these cases the ideal of $\varphi_K(C)$ is generated by quadrics and cubics.)*

Following Petri's analysis one also sees that it is possible to choose a *basis of I_2 consisting of quadrics of rank at most* 6.

It is then natural to ask whether one can always find a basis of I_2 consisting of quadrics of smaller rank. This question acquires a real significance as soon as one brings into the picture Riemann's theta function.

Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be a sympletic system of generators for $H_1(C, \mathbb{Z})$. It is then well known that one may choose a basis

$$\omega_1, \ldots, \omega_g$$

of $H^0(C, K_C)$ such that the *period matrix*

$$\left\{\int_{a_j} \omega_i, \int_{b_j} \omega_i\right\}$$

is of the form

$$(I, z)$$

where z is a symmetric matrix with positive imaginary part. Let $\Lambda \subset \mathbb{C}^g$ be the integral lattice generated by the columns of the period matrix. The divisor Θ of the *Riemann theta function*

$$\theta(u) = \sum_{n \in \mathbb{Z}^g} exp(2\pi i {}^t nu + \pi i {}^t nzn), \quad u \in \mathbb{C}^g$$

defines a principal polarization on the Jacobian of C, that is on the complex torus

$$J(C) = \mathbb{C}^g/\Lambda.$$

Let C_d denote the d-fold symmetric product of C. Fixing a base point $p_0 \in C$, one can define a mapping

$$u_d : C_d \to J(C)$$

by letting

$$u_d(p_1 + \cdots + p_d) = \left\{\sum_{i=1}^{d} \int_{p_0}^{p_i} \omega_1, \ldots, \sum_{i=1}^{d} \int_{p_0}^{p_i} \omega_g\right\}.$$

Riemann proved (see, for example [1]) that there is a point $k_{p_0} \in J(C)$ such that the image of C_{g-1}, under the map

$$\pi = u_{g-1} + k_{p_0}$$

is exactly the theta divisor:

$$\pi : C_{g-1} \to \Theta \subset J(C)$$

and that, given $D \in C_{g-1}$, then

$$\pi(K_C(-D)) = -\pi(D).$$

Moreover he proved that

$$\text{(1.3)} \qquad \begin{aligned} \text{mult.}_{\pi(D)}\Theta &= \dim H^0(C, \mathcal{O}(D)) \\ &= \dim H^0(C, K_C(-D)). \end{aligned}$$

Let us denote by Θ_{sg} the *singular locus* of Θ. It can also be proved (see, for example [1], p. 209) that

$$\dim \Theta_{sg} = g - 4$$

and that

The general point of every component of Θ_{sg} is a double point for Θ.

Andreotti and Mayer in [1], and Kempf in [6] proved the following

(1.4) THEOREM: *Let $|D| = g^1_{g-1}$ be a complete linear series of degree $g-1$ and dimension 1 on C and consider the corresponding double point of Θ*

$$\pi(D) = \pi(K_C(-D)) \in \Theta_{sg}.$$

Then the projectified tangent cone to Θ at $\pi(D)$ is a quadric of rank less than or equal to 4 which contains the curve and which can be described as follows

$$\text{(1.5)} \qquad \mathbb{P}TC_{\pi(D)}(\Theta = \bigcup_{\Delta \in |D|} \overline{\Delta'} = \bigcup_{\Delta' \in |K_C(-D)|} \overline{\Delta'} \subset \mathbb{P}^{g-1}.$$

Moreover the quadric (1.5) is of rank 3 precisely when $|2D| = |K_C|$.

Vice versa a quadric Q of rank less than or equal to four, passing through C, comes from a tangent cone to Θ (i.e. is of the form (1.5)) *if (and only if) one of its rulings cuts out on C a complete linear series of degree $g-1$ and dimension* 1, *(the base locus of this series being contained in the vertex of Q).*

Let now

$$|\mathcal{I}_C(2)| = \mathbb{P}^{\binom{g-2}{2}-1}$$

be the linear system of quadrics through the canonical curve $\varphi_K(C)$. Let

$$\mathcal{W}_{C,\theta}$$

be the subvariety of $|\mathcal{I}_C(2)|$ whose points correspond to the projectivized tangent cones to the double points of Θ. Let

$$\mathcal{W}_C(4) \supseteq \mathcal{W}_{C,\theta}$$

be the subvariety at $|\mathcal{I}_C(2)|$ whose points correspond to quadrics of rank less than or equal to 4. Denote by

$$\overline{\mathcal{W}_C(4)}, \overline{\mathcal{W}_{C,\theta}}$$

the *linear spans* at $\mathcal{W}_C(4)$ and $\mathcal{W}_{C,\theta}$, respectively in $|\mathcal{I}_C(2)|$.

It may be remarked here that, a priori, there is a significant distinction between the loci $\mathcal{W}_C(4)$ and $\mathcal{W}_{C,\theta}$. The former is defined in terms of the geometry of the curve C, while the latter is determined solely by the principally polarized Jacobian $(J(C), \Theta)$ of C. Thus, for example, if it were the case that $\overline{\mathcal{W}_{C,\theta}} = |\mathcal{I}_C(2)|$ for *every* curve, the Torelli theorem for non-hyperelliptic, non-trigonal curves would be an immediate consequence: such curve C would simply be the intersection at the tangent cones to its theta-divisor at double points

Andreotti and Mayer proved that

(1.6) THEOREM: *If C is a general curve of genus g then*

$$|\mathcal{I}_C(2)| = \overline{\mathcal{W}_C(4)} = \overline{\mathcal{W}_{C,\theta}}.$$

In this paper we undertake a general analysis of the locus $\mathcal{W}_C(4)$ and, in particular, of its relation with $\mathcal{W}_{C,\theta}$. We obtain the following two principal results.

(1.7) THEOREM: *Let C be a non-hyperelliptic curve of genus* $g > 2$, *then*

$$\overline{\mathcal{W}_{C,\theta}} = \overline{\mathcal{W}_C(4)}.$$

(1.8) THEOREM: *Let C be a non-hyperelliptic curve of genus* $g \leqq 6$ *then*

$$\overline{\mathcal{W}_{C,\theta}} = |\mathcal{I}_C(2)|,$$

(*the case* $g = 6$ *being the first non-trivial case*).

The approach taken here is to introduce a family of varieties containing a canonical curve, called *rational normal scrolls.* They serve effectively as intermediaries between the curve and the quadrics containing it, in the sense that one can describe fairly completely the linear system of quadrics containing a scroll, and that every quadric of rank less than or equal to 4 containing the canonical curve contains one of these scrolls.

The next two sections of this paper are in fact devoted to a study of rational normal scrolls and the quadrics containing them. In the following sections we apply these results to scrolls containing the canonical curve, to prove our first main result.

Finally in the last section we analyze completely the geometry of the locus $\mathcal{W}_C(4)$ for any canonical curve of genus 6 and prove our second result.

We end this introduction by establishing notation and terminology.

Let X be an algebraic variety. We shall make no distinction between line bundles and invertible sheaves on X.

If X is non-singular we shall denote by K_X the canonical sheaf on X.

Given a sheaf $\mathcal{F}$ and a divisor D on X we shall set

$$\mathcal{F}(D) = \mathcal{F} \otimes \mathcal{O}_X(D)$$

and

$$h^0(X, \mathcal{F}(D)) = \dim H^0(X, \mathcal{F}(D)).$$

If

$$X \subseteq \mathbb{P}^n$$

is a subvariety we shall let

$$\overline{X} \subseteq \mathbb{P}^n$$

be the *linear span of* X in $\mathbb{P}^n$ and we shall say that X is *non-degenerate* if $\overline{X} = \mathbb{P}^n$. We shall also let

$$\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}^n}$$

be the ideal sheaf of X and

$$|\mathcal{I}_X(r)|$$

will denote the linear system of hypersurfaces of degree r containing X. We also set

$$I_X(r) = H^0(\mathbb{P}^n, \mathcal{I}_X(r)). \tag{1.9}$$

A quadric hypersurface in $\mathbb{P}^n$ will be simply called a *quadric* in $\mathbb{P}^n$. Consider the linear system of quadrics in $\mathbb{P}^n$

$$|\mathcal{O}_{\mathbb{P}^n}(2)| \cong \mathbb{P}^{\binom{n+2}{2}-1}.$$

For any positive integer r we shall let

$$\mathcal{W}(r) \subseteqq \mathbb{P}^{\binom{n+2}{2}-1} \tag{1.10}$$

denote the subvariety whose points correspond to quadrics in $\mathbb{P}^n$ of rank less than or equal to r. If

$$X \subset \mathbb{P}^n$$

is a subvariety we set

$$\mathcal{W}_X(r) = \mathcal{W}(r) \bigcap |\mathcal{I}_X(2)|. \tag{1.11}$$

Finally by a k-plane in $\mathbb{P}^n$ we shall mean a k-dimensional linear sub-space in $\mathbb{P}^n$.

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2. The geometry of rational normal scrolls

A rational normal scroll of dimension k in $\mathbb{P}^n$ may be described in three ways.

First, take k complementary linear subspaces

$$V_i \subset \mathbb{P}^n \; i = 1, \ldots, k$$

with

$$\dim V_i = a_i$$

and such that not all the a_i's are equal to zero. If $a_i \neq 0$ choose a rational normal curve

$$C_i \subset V_i$$

and an isomorphism

$$\varphi_i : \mathbb{P}^1 \to C_i.$$

If $a_i = 0$, set

$$C_i = V_i$$

and let

$$\varphi_i : \mathbb{P}^1 \to C_i$$

be the constant map. The variety

$$X_{a_1, \ldots, a_k} = \bigcup_{t \in \mathbb{P}^1} \overline{\varphi_1(t), \ldots, \varphi_k(t)} \tag{2.1}$$

swept out by the $(k-1)$-planes spanned by the corresponding points of the C_i's is then called a *rational normal scroll.*

Alternatively the variety $X_{a_1, \ldots, a_k}$ may be described as the image of the projective bundle

$$\mathbb{P}(E) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_k)) \to \mathbb{P}^1$$

under the map given by the dual of the tautological bundle on $\mathbb{P}(E)$. Then, for each i, the image of the direct summand $\mathcal{O}_{\mathbb{P}^1}(-a_i)$ of E maps to the rational curve C_i.

It is not hard to see that the degree of the scroll $X_{a_1,\dots,a_k}$ is given by

$$\deg X_{a_1,\dots,a_k} = n - k + 1. \tag{2.2}$$

This is the smallest possible degree of an irreducible non-degenerate k-fold in $\mathbb{P}^n$. Conversely in [9], p. 607, and [5] it is proved that

(2.3) THEOREM: *Any irreducible non-degenerate k-fold of degree $n-k+1$ in $\mathbb{P}^n$ is either a rational normal scroll, a cone over the Veronese surface in $\mathbb{P}^5$, or a quadric of rank greater than* 4.

There is, finally, a third way of describing the scroll $X_{a_1,\dots,a_k}$ which has the advantage of very clearly exhibiting the defining ideal of $X_{a_1,\dots,a_k}$ in $\mathbb{P}^n$. Assume that $a_1 = \cdots = a_h = 0$, $a_i \neq 0$, $i = h,\dots,k$, where h is less than k. Choose in $\mathbb{P}^n$ homogeneous coordinates

$$X_0^{(1)},\dots,X_{a_k}^{(1)}, X_0^{(2)},\dots,X_{a_k}^{(2)},\dots,X_0^{(k)},\dots,X_{a_k}^{(k)}$$

in such a way that

$$X_0^{(i)},\dots,X_{a_i}^{(i)}$$

are homogeneous coordinates in V_i, $i = 1,\dots,k$. Consider the matrix

$$M_{a_1,\dots,a_k} = \begin{pmatrix} X_0^{(h)} \dots X_{a_h-1}^{(h)} \dots X_0^{(k)} \dots X_{a_k-1}^{(k)} \\ X_1^{(h)} \dots X_{a_h}^{(h)} \quad \dots X_1^{(k)} \dots X_{a_k}^{(k)} \end{pmatrix}. \tag{2.4}$$

We claim that

$$X_{a_1,\dots,a_k} = \{\operatorname{rank}(M_{a_1,\dots,a_k}) \leqq 1\} \tag{2.5}$$

in the sense that

(2.6) *The ideal of $X_{a_1,\dots,a_k}$ is generated by the* 2 *by* 2 *minors of the matrix $M_{a_1,\dots,a_k}$.*

To show this we first notice that equality (2.5) holds in the set-theoretical sense. This follows immediately from the definition (2.1) and from the very well known fact that, up to a change of coordinates, the rational normal curve

$$C_i \subset V_i \cong \mathbb{P}^{a_i} \quad i = h, \ldots, k$$

is given by

$$C_i = \left\{ \operatorname{rank} \begin{pmatrix} X_0^{(i)} \ldots X_{a_i-1}^{(i)} \\ X_1^{(i)} \ldots X_{a_i}^{(i)} \end{pmatrix} \leqq 1 \right\}.$$

The set theoretical equality in (2.5) implies, in particular, that the determinantal variety

$$Y = \left\{ \operatorname{rank} \left(M_{a_1, \ldots, a_k} \right) \leqq 1 \right\}$$

has the "correct" codimension and this in turn implies that Y is Cohen-Macaulay (see, for example [3] p. 1022). In order to establish (2.5) it then suffices to show that the degree of Y is equal to the degree of X (i.e. equal to $n - k + 1$). To show this it suffices to check that, if Y_{ij}, $i = 1, 2, j = 1, \ldots, n - k + 1$ are homogeneous coordinates in $\mathbb{P}^{2n-2k+1}$, then the determinantal variety

$$\{\operatorname{rank}(Y_{ij}) \leqq 1\}$$

has degree equal to $n - k + 1$. This is a straightforward computation (see, for instance [6], p. 184).

It may be instructive to introduce the matrix (2.4) in a more intrinsic way. For this set

$$X = X_{a_1, \ldots, a_k} \tag{2.7}$$

and let L be the restriction to X of the hyperplane bundle on $\mathbb{P}^n$. The scroll X is ruled by a pencil of $(k-1)$-planes which we denote by $|E|$. We then have

$$h^0(X, \mathcal{O}(E)) = 2, \quad h^0(X, L(-E)) = n - k + 1.$$

The second equality follows from the first and from the linear normality of X (see (2.9)). Let us consider the multiplication map

$$\mu : H^0(X, \mathcal{O}(E)) \otimes H^0(X, L(-E)) \to H^0(X, L).$$

It is then easy to show that, with a suitable choice of bases, *the transpose of the matrix* (2.4) *represents the dual map*

$$\mu^*: H^0(X, L)^* \to \mathrm{Hom}(H^0(X, \mathcal{O}(E)), H^0(X, L(X, L(-E))^*).$$

From now on we shall write the matrix (2.4) in the following simpler form

$$M_{a_1,\dots,a_k} = M = \begin{pmatrix} \ell_1 \dots \ell_{n-k+1} \\ \ell'_1 \dots \ell'_{n-k+1} \end{pmatrix} \tag{2.8}$$

One of the basic properties of rational normal scrolls is given by the following

(2.9) PROPOSITION: *A rational normal scroll is projectively normal.*

PROOF: Let $X \subset \mathbb{P}^n$ be a k-dimensional rational normal scroll. The case $k = 1$ is well known and we proceed by induction on k. Given an integer ν, the cohomology sequency of

$$0 \to \mathcal{I}_X(\nu) \to \mathcal{O}_{\mathbb{P}^n}(\nu) \to \mathcal{O}_X(\nu) \to 0$$

shows that the Proposition is equivalent to the statement

$$H^1(\mathbb{P}^n, \mathcal{I}_X(\nu)) = (0). \tag{2.10}$$

Let H be a general hyperplane section. According to (2.3), $X \cap H$ is again a scroll. The vanishing statement (2.10) follows then from the induction hypothesis by looking at the cohomology sequence of

$$0 \to \mathcal{I}_X(\nu - 1) \to \mathcal{I}_X(\nu) \to \mathcal{I}_{X \cap H}(\nu) \to 0 \tag{2.11}$$

Q.E.D.

Let us consider the particular case $\nu = 2$. Since X is non-degenerate we have that $h^0(X, \mathcal{I}_X(1)) = 0$. From (2.10) and (2.11) we get an *injection*

$$I_X(2) \hookrightarrow I_{X \cap H}(2). \tag{2.12}$$

By taking the intersection of X with a general $(n-k)$-plane we then see that the dimension of $I_X(2)$ equals the number of linearly independent quadrics in $\mathbb{P}^{n-k}$ passing through $n - k + 1$ points in general position. Hence

(2.13) $$\dim I_X(2) = \binom{n-k+1}{2}.$$

Combining this with (2.6) we obtain the following

(2.14) PROPOSITION: *Let X be a k-dimensional rational normal scroll contained in $\mathbb{P}^n$. Then there exists a matrix of linear forms in $\mathbb{P}^n$.*

$$M = \begin{pmatrix} \ell_1 \dots \ell_{n-k+1} \\ \ell'_1 \dots \ell'_{n-k+1} \end{pmatrix}$$

such that the ideal I_X of X is generated by the 2 by 2 minors of M. Moreover the $\binom{n-k+1}{2}$ quadrics

(2.15) $$\ell_\alpha \ell'_\beta - \ell_\beta \ell'_\alpha = 0, \quad \alpha < \beta, \quad \alpha, \beta = 1, \dots, n-k+1$$

are linearly independent and (therefore) from a basis of $I_X(2)$.

The linear independent quadrics (2.15) are *quadrics of rank less than or equal to 4 containing the scroll X*. We would now offer a geometrical picture of how these quadrics sit inside the linear system $|I_X(2)|$ of quadrics through X.

As usual we let L be the hyperplane bundle on X and $|E|$ the pencil of $(k-1)$-planes sweeping out X. Consider the projective space

$$\mathbb{P}^{\binom{n+2}{2}-1} \cong \mathbb{P}S^2H^0(X, L)$$

of all quadrics in $\mathbb{P}^n$. Let

$$\mathcal{W}_X(4)$$

be defined as in (1.11).

Now set

$$V = H^0(X, L(-E))$$

and let E_0 and E_1 be the divisors on $|E|$ defined by

$$E_0 = \{\ell_1 = \dots = \ell_{n-k+1} = 0\}$$

$$E_1 = \{\ell'_1 = \dots = \ell_{n-k+1} = 0\}.$$

Consider the vector spaces

$$V_0 = \{s \in H^0(X, L) : (s) > E_0\}$$

$$V_1 = \{s \in H^0(X, L) : (s) > E_1\}.$$

Of course $\{\ell_1, \ldots, \ell_{n-k+1}\}$ is a basis of V_0 and $\{\ell'_1, \ldots, \ell'_{n-k+1}\}$ is a basis of V_1. Fix sections $m_0 \in V_0$ and $m_1 \in V_1$. The multiplication by m_0 and m_1, respectively, gives *isomorphisms*

$$\alpha_0 : V \to V_0 \subset H^0(X, L)$$

$$\alpha_1 : V \to V_1 \subset H^0(X, L).$$

We then define a linear map

$$\alpha : \wedge^2 V \to S^2 H^0(X, L)$$

by setting

$$\alpha(v \wedge w) = \alpha_0(v) \otimes \alpha_1(w) - \alpha_0(w) \otimes \alpha_1(v).$$

From the definition of α_0 and α_1 it follows that

$$\alpha_0(v)\alpha_1(w) - \alpha_0(w)\alpha_1(v) = 0$$

is a quadric of rank less than or equal to 4, in $\mathbb{P}^n$, containing X, and in fact Proposition (2.14) exactly says that α *induces an isomorphism*

(2.16) $$\alpha : \wedge^2 V \xrightarrow{\cong} I_X(2) \subset S^2 H^0(X, L).$$

Consider now the Grassmannian Gr(2, V) of lines in $\mathbb{P}V$. If v and w are independent vectors in V we let $\overline{vw}$ denote the line in $\mathbb{P}V$ joining the points $[v]$ and $[w]$. We then define a map

(2.17) $$q : \mathrm{Gr}(2, V) \to \mathcal{W}_X(4) \subset \underset{\mathbb{P}^{\binom{n-k+1}{2}-1}}{\underset{\wr\|}{|\mathcal{I}_X(2)|}} \subset \underset{\mathbb{P}^{\binom{n+2}{2}}}{\underset{\wr\|}{\mathbb{P}S^2 H^2(X, L)}}$$

by letting

$$q(\overline{vw}) = [\alpha(v \wedge w)].$$

By definition q is the composition of the Plücker embedding of $\mathrm{Gr}(2, V)$ in $\mathbb{P}\wedge^2 V$ and the linear isomorphism

$$\mathbb{P}\wedge^2 V \to |\mathcal{I}_X(2)|$$

induced by α.

We shall call the subvariety

$$q(\mathrm{Gr}(2, V)) \subseteq \mathcal{W}_X(4) \subset \mathbb{P}^{\binom{n-k+1}{2}-1}$$

the *principal locus of quadrics of rank less than or equal to* 4 *containing X, or simply the principal locus of* $\mathcal{W}_X(4)$. We just proved that

(2.18) PROPOSITION: *Let X be a rational normal scroll of dimension k contained in* $\mathbb{P}^n$. *Let L be the hyperplane bundle on X and* $|E|$ *the pencil of* $(k-1)$*-planes on X. Set* $V = H^0(X, L(-E))$. *Then the principal locus of quadrics of rank less than or equal to* 4 *containing X sits, inside* $|\mathcal{I}_X(2)| \cong \mathbb{P}^{\binom{n-k+1}{2}-1}$, *as the image of the Grassmannian* $\mathrm{Gr}(2, V)$ *under the Plücker embedding. Moreover the quadrics of the principal locus generate the ideal of X.*

We finally wish to characterize, in an intrinsic way, the quadrics of the principal locus at $\mathcal{W}_X(4)$ among all quadrics in $\mathcal{W}_X(4)$.

Let Q be a quadric of rank 4 through X. The two rulings of Q cut out on X, away from the vertex of Q, two pencils of divisors

$$\mathbb{P}V \subset |D| \quad \text{and} \quad \mathbb{P}V' \subset |D'|$$

where V (resp. V') is a 2-dimensional subspace of $H^0(X, \mathcal{O}(D))$ (resp. $H^0(X, \mathcal{O}(D'))$), such that

$$|D + D' + F| = |L|$$

where F is the divisor cut out on X by the vertex of Q. If Q is of rank 3 then *the* ruling of Q cuts out on X a pencil of divisors

$$\mathbb{P}V \subset |D|$$

such that

$$|2D + F| = |L|.$$

Also, since X is non-degenerate, there is no quadric of rank less than 3 through X.

It then follows from the definition of the map q (see (2.17)) that

(2.19) *The quadrics of rank* 4 *belonging to the principal component of* $\mathcal{W}_X(4)$ *are exactly those for which*

$$\mathbb{P}V = |E|$$

and

$$\mathbb{P}V' \textit{ is a pencil in } |L(-E)|$$

(*or vice versa*). *The quadrics of rank* 3 *belonging to the principal component of* $\mathcal{W}_X(4)$ *are exactly those for which*

$$\mathbb{P}V' = \mathbb{P}V = |E|.$$

3. Examples of scrolls

The geometry of scrolls, specifically of the quadrics containing them, does not become interesting until the codimension of the scroll is 3 or more.

A scroll of codimension 1 is just a quadric of rank three ($X_{20\ldots0}$) or four ($X_{110\ldots0}$).

In codimension 2, if we disregard the operation of coning, there are three scrolls. The twisted cubic $X_3 \subset \mathbb{P}^3$, the Steiner surface $X_{21} \subset \mathbb{P}^4$, (classically the Steiner surface is actually the projection of X_{21} in $\mathbb{P}^3$) and the Segre threefold $X_{111} \subset \mathbb{P}^5$. Each of these three lies on a 2-dimensional system of quadrics, all of which are of rank 4 or less, (what distinguishes these three linear systems is the locus of quadrics of rank 3, which is a plane conic, a point and the empty set, respectively).

As examples of the general phenomenon described in Section 2, we want to offer a brief and informal discussion of the geometry of the surfaces X_{22} and X_{31} in $\mathbb{P}^5$.

To begin with

$$X = X_{22} \subset \mathbb{P}^5$$

is described as the locus of lines joining corresponding points on two conics C_1 and C_2, lying in complementary 2-planes in $\mathbb{P}^5$. Alternatively, it may be described as the image of a non-singular quadric surface in $\mathbb{P}^3$

$$Q \cong \mathbb{P}^1 \times \mathbb{P}^1$$

under the embedding in $\mathbb{P}^5$ given by the linear system

$$|L| = |\mathcal{O}(2, 1)|$$

of curves of type (2, 1) on Q. In these terms the fibers over the first factor map to the lines of the ruling

$$|E|$$

of S, while the fibers over the second factor map to a pencil of conics

$$\mathbb{P}V \subset |L(-E)|$$

of which the coincs C_1 and C_2 are members.

By Proposition (2.14) and (2.19) the quadrics of the principal locus of $\mathcal{W}_X(4)$ correspond to the pencils in the system $|\mathcal{O}(1, 1)| = |L(-E)|$. Equivalently, once we realize X as the quadric $Q \subset \mathbb{P}^3$, the quadrics of the principal locus correspond to the pencil of hyperplanes in $\mathbb{P}^3$ or, what is the same, to the lines in $\mathbb{P}^3$. Alternatively, a quadric of rank 4 of the principal locus can be described as follows. Take two points p and q on X not lying on one line at the ruling $|E|$. Projection from the line $\overline{pq}$ maps X birationally to a quadric surface $T \subset \mathbb{P}^3$. The cone projecting T from $\overline{pq}$ is then a quadric of the principal component.

These, however, are not the only quadrics of rank 4 through X. Consider the variety

$$Y = \left(\bigcup_{C \in \mathbb{P}V} \overline{C}\right) \subset \mathbb{P}^5$$

This variety is the scroll

$$Y \cong X_{111}$$

based on any three of the lines of $|E|$. Therefore Y lies on a 2-plane of quadrics, all of which are of rank 4. These quadrics are readily described: Y is abstractly $\mathbb{P}^1 \times \mathbb{P}^2$ and, under the embedding in $\mathbb{P}^5$, the fibers over the second factor are carried into lines (some of which are the lines of X). The cone over Y through any such line is a quadric of rank 4 through Y. Any such quadric belongs to the principal locus of $\mathcal{W}_Y(4)$ and therefore one of its rulings cuts out, on Y, the pencil $\{\overline{C}: C \in \mathbb{P}V\}$ and hence, on X, the pencil $\mathbb{P}V$. The second ruling will then cut out on X a pencil consisting of pairs of lines. We will call this 2-plane the *secondary* locus of $\mathcal{W}_X(4)$. Its intersection with the principal locus consists of a conic Γ whose points correspond to quadrics of rank 4 through X whose vertex is a line of X.

If we realize X as a quadric Q in $\mathbb{P}^3$ and view the quadrics of the principal locus as the lines in $\mathbb{P}^3$, then the points of Γ correspond to the lines of one ruling at Q.

In this context, it may be noted that the quadrics of rank 3 through X are just the cones over X through the plane $\overline{C}$ of one of the conics C. These also form a conic curve on the principal component. They correspond, in the above picture, to the lines of the other ruling of Q.

Using this picture we may say what happens when the surface X_{22} degenerates into the surface X_{31} (cf. [5], p. 34 for a description of this degeneration):

—The quadric $Q \subset \mathbb{P}^3$ becomes a quadric cone.
—The locus of rank 3 quadrics and the intersection of the principal and secondary component come together.
—The secondary component, in the limit, lies on the principal component.

The scheme of quadrics of rank less than or equal to 4 through $X_{31} \subset \mathbb{P}^5$ is thus a quadric hypersurface in $\mathbb{P}^5$ (or equivalently a Grassmannian $G(2,4)$) with an embedded 2-plane.

Finally, because of the lack of a suitable reference, we wish to describe the locus of quadrics of rank less than or equal to 4 containing a Veronese surface

$$S \subset \mathbb{P}^5.$$

First of all we show that

(3.1) *Every singular quadric Q containing S has rank less than or equal to* 4.

To see this, let p be a singular point of Q, and π_p the projection from p to a hyperplane $H \subset \mathbb{P}^5$ not passing through p. Let $\tilde{Q}$ and $\tilde{S}$ be the images under π of Q and S, respectively.

A priori three cases are possible.

a) p does not lie on the chordal variety of S. This implies that $\tilde{S}$ is smooth. Suppose $\tilde{Q}$ non-singular. Then $\tilde{S}$ would be a smooth divisor on the non-singular hypersurface $\tilde{Q}$, and hence $\tilde{S}$ would be a complete intersection. This is absurd since $\tilde{S}$ is the regular projection of another variety of the same degree. Therefore $\tilde{Q}$ is singular. But now projecting $\tilde{S}$ from the vertex of $\tilde{Q}$ would give a regular 2–1 map of $\tilde{S} = \tilde{\mathbb{P}}^2$ to a quadric surface. This is again absurd so that case (a) does not occur.

b) p lies on the chordal variety of S but not on S. In this case $\tilde{S}$ has a double line L which is the image under π_p of a conic in S lying on a 2-plane containing p. Moreover $\tilde{S}$ is the intersection of a pencil of quadrics, all of which are singular, being the cones over $\tilde{S}$ through the points $q \in L$.

c) p lies on S. Here $\tilde{S}$ is the Steiner surface,

$$\tilde{S} = X_{21} \subset \mathbb{P}^4$$

and, as we have noticed, X_{21} lies on a net of quadrics, all of which are singular. In conclusion $\tilde{Q}$ is singular and therefore Q is of rank less than or equal through 4. Q.E.D.

We can now prove the following

(3.2) Proposition: *Let $S \subset \mathbb{P}^5$ be the Veronese surface. Then $\mathcal{W}_S(4) \subset |\mathcal{I}_S(2)|$ is a cubic hypersurface.*

Proof: Recall that in the linear system

$$|\mathcal{O}_{\mathbb{P}^5}(2)| \cong \mathbb{P}^{20}$$

the locus $\mathcal{W}(5)$ of singular quadrics is a sextic which is singular along the locus $\mathcal{W}(4)$. Moreover a quadric Q of rank 4 is a *double* point for $\mathcal{W}(5)$ and the tangent cone to $\mathcal{W}(5)$ at Q consists of the quadrics tangent to the vertex of Q. By (3.1) we have

$$|\mathcal{I}_S(2)| \bigcap \mathcal{W}(5) \subseteq \mathcal{W}(4)$$

so that

$$|\mathcal{I}_S(2)| \bigcap \mathcal{W}(4) = \mathcal{W}_S(4).$$

Moreover by (a) and (b) above we know that

(3.3) $$\mathscr{W}_S(4) \underset{\neq}{\supset} \mathscr{W}_S(3).$$

Since $\mathscr{W}(5)$ is a sextic double along $\mathscr{W}(4)\setminus\mathscr{W}(3)$ it now suffices to show that a general line in $|\mathscr{I}_S(2)|$ meets $\mathscr{W}(5)$ in $\mathscr{W}(4)\setminus\mathscr{W}(3)$ and transversly there. Because of (3.3) a general line in $|\mathscr{I}_S(2)|$ meets $\mathscr{W}_S(4)$ outside $\mathscr{W}_S(3)$. Finally since there is at most one quadric through S with a given line as a vertex, the quadrics whose vertices are tangent lines to S form a family of dimension at most 3. Therefore a general line in $|\mathscr{I}_S(2)|$ will not be contained in any tangent cone to $\mathscr{W}(5)$ at a point of $\mathscr{W}(4)\setminus\mathscr{W}(3)$, proving the transversality statement.

4. Tangent cones to theta-divisors

In this section we are going to prove Theorem (1.7). Before doing this we need to make some preliminary remarks. Let

$$C \subset \mathbb{P}^{g-1}$$

be a non-hyperelliptic canonical curve of genus g. Let D be a divisor on C of degree $d \leqq g-1$ with $h^0(C, \mathcal{O}(D)) = r+1 \geqq 2$. Consider a 2-dimensional subspace

$$V \subseteq H^0(C, \mathcal{O}(D))$$

and the corresponding pencil

$$\mathbb{P}V \subseteq |D| = \mathbb{P}H^0(C, \mathcal{O}(D))$$

we then define

(4.1) $$X_V = \bigcup_{D' \in \mathbb{P}V} \overline{D'} \subset \mathbb{P}^{g-1}$$

and

(4.2) $$X_D = \bigcap_{[V] \in \mathrm{Gr}(2,\, r+1)} X_V \subset \mathbb{P}^{g-1}.$$

Clearly

(4.3) $$X_V \supseteq X_D \supset C$$

and, if $h^0(C, \mathcal{O}(D)) = 2$

(4.4) $$X_V = X_D.$$

It is shown in [4], p. 345, that, in this case X_D is a rational normal scroll of dimension $d-1$.

The proof of Theorem (1.7) is based on the following two lemmas.

(4.5) LEMMA: *Let $|\Delta|$ be a complete linear series on C of dimension $r \geqq 1$. Let*

$$F_{\mathcal{O}(\Delta)} = \{[V] \in \mathrm{Gr}(2, H^0(C, \mathcal{O}(\Delta))) : \mathbb{P}V \subset |D| \text{ is a pencil with } r-1 \text{ base points } p_1, \ldots, p_{r-1} \text{ and } h^0(C, \mathcal{O}(\Delta - \Sigma p_1)) = 2\}.$$

Then the image of $F_{\mathcal{O}(\Delta)}$ under the Plücker embedding of $\mathrm{Gr}(2, r+1)$ *is non-degenerate.*

PROOF: For any divisor $E \in |\Delta|$ the pencils containing E form a linear subspace

$$\Lambda_E \subset \mathrm{Gr}(2, r+1), \quad \Lambda_E \cong \mathbb{P}^{r-1}.$$

It will then suffice to show that for a general $E \in |\Delta|$ the intersection of Λ_E with $F_{\mathcal{O}(\Delta)}$ is non-degenerate. Let B be the fixed divisor of $|\Delta|$ and set

$$\deg B = d - k$$

so that

$$k \geqq r + 1.$$

Let E be a general divisor in $|\Delta|$ so that

$$E = B + p_1 + \cdots + p_k.$$

Since any r of the points p_α impose independent conditions on $|\Delta|$ the intersection $\Lambda_E \cap F_{\mathcal{O}(\Delta)}$ consists exactly of the pencils in $|\Delta|$ with $r-1$ base points from among the p_α's. Finally if the intersection $\Lambda_E \cap F_{\mathcal{O}(\Delta)}$ were degenerate there would exist a proper linear subsystem

$$H \subset |\Delta|$$

containing every divisor in $|\Delta|$ which contains $r-1$ of the points p_α.

But this is not possible. In fact since the points p_α's impose independent conditions on $|\Delta|$, for at least one p_α the series $|\Delta - p_\alpha|$ does not lie in H. Since any $r-1$ of the points $\{p_\beta\}_{\beta\neq 2}$ impose independent conditions on $|\Delta - p_\alpha|$, for at least one p_β the series $|\Delta - p_\alpha - p_\beta|$ does not lie in H, and so on. Q.E.D.

(4.6) LEMMA: *Let $|D|$ be a complete linear series on C with $h^0(C, \mathcal{O}(D)) = 2$. Then the linear system of quadrics containing the scroll X_D is spanned by projective tangent cones to $\Theta \subset T(C)$ at double points, i.e.*

$$\mathcal{W}_{C,\theta} \bigcap |\mathcal{I}_{X_D}(2)| \quad \textit{spans} \quad |\mathcal{I}_{X_D}(2)|.$$

PROOF: We denote by L *the hyperplane bundle on* X_D and let

$$|E| = |\overline{D}|$$

be the pencil of $(d-2)$-planes at X_D. We then have an identification of $(g-d+1)$-dimensional vector spaces

$$H^0(X_D, L(-E)) = H^0(C, K_C(-D)).$$

Recalling (2.17), Proposition (2.18) and (2.19) the natural map

$$q : \mathrm{Gr}(2, g-d+1) \to |\mathcal{I}_{X_D}(2)| \subset |\mathcal{I}_C(2)| \tag{4.7}$$

coincides with the Plücker embedding and the image is the *principal locus* of $\mathcal{W}_{X_D}(4)$. On the other hand, by Theorem (1.4), given a point $[V] \in \mathrm{Gr}(2, g-d+1)$ then

$$q[V] \in \mathcal{W}_{C,\theta}$$

if and only if the pencil $\mathbb{P}V \subset |K_C(-D)|$ has $g-d-1$ base points $p_1, \ldots, p_{g-d-1}$ and

$$h^0\Big(C, \mathcal{O}\Big(D + \sum p_i\Big)\Big) = 2.$$

Setting $\mathcal{O}(\Delta) = K_C(-D)$, the lemma follows now from Lemma (4.5).

We are now going to prove Theorem (1.7).

PROOF OF THEOREM (1.7): We must prove that

$$\overline{\mathscr{W}_{C,\theta}} \supseteq \overline{\mathscr{W}_C(4)}. \tag{4.8}$$

Given a quadric Q of rank 4 (resp. 3) one of its rulings (resp. its ruling) cuts out on C a pencil

$$\mathbb{P}V \subseteq |D|$$

where D is a divisor of degree $d \leqq g-1$ and V a 2-dimensional subspace of $H^0(C, \mathcal{O}(D))$. Recalling the definitions (4.1) and (4.2) we have

$$Q \supseteq X_V \supseteq X_D \supset C.$$

It therefore suffices to show that

$$\overline{\mathscr{W}_{C,\theta}} \supset \overline{\mathscr{W}_C(4)} \bigcap |\mathscr{I}_{X_D}(2)|. \tag{4.9}$$

Exactly as in Section 1, given two linearly independent sections s_0 and s_1 of $\mathcal{O}(D)$ the multiplication by s_0 and s_1, respectively, gives injective homomorphisms

$$\alpha_{s_0}: H^0(C, K_C(-D)) \to H^0(C, K_C)$$

$$\alpha_{s_1}: H^0(C, K_C(-D)) \to H^0(C, K_C).$$

We also have a linear map

$$\alpha_{s_0,s_1}: \wedge^2 H^0(C, K_C(-D)) \to S^2 H^0(C, K_C)$$

defined by

$$\alpha_{s_0,s_1}(t_0 \wedge t_1) = \alpha_{s_0}(t_0) \otimes \alpha_{s_1}(t_1) - \alpha_{s_0}(t_1) \otimes \alpha_{s_1}(t_0). \tag{4.10}$$

Clearly

$$\operatorname{Im} \alpha_{s_0 s_1} \subset \mathscr{W}_C(4) \bigcap |\mathscr{I}_{X_D}(2)|$$

and we can define a morphism

$$h: \operatorname{Gr}(2, \mathrm{r}+1) \times \operatorname{Gr}(2, g-d+r+1) \to |\mathscr{I}_{X_D}(2)|$$

by letting

$$h\left(\overline{s_0 s_1}, \overline{t_0 t_1}\right) = [\alpha_{s_0,s_1}(t_0 \wedge t_1)]$$

where $\overline{s_0 s_1}$ (resp. $\overline{t_0 t_1}$) denotes the 2-plane generated by s_0 and s_1 (resp. t_0, t_1). From the definition of X_D it follows that

$$\text{(4.11)} \qquad \operatorname{Im} h = W_C(4) \bigcap |\mathcal{I}X_D(2)|.$$

Of course starting from two linearly independent sections t_0, t_1 of $K_C(-D)$ and from the multiplication maps

$$\beta_{t_0}: H^0(C, \mathcal{O}(D)) \to H^0(C, K_C)$$

$$\beta_{t_1}: H^0(C, \mathcal{O}(D)) \to H^0(C, K_C)$$

we could have defined, in complete analogy with (4.12), a linear map

$$\beta_{t_0,t_1}: \wedge^2 H^0(C, \mathcal{O}(D)) \to S^2 H^0(C, K_C),$$

and it is immediate to check that

$$h(\overline{s_0 s_1}, \overline{t_0 t_1}) = [\alpha_{s_0,s_1}(t_0 \wedge t_1)] = [\beta_{t_0,t_1}(s_0 \wedge s_1)].$$

This shows that the restriction maps

$$h \Big| \{\overline{s_0, s_1}\} \times \operatorname{Gr}(2, r+1), \quad h \Big| \operatorname{Gr}(2, g-d+r-1) \times \{\overline{t_0, t_1}\}$$

are obtained by composing the Plücker embedding with a linear map. It then follows from Lemma (4.5) that

$$\text{(4.13)} \qquad \overline{h(F_{\mathcal{O}(D)} \times F_{K_C(-D)})} = \overline{\operatorname{Im} h}.$$

On the other hand from Lemma (4.6) it follows that

$$\text{(4.16)} \qquad h(F_{\mathcal{O}(D)} \times F_{K_C(-D)}) \subset \overline{\mathcal{W}_{C,\theta}}.$$

The relation (4.9), and therefore Theorem (1.7) follows now from (4.11), (4.13) and (4.14).

Q.E.D.

5. Curves of genus $g \leqq 6$

In this section we shall prove Theorem (1.8). In view of Theorem (1.7) and of the fact that the trigonal case has been extensively studied in [1], it suffices to prove the following.

(5.1) THEOREM: *Let C be a non-hyperelliptic, non-trigonal, canonical curve of genus $g \leqq 6$, then*

$$\overline{\mathcal{W}_C(4)} = |\mathcal{I}_C(2)|. \tag{5.2}$$

PROOF: Since for $g \leqq 4$ the curve C is trigonal we only have to consider two cases, and the case $g = 5$ is more or less trivial.

Let then C be a non-hyperelliptic, non-trigonal canonical curve of genus 5. C is then a complete intersection of three quadrics, so that

$$|\mathcal{I}_C(2)| \cong \mathbb{P}^2.$$

On the other hand the locus

$$\mathcal{W}(4) \subseteq |\mathcal{O}_{\mathbb{P}^4}(2)| \cong \mathbb{P}^{14}$$

of singular quadrics in $\mathbb{P}^4$, is a quintic hypersurface. Therefore the only way that $\mathcal{W}_C(4) = \mathcal{W}(4) \cap |\mathcal{I}_C(2)|$ could fail to span $|\mathcal{I}_C(2)|$ is if $\mathcal{W}_C(4)$ were a line, and in this case there would be no point at which $\mathcal{W}(4)$ and $|\mathcal{I}_C(2)|$ would meet transversely. Let us show that this cannot happen. Since the theta-characteristics are in finite number and since $\dim \Theta_{sg} = 1$, there exists a quadric Q, through C, of rank equal to 4. It is well known, and easy to see, that the tangent hyperplane H to $\mathcal{W}(4)$ at Q is the linear system of quadrics passing through the vertex p of Q. To say that $\mathcal{W}(4)$ does not meet the 2-plane $|\mathcal{I}_C(2)|$ transversally at the point corresponding to Q means that $|\mathcal{I}_C(2)| \subset H$. This implies that every quadric through C contains P so that p lies on C. This however cannot be the case: the projection π_p, from p, would map C to a septic curve in $\mathbb{P}^3$ lying on the quadric $\overline{Q} = \pi_p(Q)$, and at least one of the rulings of $\overline{Q}$ would cut on C a pencil of degree 3 or less. This is contrary to our assumptions. Therem (5.1) is therefore proved in case $g = 5$.

Now, and for the rest of this paper, we turn our attention to curves of genus 6.

Let then

$$C \subset \mathbb{P}^5$$

be a *non-hyperelliptic, non-trigonal canonical curve of genus* 6.

From the fundamental theorem on the existence of special divisors, (see [7] and [4], p. 358) we know that there exists, on C, a complete linear series $|D|$ such that

$$\deg D = 6, \dim |D| \geqq 2. \tag{5.3}$$

Since C is non-hyperelliptic, Clifford's theorem implies that

$$\dim |D| = 2.$$

Let then

$$\varphi : C \to \varphi(C) \subset \mathbb{P}^2 \tag{5.4}$$

be the morphism defined by $|D|$. We claim that, under our hypotheses, only the following three cases can occur

a) *$\varphi(C)$ is a smooth plane quintic and φ is an isomorphism.*

b) *$\varphi(C)$ is a smooth plane cubic and φ is a 2-sheeted ramified covering.*

c) *$\varphi(C)$ is an irreducible plane sextic with no point of multiplicity greater than 2 and φ is a birational map.*

Indeed the hypothesis that C is non-hyperelliptic implies that the series $|D|$ has, at most, one fixed point. If $|D|$ has one fixed point we are obviously in case (a). Suppose then that $|D|$ has no fixed point. In this case $\varphi(C)$ could, a priori, be an irreducible conic, an irreducible cubic or an irreducible sextic. Certainly the first case cannot occur since, otherwise, φ would exhibit C as a trigonal curve. If $\varphi(C)$ is an irreducible cubic, it must also be non-singular since otherwise C would be, via φ, a 2-sheeted covering of a rational curve. Therefore if $\varphi(C)$ is a cubic we are in case (b). Suppose finally that $\varphi(C)$ is an irreducible sextic. Then φ is a birational map. By the genus formula $\varphi(C)$ cannot have points of multiplicity greater than three. On the other hand if $\varphi(C)$ had a point p of multiplicity three, the preimage under φ of the variable part of the series cut out on $\varphi(C)$ by the pencil of lines through p would be a g^1_3 on C, contrary to our hypothesis. This means that we are in case (c).

We are now going to prove Theorem (5.1) in each of the cases (a), (b), (c).

Case a: If $\varphi(C)$ is a smooth plane quintic the canonical series on

$\varphi(C)$ is cut out by conics in $\mathbb{P}^2$. The canonical map

$$\varphi(C) \to C \subset \mathbb{P}^5$$

is then the restriction to $\varphi(C)$ of the Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$. Therefore C lies on a Veronese surface S. Moreover the linear system of quadrics through C is just the linear system of quadrics through S, and, as we have seen in Proposition (3.2), $\mathcal{W}_S(4)$ is a cubic hypersurface in this linear system. This proves Theorem (5.1) in case (a).

Case b: In this case

$$\varphi : C \to \varphi(C) = E \subset \mathbb{P}^2 \tag{5.5}$$

is a 2-sheeted covering of a plane non-singular (elliptic) cubic E. Given a point $r \in E$, set

$$\varphi^*(r) = p + q$$

and let ℓ_r be the linear span, in $\mathbb{P}^5$, of the divisor $p + q$ (i.e. the line joining p and q, if p and q are distinct, the tangent line to C at p, if $p = q$). Consider the surface

$$S = \Big(\bigcup_{r \in E} \ell_r\Big) \subset \mathbb{P}^5.$$

Let r' be a point in E and set

$$\varphi^*(r') = p' + q'.$$

Notice that

$$h^0(C, \mathcal{O}(p + q + p' + q')) = h^0(E, \mathcal{O}(r + r')) = 2.$$

Therefore, by the Riemann-Roch theorem, we conclude that the points p, q, p', q' all lie in a 2-plane. This implies that any two pair of lines $\ell_r, \ell_{r'}$ must meet. Since C is non-degenerate, this can happen only if all the lines ℓ_r's issue from a common point $p \in \mathbb{P}^5 \setminus C$. Since projection from p gives a two-to-one map of C onto an elliptic curve $\tilde{E} \subset \mathbb{P}^4$ of degree

$$\deg \tilde{E} = \frac{1}{2} \deg C = 5,$$

we conclude that *S is a cone over an elliptic quintic curve $\tilde{E}$ contained in $\mathbb{P}^4$.*

Now, the linear system of quadrics through S is readily described. To begin with, any quadric containing S is singular at p, and hence a cone over a quadric $\tilde{Q} \subset \mathbb{P}^4$ containing $\tilde{E}$. On the other hand $\tilde{E}$ lies on, and is cut out by, a four dimensional linear system of quadrics (projection from any point $q \in \tilde{E}$ maps $\tilde{E}$ to the complete intersection of two quadrics in $\mathbb{P}^3$). By an argument analagous to that given in the case of a genus 5 canonical curve, the singular elements of this system, (i.e. the quadrics of rank less than or equal to 4 through $\tilde{E}$) from a non-degenerate quintic hypersurface

$$\sum{}' \subset |\mathcal{I}_{\tilde{E}}(2)| \cong \mathbb{P}^4.$$

We then see that the cone S is cut out by a *quintic threefold of quadrics of rank less than or equal to* 4:

$$\sum \subset |\mathcal{I}_S(2)| \cong \mathbb{P}^4.$$

Specifically, these are the ∞^1 2-planes of quadrics corresponding to the ∞^1 g^1_4's on C pulled back from the g^1_2's on E. We then have

$$\sum \subset \mathcal{W}_C(4) \subset |\mathcal{I}_C(2)| \cong \mathbb{P}^5 \tag{5.6}$$

$$\overline{\sum} \cong \mathbb{P}^4. \tag{5.7}$$

We now ask: are there quadrics of rank less than or equal to 4 containing C other than those containing S, or have we accounted for all special linear series on C? The answer is that there are others. Let p_1, p_2, p_3 be three general points on C (in particular no two of them lying over the same point of E). Let

$$\varphi^*(\varphi(p_i)) = p_i + q_i, \; i = 1, 2, 3.$$

The projection π from the 2-plane spanned by p_1, p_2 and p_3 maps C to a plane septic curve Γ. This plane septic must then have singularities other than the triple point $\pi(q_1) = \pi(q_2) = \pi(q_3)$. If p_4 and p_5 are two points on C, different from the q_i's, and mapping to the same point in the plane, i.e. spanning, together with p_1, p_2 and p_3 only a 3-plane, we see that the divisor $p_1+p_2+p_3+p_4+p_5$ moves in a pencil which does not factor through the map (5.4). Thus the quadric

Q through C, (of rank less than or equal to 4) corresponding to this pencil is not singular at p, and so does not contain S. Therefore this quadric correspond to a point

$$x \in \mathscr{W}_C(4)$$

such that

$$x \notin \overline{\Sigma}.$$

From (5.6) and (5.7) it follows that the point x together with Σ span $|\mathscr{I}_C(2)|$, proving the theorem in case (b).

Case c: In this case the morphism

$$\varphi : C \to \varphi(C) = \Gamma \subset \mathbb{P}^2$$

maps C birationally onto a plane irreducible sextic having no point of multiplicity greater than two. In general the curve Γ will be a sextic with four ordinary double points (nodes), $p_1, \ldots, p_4$, no three of which are collinear. Let us start by studying this general situation.

In this case Γ prossesses 5 *distinct* g^1_4's; the ones cut out by the pencils of lines through each of the points p_i's and the one cut out by the pencil of conics through $p_1, \ldots, p_4$.

Let

$$|D_1|, \ldots, |D_5|$$

denote these five g^1_4's, (by [7] we know that these are the only g^1_4's on C).

The *adjoint linear system* of Γ, i.e. the linear system of cubics through $p_1, \ldots, p_4$, cuts out on Γ the canonical series and also defines a birational map

$$\psi : \mathbb{P}^2 \to S \subset \mathbb{P}^5$$

onto a *Del Pezzo surface* S of degree 5 containing the canonical curve C, (here and in the sequel, by a Del Pezzo surface we shall mean the, possibly singular, image of a rational surface under its anticanonical map).

As in the previous cases our first question will concern the quadrics of rank less than or equal to 4 containing S. To see what these are we look at the five g_4^1's on Γ which we previously described, and consider the corresponding scrolls

$$X_{D_i} = \left(\bigcup_{\Delta \in |D_i|} \overline{\Delta} \right) \subset \mathbb{P}^5, \; i = 1, \ldots, 5.$$

We claim that

(5.8) *Each of the scrolls X_{D_i} contains the Del Pezzo surface S.*

This is readily seen: under the map ψ lines ℓ in $\mathbb{P}^2$ through p_i are carried into plane conics and the corresponding scroll X_{D_i} is swept out by the 2-planes spanned by the four points on C which correspond to the four points of intersection of ℓ with Γ other than p_i, i.e. X_{D_i} is swept out by the planes of the conics in this pencil. Likewise, the conics in $\mathbb{P}^2$ through all four of the points p_i's are mapped to conics in $\mathbb{P}^5$, and the union of their span is the scroll X_{D_5} associated to the pencil they cut out on Γ.

Our second observation is that

(5.9) *The intersection of any two of the scrolls X_{D_i} is just the surface S.*

Let X_1 and X_2 be any two of the five scrolls X_{D_i}'s. Let $V \subset \mathbb{P}^5$ be a general 3-plane containing a point $p \in \mathbb{P}^5 \setminus S$. Each of the scrolls $X_i, i = 1, 2,$ intersects V in a twisted cubic curve $F_i \subset V$. Since $X_1 \neq X_2$ these two twisted cubics are distinct. But we know that $F_1 \cap F_2$ contains the five points of intersection of V with the quintic surface S. If F_1 and F_2 had a sixth point in common they would be equal. Thus F_1 meets F_2 only in the points of $V \cap S$. Hence $p \in X_1 \cap X_2$. Q.E.D.

We then see that the linear system

$$|\mathcal{I}_S(2)| \cong \mathbb{P}^4$$

contains five 2-planes of quadrics of rank less than or equal to 4

$$(\pi_1 \cup \ldots \cup \pi_5) \subset |\mathcal{I}_S(2)|$$

these being the nets of quadrics through the five scrolls X_{D_i}'s. We next note that these are all the quadrics of rank less than or equal to 4 through S, i.e.

$$\pi_1 \cup \ldots \cup \pi_5 = \mathcal{W}_S(4). \tag{5.10}$$

In fact if $Q \subset \mathbb{P}^5$ is any quadric of rank 4 (resp. 3) through S, its two rulings (resp. its ruling) cut (resp. cuts) out on S two pencils (resp. one pencil) of divisors, the sum of whose degrees (resp. the double of whose degree) is at most five. Therefore one of these pencils (resp. this pencil) must consist of conics. But the only pencil of conics on the Del Pezzo S are the images, under ψ, of the pencils of lines through the p_i's, and of the pencil of conics through all four p_i's. If the planes of our ruling (resp. of the ruling) of Q cut out one of these pencils then Q must contain the corresponding scroll X_{D_i}, proving (5.9).

The five planes $\pi_1, \ldots, \pi_5$ meet pairwise in points. These ten points correspond to the ten quadrics that are obtained by projecting S from any of the 10 lines on S, i.e., whose vertex lies on S. From this and (5.10) it follows that

$$\mathbb{P}^5 \cong |\mathcal{I}_C(2)| \supseteq \overline{\mathcal{W}_C(4)} \supseteq \overline{\mathcal{W}_S(4)} \cong \mathbb{P}^4.$$

Finally observe that since C is of genus 6 then

$$\dim \Theta_{sg} = 2.$$

On the other hand by the second part of Theorem (1.4), by the proof of Lemma (4.5) and by our description of $\mathcal{W}_S(4)$, we see that only ∞^1 projectivized tangent cones to Θ, at double points, are among the quadrics containing S. Therefore there is a quadric Q of rank 4 containing C but not S. Let

$$x \in \mathcal{W}_C(4) \tag{5.12}$$

be the point corresponding to Q. Then

$$x \notin \overline{\mathcal{W}_S(4)}.$$

This together with (5.11) implies that

$$\overline{\mathcal{W}_C(4)} = |\mathcal{I}_C(2)|$$

proving Theorem (5.1) in this case.[1]

Consider now how the plane sextic Γ may degenerate and how this will affect our argument. Eliminating the possibility that Γ acquires a triple point, a possiblility which is excluded in case (c), we see that under any degeneration it is still the case that the adjoint system of Γ maps the plane birationally to a quintic Del Pezzo surface $S \subset \mathbb{P}^5$, containing the canonical curve C. It is also the case that the quadrics of rank less than or equal to 4 through S are exactly the quadrics containing one of the threefold scrolls X_D corresponding to the g_4^1's on C. Again, as in (5.9), any two of the scrolls X_D can intersect only in S so that: *the above argument continues to hold as long as C possesses two or more g_4^1's.*

We then see that the only curves C for which the above argument fails to work are those for which the five g_4^1's all come together or equivalently those for which the plane model Γ of C is a *sextic with four infinitely near double points three of which are collinear.*

It remains then to treat this one last case. Unfortunately, this requires a somewhat more delicate analysis than the previous ones, since the (degenerate) Del Pezzo surface S containing C is *not* cut out by quadrics of rank less than or equal to 4. Indeed by our previous analysis (see the proof of (5.8)) the only quadrics of rank less than or equal to 4 containing S contain the threefold scroll X_D associated to the unique

$$g_4^1 = |D|$$

on C.

We start with the scroll X_D. As before of the net of quadrics containing X_D only ∞^1 are actually projectivized tangent cones to Θ, so that there must be a projectivized tangent cone

$$Q = \mathbb{P}TC_\lambda(\Theta)$$

for some (double) point $\lambda \in \Theta_{sg}$, *not* containing the scroll X_D. Let

$$T = X_D \cap Q \tag{5.13}$$

[1] It is amusing to note that the sixth component of $\mathcal{W}_C(4)$ (the one coming from Θ_{sg}) meets each of the planes π_i in a sextic curve, these curves are in fact the five images of C in $\mathbb{P}^2$ under the maps given by the nets $|K_C(-D_i)|$, $i = 1, \ldots, 5$.

be the surface of intersection. The procedure now will be to study T, until we know enough to conclude that the quadrics of rank less than or equal to 4 through C do indeed span $|\mathcal{I}_C(2)|$.

To begin with we claim that:

(5.14) *C does not lie on any surface Σ of degree* 4 *or* 5 *except the degenerate Del Pezzo surface S.*

To see this note that a general hyperplane section $S' = H \cap S$ of S is an elliptic normal curve which is, as mentioned above, cut out by quadrics. If the hyperplane section $\Sigma' = H \cap C$ had degree 4, every quadric containing S', and hence meeting Σ' in the 10 points of $(H \cap C) \subset (S' \cap \Sigma')$, would contain Σ'. Similarly, if Σ' had degree 5, any quadric containing S' and one point $p \in \Sigma' \setminus S$ would contain Σ'. Since S' lies on five linearly independent quadrics this means that there are four linearly independent quadrics containing both S' and Σ'. This is impossible: three quadrics in $\mathbb{P}^4$ whose intersection contains an irreducible non-degenerate curve of degree

$$\deg S' + \deg \Sigma' = 10 > 8$$

intersect in an irreducible non-degenerate surface, whose degree is necessarily 3. The fourth quadric, then, would cut this surface in a curve of degree at most 6. This contradiction proves (5.14).

Let us go back to the surface T defined in (5.13). Since every quadric of rank less than or equal to 4 containing S also contains X_D, the surface T cannot contain S. Therefore, by (5.14) we conclude that:

(5.15) *T is an irreducible surface of degree* 6.

We now look at the pencil of curves cut out on T by the 2-planes of X_D. We ask whether they may all be reducible, that is if T may be projectively ruled. If this were the case, each line would have to meet C twice. This in turn would imply that singular points of the conics of the pencil are variable (if two lines from different planes of X_D met, there would be a second g^1_4 on C) so that T would be singular along a curve. But then the general hyperplane section of T, which is, birationally, the base of the ruled surface T, would be a singular sextic curve in $\mathbb{P}^4$ and so a curve of genus 0 or 1. Therefore C would be either hyperelliptic or elliptic-hyperelliptic, contrary to our hypothesis.

The conclusion, then, is that the conics cut out on T by the 2-planes of X_D are generically irreducible. Therefore by Noether's Lemma we have that:

(5.16) *T is rational.*

Next we note that if T were the regular projection of a non-degenerate surface $\tilde{T} \subset \mathbb{P}^6$, the inverse image $\tilde{C} \subset \tilde{T}$ of C, would still be a canonical curve of genus 6 and hence lie in a hyperplane section of $\tilde{T}$. But the degree of $\tilde{C}$ would be again given by

$$2g - 2 = 10 > \deg T = \deg \tilde{T}$$

and this is absurd. We conclude that:

(5.17) *T is not the regular projection of a surface $\tilde{T} \subset \mathbb{P}^6$.*

Finally we may use the above properties to determine the genus of a general hyperplane section

$$E = H \cap T$$

of T. Let

$$\pi : \tilde{T} \to T$$

denote the desingularization of T, and set

$$\tilde{E} = \pi^{-1} E.$$

Consider the exact sheaf sequence

$$0 \to \mathcal{O}_{\tilde{T}} \to \mathcal{O}_{\tilde{T}}(\tilde{E}) \to \mathcal{O}_{\tilde{E}}(\tilde{E}) \to 0. \qquad (5.18)$$

The linear system $|\mathcal{O}_{\tilde{E}}(\tilde{E})|$, has degree 6 and dimension at least 4; thus by Clifford's theorem is non-special. From (5.16) we get

$$h^1(\tilde{T}, \mathcal{O}_{\tilde{T}}) = 0$$

while (5.17) gives

$$h^0(\tilde{T}, \mathcal{O}_{\tilde{T}}(\tilde{E})) = 6.$$

Therefore the long exact cohomonology sequence of (5.18) gives

$$h^0(\tilde{E}, \mathcal{O}_{\tilde{E}}(\tilde{E})) = 5.$$

Applying the Riemann-Roch theorem, we find that the genus of $\tilde{E}$ is equal to 2. It follows that

$$\deg |\mathcal{O}_{\tilde{E}}(\tilde{E})| = 6 > 2g(\tilde{E}) + 1$$

so that the complete linear series $|\mathcal{O}_{\tilde{E}}(\tilde{E})|$ gives an *embedding* of $\tilde{E}$. We can then conclude that:

(5.19) *The general hyperplane section E of T is a smooth curve of genus* 2.

At this point we may quote a result of Castelnuovo (see [2] and [8], p. 155) which gives us a complete description of T:

(5.20) *Any surface T satisfying* (5.15), (5.16), (5.17) *and* (5.19) *is the image of $\mathbb{P}^2$, in $\mathbb{P}^5$, under the rational map η given by a fixed-component-free linear system of plane quartics having a double point q and passing through six points $p_1, \ldots, p_6$.*

The surface T, being the complete intersection of X_D with the (rank 4) quadric Q, lies on a 3-dimensional linear system of quadrics. We then have the following picture

(5.21)
$$\begin{array}{ccccc} |\mathcal{I}_{X_D}(2)| & \subset & |\mathcal{I}_T(2)| & \subset & |\mathcal{I}_C(2)| \\ \wr\| & & \wr\| & & \wr\| \\ \mathbb{P}^2 & & \mathbb{P}^3 & & \mathbb{P}^5 \end{array}$$

We now ask what are the quadrics of rank less than or equal to 4 in $|\mathcal{I}_T(2)|$. Again to answer this question we look for pencils of curves of low degree on T. We easily find that

(i) T contains no pencil of lines.
(ii) T contains one pencil of conics. This is the pencil cut out by the 2-planes of X_D, or, in other terms, the images under the map η of the pencil of lines through the point q.
(iii) T contains finitely many pencils of twisted cubic curves. These are the images under η of the pencils of lines through each of the points p_i, $i = 1, \ldots, 6$, of the pencils of conics through q and three of

the points p_i's, and of the pencils of cubics, double at q and passing through five of the points p_i's.
(iv) T contains no pencil of plane cubics.

Now if Q' is any quadric of rank less than or equal to 4 containing T, the planes of at least one of its rulings must cut out on T a pencil of curves of degree 3 or less. If this pencil is the pencil of conics, then Q' simply belongs to the net of quadrics through X_D. On the other hand, if this pencil is of degree 3, then it determines Q'. The conclusion then is that, apart from the net $|\mathcal{I}X_D(2)|$, T lies on only finitely many quadrics of rank less than or equal to 4. We therefore have

(5.22) $$\mathcal{W}_T(4) = |\mathcal{I}X_D(2)| \cup \{\text{finite set}\}$$

(5.23) $$\overline{\mathcal{W}_T(4)} = |\mathcal{I}_T(2)| \cong \mathbb{P}^3.$$

In particular we also have that:

(5.24) *The point*

$$x \in \mathcal{W}_{C,\theta}(4) \cap |\mathcal{I}_T(2)|$$

corresponding to the quadric Q is one of the isolated points of $\mathcal{W}_T(4)$.

In view of (5.21) and (5.23) in order to conclude the proof of Theorem (5.1) it suffices to show that $\mathcal{W}_{C,\theta}$ and $|\mathcal{I}_T(2)|$ are not both contained in a hyperplane of $|\mathcal{I}_C(2)| \cong \mathbb{P}^5$. But this is clear, since otherwise $\mathcal{W}_{C,\theta}$, which is of pure dimension 2, would not intersect the 3-plane $|\mathcal{I}_T(2)|$ in any isolated point, contrary to what we just proved in (5.24). The proof of Theorem (5.1) is now complete.

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Note: During the preparation of this manuscript, the authors found out that theorems (1.7) and (1.8) for curves of genus $g = 5$ had been proved independently by Makoto Namba. These appear in his book *Families of Meromorphic Functions on Compact Riemann Surfaces.* (Lecture notes #767, Springer-Verlag, 1979) as Propositions 2.5.10 (p. 105) and Theorem 2.6.4.

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